A Structure of KK-Algebras and its Properties

S. Asawasamrit

Department of Mathematics, Faculty of Applied Science,
King Mongkut’s University of Technology North Bangkok, Bangkok 10800, Thailand
Centre of Excellence in Mathematics, CHE,
Sri Ayutthaya Road, Bangkok 10400, Thailand
suphawata@kmutnb.ac.th

A. Sudprasert

University of the Thai Chamber of Commerce, Bangkok, Thailand
aisuriya_sud@utcc.ac.th

Abstract

In this paper, we define the notions of KK-algebras, quotient KK-algebras and investigate its properties. Moreover we show the relation between ideals and congruences on KK-algebras.

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1 Introduction and Preliminaries

By an algebra \( X = (X, *, 0) \), we mean a non-empty set \( X \) together with a binary operation \( * \) and a some distinguished 0. H. Yisheng [6] studied an algebraic structure called a BCI-algebra which is an algebra \( (X, *, 0) \) with a binary operation \( * \) such that for all \( x, y, z \in X \), satisfies the following properties:

(B-1) \( ((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0 \);
(B-2) \( x \ast x = 0 \);
(B-3) \( x \ast y = 0 \) and \( y \ast x = 0 \) imply \( x = y \), for any \( x, y, z \in X \).

In this paper is introduction to the general theory of KK-algebra. We will first give the notion of KK-algebra, quotient KK-algebra and investigate elementary and fundamental properties, and we will show relation between ideals and congruences.
2 KK-algebras

In this section, we do define some familiar concepts as KK-algebras, both for illustration and for review of the concept. First we give a few definitions and some notation.

**Definition 2.1.** An algebra \((X; \ast, 0)\) with a binary operation \(\ast\) and a nullary operation \(0\). Then \(X\) is called KK-algebra if it satisfies for all \(x, y, z \in X\):

\[(KK-1) \quad (x \ast y) \ast ((y \ast z) \ast (x \ast z)) = 0;\]
\[(KK-2) \quad 0 \ast x = x;\]
\[(KK-3) \quad x \ast y = 0 \text{ and } y \ast x = 0 \text{ if and only if } x = y.\]

First, give example of KK-algebra.

**Examples 2.2.** Let \(\ast\) be defined on an abelian group \(G\) by letting \(x \ast y = x^{-1} \cdot y\), where \(x, y \in G\), with \(e\) is unity element of \(G\). Then \((G; \ast, e)\) is a KK-algebra.

**Examples 2.3.** Let \(X = \{0, 1\}\) and let \(\ast\) be defined by

\[
\begin{array}{ccc}
0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{array}
\]

Then \((G; \ast, 0)\) is a KK-algebra.

**Theorem 2.3.** Let \((X; \ast, 0)\) be a KK-algebra if and only if it satisfies the following conditions: for all \(x, y, z \in X\),

1. \(x \ast y) \ast ((y \ast z) \ast (x \ast z)) = 0;\)
2. \(x \ast (x \ast y) \ast y = 0;\)
3. \(x \ast x = 0;\)
4. \(x \ast y = 0 \text{ and } y \ast x = 0 \text{ if and only if } x = y.\)

**Proof.** Assume that \((X; \ast, 0)\) is a KK-algebra. From definition of KK-algebra, (1) and (4) holds. Then we see that \(x \ast ((x \ast y) \ast y) = 0\), and \(x \ast x = 0 \ast (x \ast x) = 0 \ast 0 \ast (0 \ast x) = 0 \ast 0 \ast (0 \ast x) = 0\), so (2) and (3) holds.

Conversely, we need to show KK-2. By (1), (2) and (3), we see that \(((0 \ast x) \ast x) \ast 0 = ((0 \ast x) \ast x) \ast (0 \ast (0 \ast x) \ast x) = 0\). And since \(0 \ast ((0 \ast x) \ast x) = 0\). From (4), it follows that \(0 \ast x = 0\) and \(x \ast (0 \ast x) = x \ast ((x \ast x) \ast x) = 0\). Therefore \(0 \ast x = x\), proving our theorem. 

