On the Dihedral and Quaternion Homology of Banach Space

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Abstract

In this article we study some relation on the cyclic, dihedral and quaternion homology for a unital involutive Banach algebra.

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1 Introduction

In 1987 Loday [8], and Krassawkas, Lapin and Solovev [7] introduced and studied the dihedral (co)homology of involutive unital algebras. The authors studied the dihedral (co)homology of certain classes of operator algebras [3], [4], [5] and [4]. The dihedral homology of algebras and its relation with quaternion homology has been studied by Loday [8]. The homology theory of some classes of C*-algebras has been studied in [1]. In this article we are concerned with the dihedral and quaternion Banach spaces and their homology.

2 Dihedral Homology of Banach algebra

We recall the definition and properties of of Banach algebra and its homology from [7] and [5]. Let A be an unital Banach algebra over a commutative ring $K (K = C)$. Define the complex $C(A) = (C_*(A), b_*)$, where $C_n(A) = A \otimes \cdots \otimes A$ is the tensor product of algebra $(n + 1 \times)$ and, $b_* : C_n(A) \rightarrow C_{n-1}(A)$ is the boundary operator

$$b_n(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i-1} \otimes \cdots \otimes a_{n-1}.$$

(1)
It is well known that $b_{n-1}b_n = 0$, and hence $\ker b_n \supset \text{Im } b_{n+1}$. The group

$$H_n(A) = H(C(A)) = \frac{\ker b_n}{\text{Im } b_{n-1}}$$

(2)

is called the Hochschild homology of unital Banach algebras $A$ with involutive and denote by $(HH*(A))$.

If $A$ is an unital Banach algebra, one acts on the complex $C(A)$, by the cyclic group of order $(n+1)$ by means of the operator $t_n : C_n(A) \rightarrow C_n(A)$ such that:

$$t_n(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

(3)

The quotient complex $CC_n(A) = C_n(A)/\text{Im}(1 - t_n)$ is a subcomplex of a complex $C_n(A)$. Following [9] the cyclic homology of algebra $A$ is the homology of the complex $CC_*(A)$.

If $A$ is endowed with an involution $*$, acts on $C_n(A)$ by $t_n$ and $r_n$ as

$$r_n(a_0 \otimes \ldots \otimes a_n) = \alpha(-1)^{n(n+1)/2} a_0^* \otimes a_n^* \otimes \ldots \otimes a_1^*, \alpha = \pm 1$$

(4)

The quotient complex $CD_n(A) = [C_n(A)/\text{Im}(1 - t_n) + \text{Im}(1 - r_n)]$ is a subcomplex of a complex $C_n(A)$. Following [8] the dihedral homology of algebra $A$ is the homology of the complex $CD_*(A)$.

**Definition 2.1** Let $A$ be an unital Banach algebra. Then the dihedral homology of $A$ is given by

$$^\alpha HD_n(A) = H_n(^\alpha \text{Tot } CC(A))$$

(5)

**Theorem 2.1** For a unital Banach algebra $A$ with involutive over ring $K(K = C)$ (with $\frac{1}{2} \in K$), there exists a long exact sequence

$$
\cdots \longrightarrow H_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(E) \xrightarrow{B} H_{n-1}(A) \longrightarrow \cdots
$$

(6)

**Theorem 2.2** For given algebra $A$, the long exact sequence

$$
\cdots \longrightarrow ^{-\alpha} HD_n(A) \xrightarrow{j_*} HC_n(A) \xrightarrow{i_*} ^\alpha HD_n(A) \longrightarrow ^\alpha HD_{n-1}(A) \longrightarrow \cdots
$$

(7)

when $j_*$ is connecting homeomorphism gives the short following exact sequence

$$
0 \longrightarrow ^{-\alpha} HD_n(A) \xrightarrow{j_*} HC_n(A) \xrightarrow{i_*} ^\alpha HD_n(A) \longrightarrow ^{-\alpha} HD_{n-1}(A) \longrightarrow 0
$$

(8)

where $i_*j_* = id$, which, under condition ($\frac{1}{2} \in K$), get the following assertion
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Proposition 2.3  If \( \frac{1}{2} \in K \), for given unital Banach algebra, we have natural isomorphisms

\[ HC_n(A) \cong HD_n(A) \oplus - \alpha HD_n(A). \quad (9) \]

