Variational Iteration Method for Special Nonlinear Partial Differential Equations

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Abstract

In this paper, we applied the Variational iteration method (VIM) is applied for solving nonlinear parabolic-hyperbolic partial differential equations. Some illustrative examples of one-dimensional and two-dimensional are presented to show the efficiency of the method.

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1 Introduction

In 1999, the variational iteration method (VIM) was first proposed by Ji-Huan He [1,2]. The idea of the VIM is to construct an iteration method based on a correction functional that includes a generalized Lagrange multiplier. The value of the multiplier is chosen using variational theory so that each iteration improves the accuracy of the solution. The initial approximation (trial function) usually includes unknown coefficients which can be determined to satisfy any boundary and initial conditions. This method is now widely used by many researchers to study linear and nonlinear partial differential equation [8-17]. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise a few approximations can be used for numerical purposes.

In this paper, we applied the VIM for solving Cauchy problem for the nonlinear parabolic-hyperbolic partial differential equation of the following
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{\partial^2}{\partial t^2} - \Delta \right) u = F(u)
\]  
(1)

where the nonlinear term is represented by \( F(u) \), and \( \Delta \) is the Laplace operator in \( \mathbb{R}^n \).

2 Variational Iteration Method

Consider the following differential equation

\[
Lu + Nu = g(x,t)
\]  
(2)

where \( L \) is a linear operator, \( N \) is a nonlinear operator and \( g(x,t) \) is a known real function. According to VIM [1-7], we can construct a correction functional, \( u(x,t) \), as follows:

\[
u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left( Lu_n(x,s) + Nu_n(x,s) - g(x,s) \right) ds
\]  
(3)

where \( \tilde{u}_n(x,t) \) is a correction functional but \( \tilde{u}_n(x,t) \) is considered as a restricted variation [7,8], i.e. \( \delta \tilde{u}_n(x,t) = 0 \). The subscript \( n \) denotes the \( n \)th-order approximation. The optimal value of the general Lagrange multipliers \( \lambda \) [8] can be identified by using the stationary conditions of the variational theory.


Consider nonlinear parabolic-hyperbolic partial differential equation.

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{\partial^2}{\partial t^2} - \Delta \right) u = F(u)
\]  
(4)

To solve Eq.(4) by means of VIM, we construct a correction functional,

\[
u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s) \left( \frac{\partial}{\partial s} - \Delta \right) \left( \frac{\partial^2}{\partial s^2} - \Delta \right) u - F(u) \right) ds
\]  
(5)
Variational iteration method

\[ \delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(s) \left[ \left( \frac{\partial}{\partial s} - \Delta \right) \left( \frac{\partial^2}{\partial s^2} - \Delta \right) u - F(u) \right] ds \]

\[ \delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(s) \left[ (u_n)_s - c_1 u_n \right] ds \]  

This yields the stationary conditions

\[ \lambda'(s) + c_1 \lambda(s) = 0, \quad 1 + \lambda(s) \bigg|_{\text{at} \ t} \]

Consider two cases. If \( c_1 = 0 \) then \( \lambda = -1 \) and if \( c_1 \neq 0 \) then \( \lambda = -e^{-c_1(x-t)} \).

Substituting these values of the Lagrange multipliers into the functional (5) give the iteration formulas

\[ u_{n+1}(x,t) = u_n(x,t) - \int_0^t e^{-c_1(x-t)} \left[ \left( \frac{\partial}{\partial s} - \Delta \right) \left( \frac{\partial^2}{\partial s^2} - \Delta \right) u - F(u) \right] ds \]  

4. Illustrative Example

Example 1. Consider the following equation

\[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} + u \]  

with the initial conditions

\[ u(x,0) = e^x \]

The exact solution is

\[ u(x,t) = e^{x+t} \]

In VIM we can construct a correction functional for Eq. (9) as follows

\[ u_{n+1}(x,t) = u_n(x,t) - \int_0^t e^{-c_1(x-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u_n - \frac{\partial^2 u_n}{\partial x^2} - \frac{\partial^2 u_n}{\partial t^2} + \frac{\partial u_n}{\partial t} - u_n \right] ds \]  

Assuming that

\[ u_0(x,t) = u(x,0) = e^x \]

By means of the above iteration formula (10), we obtain

\[ u_{i}(x,t) = u_{i-1}(x,t) - \int_0^t e^{-c_1(x-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u_{i-1} - \frac{\partial^2 u_{i-1}}{\partial x^2} - \frac{\partial^2 u_{i-1}}{\partial t^2} + \frac{\partial u_{i-1}}{\partial t} - u_{i-1} \right] ds \]

\[ = e^x - \int_0^t e^{-c_1(x-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) e^x - \frac{\partial^2 e^x}{\partial x^2} - \frac{\partial^2 e^x}{\partial t^2} + \frac{\partial e^x}{\partial t} - e^x \right] ds \]

\[ = e^{(x+t)} \]

which is the exact solution that satisfies Eq. (9) with the initial conditions.