**Definition 2.4.** Define a binary relation \(\leq\) on KK-algebra \(X\) by letting \(x \leq y\) if and only if \(y \ast x = 0\).
Proposition 2.5. If \((X; *, 0)\) is a KK-algebra, then \((X; \leq)\) is a partially order set.

Proposition 2.6. If \((X; *, 0)\) be a KK-algebra and \(x \leq 0\), then \(x = 0\), for any \(x \in X\). Moreover, 0 is called a minimal element in \(X\).

**Proof.** Let \(x \leq 0\), then \(0 \ast x = 0\). By KK-2, \(0 \ast x = x\), and thus \(x = 0\). □

It is easy to show that the following properties are true for a KK-algebra.

Theorem 2.7. Let \((X; *, 0)\) be a KK-algebra if and only if it satisfies the following conditions: for all \(x, y, z \in X\),

1. \(((y \ast z) \ast (x \ast z)) \leq (x \ast y)\);
2. \(((x \ast y) \ast y) \leq x\);
3. \(x \leq y\) if and only if \(y \ast x = 0\).

Proposition 2.8. Let \(x, y, z\) be any element in a KK-algebra \(X\). Then

1. \(x \leq y\) implies \(y \ast z \leq x \ast z\).
2. \(x \leq y\) implies \(z \ast x \leq z \ast y\).

Proposition 2.9. Let \(x, y, z\) be any element in a KK-algebra \(X\). Then \(x \ast (y \ast z) = y \ast (x \ast z)\).

**Proof.** Since theorem 2.7(2), \((x \ast z) \ast z \leq x\), and by proposition 2.8(1), we get that \(x \ast (y \ast z) \leq ((x \ast z) \ast z) \ast (y \ast z)\). Putting \(x = y\) and \(y = x \ast z\) in theorem 2.7(1), it follows that \(((x \ast z) \ast z) \ast (y \ast z) \leq y \ast (x \ast z)\). By the transitivity of \(\leq\) gives \(x \ast (y \ast z) \leq y \ast (x \ast z)\). And we replacing \(x\) by \(y\) and \(y\) by \(x\), we obtain \(y \ast (x \ast z) \leq x \ast (y \ast z)\). By the anti-symmetry of \(\leq\), thus \(x \ast (y \ast z) = y \ast (x \ast z)\) and finishing the proof. □

Corollary 2.10. Let \(x, y, z\) be any element in a KK-algebra \(X\). Then

1. \(y \ast z \leq x\) if and only if \(x \ast z \leq y\).
2. \((z \ast x) \ast (z \ast y) \leq x \ast y\).
3. \(x \leq y\) implies \(x \ast z \leq y \ast z\).

Proposition 2.11. Let \(x, y, z\) be any element in a KK-algebra \(X\). Then

1. \(((x \ast y) \ast y) \ast y = x \ast y\).
2. \((x \ast y) \ast 0 = (x \ast 0) \ast (y \ast 0)\).

**Proof.** (1) From theorem 2.3(2) and theorem 2.7(1), \(((x \ast y) \ast y) \ast (x \ast y) \leq x \ast ((x \ast y) \ast y) = 0\). Thus \(((x \ast y) \ast y) \ast (x \ast y) = 0\). Since \((x \ast y) \ast ((x \ast y) \ast y) = ((x \ast y) \ast y) \ast ((x \ast y) \ast y) = 0\). So, by KK-3, \((x \ast y) \ast y = x \ast y\).

2. Since \((x \ast 0) \ast (y \ast 0) = (x \ast 0) \ast (y \ast ((x \ast y) \ast (x \ast y))) = (x \ast 0) \ast (((x \ast y) \ast (x \ast y)) = (x \ast 0) \ast ((x \ast y) \ast (x \ast y))) = (x \ast y) \ast ((x \ast 0) \ast (x \ast 0)) = (x \ast y) \ast 0\).