3 Quaternion Group \( Q_m \)

Let \( H \) be algebra of quaternions \( \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \mathbb{R}k \). For every natural number \( m \geq 2 \), the generalized quaternion group \( Q_m \) is defined as a subgroup of the multiplicative group \( H^* \), generated by the elements \( x = e^{\frac{\pi}{m}} \) and \( y = j \). It is clear that the element \( x \) has order \( 2m \) and the relations \( y^2 = x^m \) and \( xy^{-1} = x^{-1} \) are fulfilled. Hence \( x^m y^{-1} = yxy^{-1}yxy^{-1}...yxy^{-1} = x^{-m} \) and we deduce that

\[ y.y^2.y^{-1} = y^{-2}, \text{ i.e } x^{2m} = y^4 = 1 \quad (10) \]

Thus, the cyclic subgroup \( C \) generated by the element \( x \), is a normal subgroup and has index tow in \( Q_m \). It follows that the group \( Q_m \) itself has order \( 4m \).

Let us list the most important properties of the generalized quaternion group \( Q_m \).

(i) The group \( Q_m \) is given by a co-representation

\[ Q_m = \{ x, y, x^m = y^2, yxy^{-1} = x^{-1} \}. \quad (11) \]

(ii) In the extension

\[ o \to C \to Q_m \to \mathbb{Z}/2 \to 0, \quad (12) \]

the renerator of \( \mathbb{Z}/2 \) acts on \( C \) as the multiplication by \(-1\)

(iii) Every element in the set \( Q_m/C \) has order 4.

Proposition 3.1  Let \( R \) be a commutative ring with unit. Then there exist 4−periodic resolution of the trivial \( Q_m \)−module \( R : \)

\[ ... \to R[Q_m] \xrightarrow{N} R[Q_m] \xrightarrow{u} R[Q_m]^2 \xrightarrow{v} R[Q_m]^2 \xrightarrow{u} R[Q_m] \xrightarrow{\varepsilon} R \to 0 \quad (13) \]

where \( \varepsilon \) is the natural augmentation,

\[ u = (1 - x, 1 - y), v = \begin{pmatrix} T & 1 + xy \\ -(1 + y) & x - y \end{pmatrix}, w = \begin{pmatrix} 1 - x \\ yx - 1 \end{pmatrix}, T = 1 + x + x^2 + ... + x^{m-1}, \]

\[ N = \sum_{g \in Q_m} g = (1 + y^2 + y^3 + y)T. \]

Proof. we use Fox derivatives[2] Let
be a group generator by the elements $g_1, ..., g_k$ with relations $r_1, ..., r_l$. The free differential $\frac{\partial r_i}{\partial g_j}$ of the group ring $Z[G]$ is defined by:

$$\frac{\partial (ab)}{\partial g} = \frac{\partial b}{\partial g}, \frac{\partial g}{\partial g} = 1, \frac{\partial h}{\partial g} = 0$$

where $h$ is any generator of $G$ not equal to $g$. Then according to Fox [2], the sequence

$$Z[G]^l \to Z[G]^k \to Z[G]^k \to Z \to 0,$$

where $\varepsilon(g) = 1, u(g_1, ..., 1 - g_k), v = \left( \frac{\partial r_i}{\partial g_j} \right), 1 \leq i \leq k \leq j \leq l$ is the first part of the free resolution of the trivial $G$-module $Z$.

By using the Fox’s derivatives of the generalized quaternion group $Q_m$ when $k = 2, l = 2, g_1 = x, g_2 = y, r_1 = x^m y^{-2}, r_2 = xy x^{-1}$ and $u(1 - x, 1 - y)$,

$$v = \left[ \begin{array}{cc} \frac{\partial (x^m y^{-2})}{\partial x} & \frac{\partial (xy x^{-1})}{\partial x} \\ \frac{\partial (x^m y^{-2})}{\partial y} & \frac{\partial (xy x^{-1})}{\partial y} \end{array} \right] = \left[ \begin{array}{c} 1 + x + ... + x^{m-1} + xy \\ -(1 + y) \\ x - 1 \end{array} \right],$$

we get the following exact sequence

$$R[Q_m]^2 \xrightarrow{v} R[Q_m]^2 \to R[Q_m] \xrightarrow{u} R \to 0$$

Considering the factor $Hom_{R[Q_m]}(-, R[Q_m])$ and modifying the $Q_m$ module structure by means of the isomorphism $f : Q_m \to Q_m, f(x) = x^{-1}, f(y) = (by)^{-1}$ we get the exact sequence:

$$Hom_{R[Q_m]}(R[Q_m]^2, R[Q_m]) \xleftarrow{\text{by}} Hom_{R[Q_m]}(R[Q_m]^2, R[Q_m]) \xrightarrow{\text{by}} 0$$