Example 2. Consider the following equation
\[
\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u = \frac{\partial^4 u}{\partial t^4} + \frac{\partial^2 u}{\partial t^2} + u \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + 2u
\]  
(13)

with the initial conditions
\[ u(x,0) = \cos x \]

The exact solution is
\[ u(x,t) = e^{2t} \cos x \]

In VIM we can construct a correction functional for Eq.(13) as follows
\[
u_{n+1}(x,t) = u_n(x,t) - \int_0^t e^{-2(s-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u_n - \frac{\partial^4 u_n}{\partial t^4} + \frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^2 u_n}{\partial x^2} - \frac{\partial u_n}{\partial t} + 2u_n \right] ds
\]
(14)

To get the iteration, we start an initial approximation
\[ u_0(x,t) = u(x,0) = \cos x \]
(15)

By means of the above iteration formula (14), we obtain
\[
u_1(x,t) = u_0(x,t) - \int_0^t e^{-2(s-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u_0 - \frac{\partial^4 u_0}{\partial t^4} + \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial u_0}{\partial t} + 2u_0 \right] ds
\]
\[ = \cos x - \int_0^t e^{-2(s-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \cos x - \frac{\partial^4 \cos x}{\partial t^4} + \frac{\partial^2 \cos x}{\partial t^2} - \frac{\partial^2 \cos x}{\partial x^2} - \frac{\partial \cos x}{\partial t} + 2\cos x \right] ds
\]
\[ = e^{2t} \cos x
\]
(16)

which is the exact solution that satisfies Eq. (13) with the initial conditions.

Example 3 Consider the following equation
\[
\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) u = \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial t} + u
\]  
(17)

with the initial conditions
\[ u(x_1,x_2,0) = e^{x_1+x_2} \]

The exact solution is
\[ u(x,t) = e^{x_1+x_2+t} \]

In VIM we can construct a correction functional for Eq.(17) as follows
\[
u_{n+1}(x,t) = u_n(x,t) - \int_0^t e^{-2(s-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) u_n - \frac{\partial u_n}{\partial x_1} + \frac{\partial u_n}{\partial t} - u_n \right] ds
\]
(18)

Assuming that
Variational iteration method

By means of the above iteration formula (18), we obtain

$$u_0(x_1, x_2, t) = u(x_1, x_2, 0) = e^{\eta_1 x_1} + e^{\eta_2 x_2}$$

(19)

which is the exact solution that satisfies the Eq. (17) with the initial conditions.

Example 4 Consider the following equation

$$\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right] u = -3 \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_1^2} + \frac{5}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + 2u$$

(21)

with the initial conditions

$$u(x_1, x_2, 0) = \sinh x_1 + e^{\eta_2}$$

The exact solution is

$$u(x, t) = \sinh x_1 + e^{\eta_2 x_2}$$

In VIM we can construct a correction functional for Eq.(21) as follows

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t e^{-2(x-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) u_n \right.
+ \frac{3}{2} \frac{\partial^2 u_n}{\partial x_2^2} + \frac{5}{2} \frac{\partial^2 u_n}{\partial x_1^2} - \frac{\partial u_n}{\partial t} - 2u_n] \right) \, ds$$

(22)

Assuming that

$$u_0(x_1, x_2, t) = u(x_1, x_2, 0) = \sinh x_1 e^{\eta_2}$$

(23)

By means of the above iteration formula (22), we obtain

$$u_1(x, t) = u_0(x, t) - \int_0^t e^{-2(x-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) u_0 \right.
+ \frac{3}{2} \frac{\partial^2 u_0}{\partial x_2^2} + \frac{5}{2} \frac{\partial^2 u_0}{\partial x_1^2} - \frac{\partial u_0}{\partial t} - 2u_0] \right) \, ds$$

$$\sinh x_1 e^{\eta_2} - \int_0^t e^{-2(x-t)} \left[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) \sinh x_1 e^{\eta_2} 
+ \frac{3}{2} \frac{\partial^2 \sinh x_1 e^{\eta_2}}{\partial x_2^2} + \frac{5}{2} \frac{\partial^2 \sinh x_1 e^{\eta_2}}{\partial x_1^2} - \frac{\partial \sinh x_1 e^{\eta_2}}{\partial t}] \, ds$$

$$= \sinh x_1 + e^{\eta_2 x_2}$$

(24)
which is the exact solution that satisfies the Eq. (21) with the initial conditions.

5. Conclusion

In this paper, we have successfully applied the VIM in finding the exact solution for the nonlinear partial differential-hyperbolic equation. Results obtained by the method confirm the easily, accurately and efficiency of the method.

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References


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