The proof is complete. □

In this paper we will denote \(\mathbb{N}\) for the set of all nonnegative integers, i.e.,
0, 1, 2, ..., and \( \mathbb{N}^* \) for the set of all natural numbers, i.e., 1, 2, 3, ..., and we will also use the following notation in brevity:

\[
y^0 \ast x = x, \\
y^n \ast x = \underbrace{y \ast (\ldots \ast (y \ast (y \ast x)))}_{n \text{ times}},
\]

where \( x, y \) are any elements in a KK-algebra and \( n \in \mathbb{N}^* \).

**Proposition 2.12.** Let \( x, y \) be any element in a KK-algebra \( X \). Then

1. \( ((y \ast x) \ast x)^n \ast x = y^n \ast x \) for any \( n \in \mathbb{N} \).
2. \( (x^n \ast 0) \ast 0 = (x \ast 0)^n \ast 0 \) for any \( n \in \mathbb{N} \).

**Proof.** Let \( X \) be a KK-algebra and \( x, y \in X \) and \( n, m \in \mathbb{N} \).

1. Proceed by induction on \( n \) and defined the statement \( P(n) \), \( ((y \ast x) \ast x)^n \ast x = y^n \ast x \). We see that \( P(0) \) is true, \( ((y \ast x) \ast x)^0 \ast x = x = y^0 \ast x \). Assume that \( P(k) \) is true for some arbitrary \( k \geq 0 \), that is \( ((y \ast x) \ast x)^k \ast x = y^k \ast x \). Since \( ((y \ast x) \ast x)^{k+1} \ast x = ((y \ast x) \ast x) \ast ((y \ast x) \ast x)^k \ast x = ((y \ast x) \ast x) \ast (y^k \ast x) = y^k \ast ((y \ast x) \ast x) \ast x = y^k \ast (y \ast x) = y^{k+1} \ast x \) This show that \( P(k+1) \) is true and by the principle of mathematical induction, \( P(n) \) is true for each \( n \in \mathbb{N}^* \).

2. Since \( (x^n \ast 0) \ast 0 = (x \ast (x^{n-1} \ast 0)) \ast 0 = (x \ast 0) \ast ((x^{n-1} \ast 0) \ast 0) = (x \ast 0) \ast ((x \ast (x^{n-2} \ast 0)) \ast 0) = (x \ast 0) \ast ((x \ast 0) \ast ((x^{n-2} \ast 0) \ast 0)) = (x \ast 0)^2 \ast ((x^{n-2} \ast 0) \ast 0) = \ldots = (x \ast 0)^n \ast 0 \)

Given \( x \in X \) if it satisfies \( x \ast 0 = 0 \), that is \( 0 \leq x \), the element \( x \) is called a positive element of \( X \). By definition, the zero element 0 of \( X \) is positive.

**Proposition 2.12.** Let \( x \) be any element in a KK-algebra \( X \). Then \( (x \ast 0) \ast 0 \) is a positive element of \( X \) for every \( x \in X \).

**Proof.** Since \( (x \ast 0) \ast 0 = ((x \ast 0) \ast 0) \ast 0 = (x \ast 0) \ast (x \ast 0) \ast 0 = 0 \). Therefore \( (x \ast 0) \ast 0 \) is a positive element of \( X \).

### 3 Ideals

**Definition 3.1.** A non-empty subset \( A \) of a KK-algebra \( X \) is called a closed of \( X \) on condition that \( x \ast y \in A \) whenever \( x, y \in A \).

**Definition 3.2.** A non-empty subset \( A \) of a KK-algebra \( X \) is called an ideal of \( X \) if it satisfies the following conditions:

1. \( 0 \in A \)
2. For any \( x, y \in X \), \( x \ast y \in A \) and \( x \in A \) imply \( y \in A \).

**Examples 3.3.** Let \( X = \{0, 1, 2, 3\} \) and let \( \ast \) be defined by the table
Thus, it can be easily shown that $X$ is a KK-algebra. And we see that $I = \{0, 1\}$ and $J = \{0, 3\}$ are closed ideals of $X$.

**Lemma 3.4.** Let $A$ be a closed of KK-algebra $X$. Then $A$ is an ideal of $X$ if and only if $x \in A$ and $z * y \notin A$ imply $z * (x * y) \notin A$ for all $x, y, z \in X$.