It is easy to verify that:

$$u^* = (1 - x, 1 - y) = \left[ \begin{array}{c} 1 - x \\ yx - 1 \end{array} \right] = w$$

$$v^* = \left[ \begin{array}{cc} T & 1 + xy \\ -(1 + y) & x - 1 \end{array} \right]^* = \left[ \begin{array}{cc} T & 1 + xy \\ -(1 + y) & x - 1 \end{array} \right] = v$$

and we get
the following exact sequence:

$$0 \longrightarrow R \xrightarrow{\varepsilon^*} R[Q_m] \xrightarrow{w} R[Q_m]^2 \xrightarrow{v} R[Q_m]^2$$  \hspace{1cm} (19)

Since the composition $\varepsilon^* \circ \varepsilon$ is a homomorphism $N$, from ??? and ??? we get the required 4-periodic resolution.

## 4 Quaternion Banach Spaces

Let $E = \mathcal{O}_{n \geq 0} E_n$ be a graded Banach space over the field of complex numbers $\mathbb{C}$.

Consider the families of continuous linear maps on $E$:

- $d_n^1 : E_n \longrightarrow E_{n-1}$
- $s_n^j : E_n \longrightarrow E_{n+1}$
- $T_n, w_n : E_n \longrightarrow E_n$

$0 \leq i \leq n, 0 \leq i \leq n$

which satisfy the following conditions

$$d_n^id_{n+1}^j = d_{n+1}^{j-1}d_n^i, \hspace{1cm} i < j$$

$$s_n^is_{n+1}^j = s_{n+1}^{j+1}s_n^i, \hspace{1cm} i \leq j$$

$$d_n^is_{n-1}^j = \begin{cases} s_{n-2}^{j-1}s_n^i, & i < j, \\ Id(E_{n-1}), & i = j, j + 1 \\ s_n^{-2}d_n^{i-1}, & i > j \end{cases}$$  \hspace{1cm} (21)

$$d_n^it_n = t_{n-1}d_n^{i-1}, \hspace{1cm} s_n^it_n = T_{n+1}^is_{n+1}^{i+1} \hspace{1cm} 1 \leq i \leq n,$$

$$d_n^iw_n = w_{n-1}d_n^{n-j}, \hspace{1cm} s_n^iw_n = w_{n+1}s_n^{n-j} \hspace{1cm} 0 \leq j \leq n,$$

$$t_n^2 = w_n^2, \hspace{1cm} w_nt_nw_n^{-1} = t_n^{-1}. \hspace{1cm} (23)$$

A graded Banach space $E = \mathcal{O}_{n \geq 0} E_n$ considered together with these families of continuous linear maps is called a quaternion Banach space. An arbitrary unital Banach algebra $A$ generates the quaternion Banach space.

Indeed, put

$$E_n = A \hat{\otimes} A \hat{\otimes} A... \hat{\otimes} A(n + 1 \text{ times})$$  \hspace{1cm} (24)

where $\hat{\otimes}$ is the continuous tensor product in the sense of Grothendieck.

Define operators:

$$d_n^1 : E_n \longrightarrow E_{n-1}, \hspace{1cm} s_n^j : E_n \longrightarrow E_{n+1}$$  \hspace{1cm} (25)
by means of the formulas
\[ d_n^i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \quad 0 \leq i \leq n \] (26)
\[ d_n^m(a_0 \otimes \ldots \otimes a_n) = a_n a_0 \otimes \ldots \otimes a_{n-1} \] (27)
\[ s_n^j(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_j \otimes e \otimes a_j a_{j+1} \otimes \ldots \otimes a_n, \quad 0 \leq j \leq n \] (28)
\[ s_n^m(a_0 \otimes \ldots \otimes a_n) = e \otimes a_0 \otimes \ldots \otimes a_n. \] (29)

Moreover, define the operators \( t_n : E_n \to E_n \), \( w_n : E_n \to E_n \) putting
\[ t_n(a_0 \otimes \ldots \otimes a_n) = (1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1} \] (30)
\[ w_n(a_0 \otimes \ldots \otimes a_n) = \alpha (-1)^{\frac{n(n+1)}{2}} a_0^* \otimes a_n^* \otimes \ldots \otimes a_1^*, \] (31)
where \( \alpha \) is a root of the 4th degree of 1, \( a_i^* \) is the image of elements \( a_i \in A \) under involution \( * : A \to A \). It is easy to verify that the family so defined of Banach spaces and continuous linear maps is a quaternion Banach space. In what follows we denote the quaternion Banach space by :
\[ Q(A) : Q(A)_n = A \otimes \ldots \otimes A \ ((n+1) \text{ times}). \] (32)