**Proof.** Let $A$ be an ideal of $X$ and let $x \in A$ whereas $z * y \notin A$. Suppose that $z * (x * y) \in A$. By proposition 2.9, we see that $x * (z * y) \in A$. Since $A$ is an ideal of $X$ and $x \in A$, $z * y \in A$, a contradiction. So $z * (x * y) \notin A$.

Conversely, assume that if $x \in A$ and $z * y \notin A$ imply $z * (x * y) \notin A$ for all $x, y, z \in X$. Since $A$ is a closed of $X$, then there is $x \in A$ which $0 = x * x \in A$. That is, $0 \in A$. Now, let $x * y \in A$ and $x \in A$. Assume that $y \notin A$. We have that $0 * y = y \notin A$. It follows that $0 * (x * y) \notin A$. Hence $x * y \notin A$, contradiction. Therefore $A$ is an ideal of $X$. This completes the proof. \qed

**Corollary 3.5.** Let $A$ be a closed of KK-algebra $X$. Then $A$ is an ideal of $X$ if and only if $x \in A$ and $y \notin A$ imply $x * y \notin A$ for all $x, y \in X$.

**Lemma 3.6.** Let $A$ be a closed of KK-algebra $X$. Then $A$ is an ideal of $X$ if and only if $x * (y * z) \in A$ and $x * z \notin A$ imply $y \notin A$ for all $x, y, z \in X$.

**Proof.** Let $A$ be an ideal of $X$ and let $x * (y * z) \in A$, $x * z \notin A$. Suppose that $y \in A$. By proposition 2.9, we have $y * (x * z) \in A$. Since $A$ is an ideal of $X$, thus $x * z \in A$, contradiction, this shows that $y \notin A$.

Conversely, assume that $x * (y * z) \in A$ and $x * z \notin A$ imply $y \notin A$ for all $x, y, z \in X$. Since $A$ is a closed of $X$, then there is $y \in A$ which $0 = y * y \in A$. Then $0 \notin A$. Let $y * z \in A$, $y \in A$ and suppose that $z \notin A$. By KK-2, $0 * (y * z) \in A$ and $0 * z \notin A$. By assumption, so $y \notin A$, a contradiction. This proves that $A$ is an ideal of $X$. \qed

**Corollary 3.7.** Let $A$ be a closed of KK-algebra $X$. Then $A$ is an ideal of $X$ if and only if $x * y \in A$ and $y \notin A$ imply $x \notin A$ for all $x, y, z \in X$.

This Lemma gives some properties of ideal of KK-algebra.

**Lemma 3.8.** If $A$ is an ideal of KK-algebra $X$ and $B$ is an ideal of $A$, then $B$ is an ideal of $X$.

**Proof.** Since $B$ is an ideal of $A$, then $0 \in B$. Let $x, y \in X$ such that $x * y \in B$ and $x \in B$. It follows that that $x * y \in A$ and $x \in A$. By assumption, $A$ is an ideal of $X$, so $y \in A$ and $x \in B$. From $B$ is an ideal of $A$, so $y \in B$. Therefore, $B$ is an ideal of $X$. \qed
Theorem 3.9. Let \( \{ J_i : i \in \mathbb{N} \} \) be a family of ideals of a KK-algebra \( X \) where \( J_n \subseteq J_{n+1} \) for all \( n \in \mathbb{N} \). Then \( \bigcup_{n=1}^{\infty} J_n \) is an ideal of \( X \).

**Proof.** Let \( \{ J_i : i \in \mathbb{N} \} \) be a family of ideals of \( X \). It can be proved easily that \( \bigcup_{n=1}^{\infty} J_n \subseteq X \). Since \( J_i \) is an ideal of \( X \) for all \( i \), so \( 0 \in \bigcup_{n=1}^{\infty} J_n \). Let \( x \ast y \in \bigcup_{n=1}^{\infty} J_n \) and \( x \in \bigcup_{n=1}^{\infty} J_n \). It follows that \( x \ast y \in J_j \) for some \( j \in \mathbb{N} \) and \( x \in J_k \) for some \( k \in \mathbb{N} \). Furthermore, let \( J_j \subseteq J_k \). Hence \( x \ast y \in J_k \) and \( x \in J_k \). By assumption, \( J_k \) is an ideal of \( X \), it follows that \( y \in J_k \). Therefore, \( y \in \bigcup_{n=1}^{\infty} J_n \), proving that \( \bigcup_{n=1}^{\infty} J_n \) is an ideal of \( X \). \( \square \)

Theorem 3.10. Let \( \{ J_i : i \in \mathbb{N} \} \) be a family of closed ideals of a KK-algebra \( X \) where \( J_n \subseteq J_{n+1} \) for all \( n \in \mathbb{N} \). Then \( \bigcup_{n=1}^{\infty} J_n \) is a closed ideal of \( X \).

**Proof.** Let \( \{ J_i : i \in \mathbb{N} \} \) be a family of closed ideals of \( X \). By theorem 3.9, \( \bigcup_{n=1}^{\infty} J_n \) is an ideal of \( X \). We will show that \( \bigcup_{n=1}^{\infty} J_n \) is a closed of \( X \). Let \( x, y \in \bigcup_{n=1}^{\infty} J_n \). It follows that \( x \in J_j \) for some \( j \in \mathbb{N} \) and \( y \in J_k \) for some \( k \in \mathbb{N} \). WLOG, we assume that \( j \leq k \), we obtain \( J_j \subseteq J_k \). That is, \( x \in J_k \) and \( x \in J_k \). Since \( J_k \) is a closed of \( X \), we get \( x \ast y \in J_k \subseteq \bigcup_{n=1}^{\infty} J_n \). This proves that \( \bigcup_{n=1}^{\infty} J_n \) is a closed ideal of \( X \). \( \square \)

Theorem 3.11. Let \( \{ I_j : j \in J \} \) be a family of ideals of a KK-algebra \( X \). Then \( \bigcap_{j \in J} I_j \) is an ideal of \( X \).

**Proof.** Let \( \{ I_j : j \in J \} \) be a family of ideals of \( X \). It is obvious that \( \bigcap_{j \in J} I_j \subseteq X \). Since \( 0 \in I_j \) for all \( j \in J \), it follows that \( 0 \in \bigcap_{j \in J} I_j \). Let \( x \ast y \in \bigcap_{j \in J} I_j \) and \( x \in \bigcap_{j \in J} I_j \). We get that \( x \ast y \in I_j \) and \( x \in I_j \) for all \( j \in J \), then \( y \in I_j \) for all \( j \in J \). Because \( I_j \) is an ideal of \( X \). So \( y \in \bigcap_{j \in J} I_j \), proving our theorem. \( \square \)

Theorem 3.12. Let \( \{ I_j : j \in J \} \) be a family of closed ideals of a KK-algebra \( X \). Then \( \bigcap_{j \in J} I_j \) is a closed ideal of \( X \).

**Proof.** Let \( \{ I_j : j \in J \} \) be a family of closed ideals of \( X \). By theorem 3.11, \( \bigcap_{j \in J} I_j \) is an ideal of \( X \). We will show that \( \bigcap_{j \in J} I_j \) is a closed of \( X \). Let \( x, y \in \bigcap_{j \in J} I_j \). It follows that \( x, y \in I_j \) for all \( j \in J \). Since \( I_j \) is a closed of \( X \) and \( x \ast y \in I_j \) for all \( j \in J \), then \( x \ast y \in \bigcap_{j \in J} I_j \). This show that \( \bigcap_{j \in J} I_j \) is a closed ideal of \( X \). \( \square \)
4 Quotient KK-Algebras

In this section, we describe congruence on KK-algebras.

**Definition 4.1.** Let $I$ be an ideal of a KK-algebra $X$. Define a relation $\sim$ on $X$ by

$$x \sim y \text{ iff } x \ast y \in I \text{ and } y \ast x \in I.$$

**Theorem 4.2.** If $I$ is an ideal of KK-algebra $X$, then the relation $\sim$ is an equivalence relation on $X$.

**Proof.** Let $I$ be an ideal of $X$ and $x, y, z \in X$. By Theorem 2.3, $x \ast x = 0$ and assumption, $x \ast x \in I$. That is, $x \sim x$. Hence $\sim$ is reflexive. Next, suppose that $x \sim y$. It follows that $x \ast y \in I$ and $y \ast x \in I$. Then $y \sim x$, so $\sim$ is symmetric. Finally, let $x \sim y$ and $y \sim z$. Then $x \ast y, y \ast x, y \ast z, z \ast y \in I$ and $(y \ast x) \ast ((z \ast y) \ast (z \ast x)) = 0 \in I$. It follows that $(z \ast y) \ast (z \ast x) \in I$, and since $z \ast y \in I$, so $z \ast x \in I$. Similarly, $x \ast z \in I$. Thus $\sim$ is transitive. Therefore $\sim$ is an equivalence relation. $\Box$

**Lemma 4.3.** Let $I$ be an ideal of KK-algebra $X$. For any $x, y, u, v \in X$, if $u \sim v$ and $x \sim y$, then $u \ast x \sim v \ast y$.

**Proof.** Assume that $u \sim v$ and $x \sim y$, for any $x, y, u, v \in X$, then $u \ast v, v \ast u, x \ast y, y \ast x \in I$ and by K-1, we see that $(u \ast v) \ast ((v \ast x) \ast (u \ast x)) = 0$ and $(v \ast u) \ast ((u \ast x) \ast (v \ast x)) = 0$. From assumption and $I$ is an ideal of $X$, these imply that $(v \ast x) \ast (u \ast x) \in I$ and $(u \ast x) \ast (v \ast x) \in I$. This shows that $v \ast x \sim u \ast x$.

On the other hand, by corollary 2.10, we have that $(y \ast x) \ast ((v \ast y) \ast (v \ast x)) = 0$ and $(x \ast y) \ast ((v \ast x) \ast (v \ast y)) = 0$. From assumption and $I$ is an ideal of $X$, these imply that $(v \ast y) \ast (v \ast x) \in I$ and $(v \ast x) \ast (v \ast y) \in I$. Thus $v \ast x \sim v \ast y$. Since $\sim$ is symmetric and transitive, so $u \ast x \sim v \ast y$. $\Box$

**Corollary 4.4.** If $I$ is an ideal of KK-algebra $X$, then the relation $\sim$ is a congruence relation on $X$.

**Proof.** By theorem 4.2 and lemma 4.3. $\Box$

**Definition 4.5.** Let $I$ be an ideal of a KK-algebra $X$. Given $x \in X$, the equivalence class $[x]_I$ of $x$ is defined as the set of all element of $X$ that are equivalent to $x$, that is,

$$[x]_I = \{y \in X : x \sim y\}.$$

We define the set $X/I = \{[x]_I : x \in X\}$ and a binary operation $\circ$ on $X/I$ by

$$[x]_I \circ [y]_I = [x \ast y]_I.$$

**Theorem 4.6.** If $I$ is an ideal of KK-algebra $X$ with $X/I = \{[x]_I : x \in X\}$ where a binary operation $\circ$ on a set $X/I$ is defined by $[x]_I \circ [y]_I = [x \ast y]_I$, then the binary operation $\circ$ is a mapping from $X/I \times X/I$ to $X/I$.

**Proof.** Let $[x_1]_I, [x_2]_I, [y_1]_I, [y_2]_I \in X/I$ such that $[x_1]_I = [x_2]_I$ and $[y_1]_I = [y_2]_I$. Then $x_1 \sim x_2$ and $y_1 \sim y_2$. Hence $x_1 \ast y_1 \sim x_2 \ast y_2$. Thus $[x_1]_I \circ [y_1]_I = [x_2]_I \circ [y_2]_I$. $\Box$
[y_2]_I$. It follows that $x_1 \sim x_2$ and $y_1 \sim y_2$. By lemma 4.3, $x_1 \ast y_1 \sim x_2 \ast y_2$, proving that $[x_1 \ast y_1]_I = [x_2 \ast y_2]_I$. \hfill \Box

**Theorem 4.7.** If $I$ is an ideal of KK-algebra $X$, then $(X/I; \circ, [0]_I)$ is a KK-algebra. Moreover, the set $X/I$ is called the quotient KK-algebra.

**Proof.** Let $[x]_I, [y]_I, [z]_I \in X/I$. Then $([z]_I \circ [x]_I) \circ ([y]_I \circ [z]_I) = [z \ast x]_I \circ ([x \ast y]_I \circ [z \ast y]_I) = [z \ast x]_I \circ [x \ast y]_I \circ (z \ast y)_I = ([z \ast x] \ast (x \ast y) \ast (z \ast y))_I = [0]_I$. It is clear that $[0]_I \circ [x]_I = [0 \ast x]_I = [x]_I$. Now, let $[x]_I \circ [y]_I = [0]_I$ and $[y]_I \circ [x]_I = [0]_I$. It follows that $x \ast y \sim 0$ and $y \ast x \sim 0$, that is $0 \ast (x \ast y), 0 \ast (y \ast x) \in I$. Since $I$ is an ideal of $X$ and $0 \in I$, we get that $x \ast y, y \ast x \in I$. Consequently, $x \sim y$, proving that $[x]_I = [y]_I$. Therefore, $(X/I; \circ, [0]_I)$ is a KK-algebra. \hfill \Box

**Example 4.8.** According to example 3.3, we can get that $X/I = \{[0]_I, [2]_I\}$, where $[0]_I = [1]_I = \{0, 1\}$ and $[2]_I = [3]_I = \{2, 3\}$. Let $\circ$ be defined on $X/I$ by

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>$[0]_I$</th>
<th>$[2]_I$</th>
</tr>
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<tbody>
<tr>
<td>$[0]_I$</td>
<td>$[0]_I$</td>
<td>$[2]_I$</td>
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<tr>
<td>$[2]_I$</td>
<td>$[2]_I$</td>
<td>$[0]_I$</td>
</tr>
</tbody>
</table>

Then $(X/I; \circ, [0]_I)$ is a KK-algebra.

**Lemma 4.9.** Let $X$ be a KK-algebra and $I, J$ be any sets such that $I \subseteq J \subseteq X$. Suppose that $I$ is an ideal of $X$, then $J$ is an ideal of $X$ if and only if $J/I$ is an ideal of $X/I$.

**Proof.** Let $I$ be an ideal of $X$ with $I \subseteq J \subseteq X$. Suppose firstly that $J$ is an ideal of $X$, then $J/I = \{[x]_I : x \in J\}$, where $[x]_I = \{y \in J : x \sim y\}$, and $X/I = \{[x]_I : x \in X\}$, where $[x]_I = \{y \in X : x \sim y\}$. Obviously, $J/I \subseteq X/I$ and $[0]_I \in J/I$. Now, let $[x]_I \circ [y]_I \in J/I$ and $[y]_I \in J/I$. Then $[x \ast y]_I = [x]_I \circ [y]_I \in J/I$, it follows that $x \ast y \in J$ and $x \in J$ by assumption, $y \in J$. Accordingly, $[y]_I \in J/I$, this shows that $J/I$ is an ideal of $X/I$.

On the other hand, suppose that $J/I$ is an ideal of $X/I$ and $I$ is an ideal of $X$ with $I \subseteq J \subseteq X$. Thus, $0 \in J$. Let $x \ast y \in J$ and $x \in J$. It follows that $[x \ast y]_I, [x]_I \in J/I$. Since $[x \ast y]_I \circ [y]_I = [x]_I \circ [y]_I$, so $[x]_I \circ [y]_I \in J/I$. By hypothesis, $[y]_I \in J/I$ implies $y \in J$, proving our lemma. \hfill \Box

**Lemma 4.10.** Let $X$ be a KK-algebra and $I, J$ be any sets such that $I \subseteq J \subseteq X$. Suppose that $I$ is a closed ideal of $X$. Then $J$ is a closed ideal of $X$ if and only if $J/I$ is a closed ideal of $X/I$.

**Proof.** Similar to that of lemma 4.9 \hfill \Box

**Lemma 4.11.** Let $I, J$ be ideals of a KK-algebra $X$ with $I \subseteq J$, then $I$ is an ideal of $X$.

**Proof.** Obvious. \hfill \Box
Next, the basic properties of equivalence classes are considered are as the following Theorem.

**Theorem 4.12.** Let $I$ be a closed ideal of a KK-algebra $X$ and $a, b \in X$. Then

(1) $[a]_I = I$ iff $a \in I$.

(2) $[a]_I = [b]_I$ or $[a]_I \cap [b]_I = \emptyset$.

**Proof.** Let $I$ be a closed ideal of $X$ and $a, b \in X$.

(1) It is clear due to the fact that $a \sim a$ for all $a \in X$ and $a \ast a = 0 \in I$, so we get that $a \in [a]_I = I$. Conversely, let $x \in [a]_I$. Then $x \sim a$, it follows that $x \ast a, a \ast x \in I$. By hypothesis, $x \in I$. Hence, $[a]_I \subseteq I$. To show that $I \subseteq [a]_I$, choose $x \in I$. Since $I$ is a closed of $X$, we have $x \ast a, a \ast x \in I$. Thus, $x \sim a$, this means that $x \in [a]_I$ and shows that $I \subseteq [a]_I$. Consequently, $[a]_I = I$.

(2) Assume that $[a]_I \cap [b]_I \neq \emptyset$. Then there is $x \in [a]_I \cap [b]_I$ such that $x \in [a]_I$ and $y \in [a]_I$. It follows that $x \sim a$ and $x \sim b$, so $a \sim b$ by the symmetric and transitive properties. Thus $[a]_I = [b]_I$. \hfill \Box

**Theorem 4.13.** If $I$ is a closed ideal of a KK-algebra $X$ and $y \in I$, then $[y]_I$ is closed ideal of $X$.

**Proof.** Let $I$ be a closed ideal of $X$ and $y \in I$. It is clear that $0 \in [y]_I$.

Now, suppose that $a \ast b \in [y]_I$ and $a \in [y]_I$. We will show that $b \in [y]_I$.

Then $a \ast b \sim y$ and $a \sim y$, it follows that $y \ast (a \ast b) \in I$ and $y \ast a \in I$. By assumption, $a \in I$. From proposition 2.9, $a \ast (y \ast b) = y \ast (a \ast b) \in I$, and $I$ is a closed ideal of $X$ and $a \in I$, therefore, $y \ast b \in I$. By properties of $X$, we get that $(a \ast 0) \ast (((a \ast b) \ast y) \ast (b \ast y)) = (a \ast (b \ast b)) \ast (((a \ast b) \ast y) \ast (b \ast y)) = (b \ast (a \ast b)) \ast (((a \ast b) \ast y) \ast (b \ast y)) = 0$. By hypothesis, $(a \ast 0) \ast (((a \ast b) \ast y) \ast (b \ast y)) \in I$, and $I$ is closed and $a \in I$, then $a \ast 0 \in I$. Thus $((a \ast b) \ast y) \ast (b \ast y) \in I$. From $a \ast b \sim y$ and $I$ is closed, then $b \ast y \in I$. Hence, $b \sim y$, this means $a \in [y]_I$.

Accordingly, $[y]_I$ is an ideal of $X$.

Finally, let $a, b \in [y]_I$. Then $a \sim y$ and $b \sim y$, by lemma 4.3, $a \ast b \sim y \ast y$. By theorem 2.3, it follows that $a \ast b \sim 0$. Thus $a \ast b \in [0]_I$. Now, we have $0, y \in I$ and $I$ is closed, so $0 \ast y \in I$ and $y \ast 0 \in I$. That is, $0 \sim y$. Hence, $[0]_I = [y]_I$. By transitive, $a \ast b \in [y]_I$, proving our theorem. \hfill \Box

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**References**


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