5 Continuous Quaternion Homology

Proposition 5.1. Let \( E = \otimes_{n \geq 0} E_n \) be a quaternion Banach space. Put
\[ t_n = (-1)^n t_n, \quad r_n = (-1)^{\frac{n(n+1)}{2}} \alpha w_n, \] (33)
where \( \alpha = 1, -1, i, -i \). Then there exists a bicomplex \( \partial \epsilon(w) \) with 4-periodic rows :
\[
\begin{array}{cccccc}
\ldots & & \ldots & & \ldots & \ldots \\
E_n & \leftarrow & E_n \otimes E_n & \leftarrow & E_n \otimes E_n & \leftarrow \quad E_n \otimes E_n & \leftarrow & E_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
-(b^* \otimes b) & \leftarrow & (b^* \otimes b) & \leftarrow & b & \leftarrow & b & \leftarrow & b \\
E_{n-1} & \leftarrow & E_{n-1} \otimes E_{n-1} & \leftarrow & E_{n-1} \otimes E_{n-1} & \leftarrow & E_{n-1} \otimes E_{n-1} & \leftarrow & E_{n-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & & \ldots & & \ldots & \ldots & & \ldots & & \ldots \\
\end{array}
\] (34)
where \( u = (1 - t, 1 - r) \), \( v = \begin{bmatrix} T & 1 - tr \\ -1 + r & t - 1 \end{bmatrix} \), \( w = \begin{bmatrix} 1 - t \\ -rt - 1 \end{bmatrix} \), \( T = 1 + t + \ldots + t^{n-1} \), 
\( N = (1 + r + r^2 + r^3) T \).

Proof. This statement follows immediately from the following:

\[
\begin{align*}
    b(1 - t) &= (1 - t)b, \\
    br &= rb, \\
    b'br &= brb, \\
    b'T &= Tb, \\
    b'N &= Nb
\end{align*}
\]

**Definition 5.1** Let \( E = \bigotimes_{n \geq 0} E_n \) be a quaternion Banach space. Define the quaternion homology of \( E \) by the formula:

\[
\alpha HQ_n(E) = H_n(\text{Tot}^{\alpha} \varepsilon (E)).
\] (35)

Let \( E = Q_n(A) \), then \( \alpha HQ_n(E) = H_n(\text{Tot}^{\alpha} \varepsilon (A)) \).

Consider now the bicomplex consisting the first four columns of the bicomplex \( \alpha \varepsilon (E) \) we shall denote it by \( \alpha p(E) \), and suppose the following exact sequence

\[
o \rightarrow \alpha p(E) \rightarrow \text{Tot}^{\alpha} \varepsilon (EA) \rightarrow \text{Tot}^{\alpha} \varepsilon (EA) \rightarrow 0
\] (36)

following [8] the homology of the complex \( \alpha P(E) \) is periodic and given by:

\[
\alpha HP_n(E) = H_n(\text{Tot}^{\alpha} P(E)).
\] (37)

Since the bicomplex \( \alpha \varepsilon (E) \) has 4-periodic rows, we get the following exact sequence relating the periodic homology with quaternion homology.

**Theorem 5.2** there exists the exact sequence

\[
\begin{align*}
    &\rightarrow \alpha HP_n(E) \rightarrow \alpha HQ_n(E) \rightarrow \alpha HQ_{n-4}(E) \rightarrow \\
    &\rightarrow \alpha HP_{n-1}(E) \rightarrow \alpha HQ_{n-1}(E) \rightarrow \alpha HQ_{n-5}(E) \rightarrow
\end{align*}
\] (38)

following [7] the relation between \( \alpha HP_n(E) \) and the dihedral homology is given by:

\[
\begin{align*}
    &\rightarrow \alpha HP_n(E) \rightarrow \alpha HD_n(E) \rightarrow \alpha HD_{n-4}(E) \rightarrow \\
    &\rightarrow \alpha HP_{n-1}(E) \rightarrow \alpha HD_{n-1}(E) \rightarrow \alpha HD_{n-5}(E) \rightarrow
\end{align*}
\] (39)

Comparing the relations 39 and 38 we get the relation between dihedral and quaternion homology in the following.
Proposition 5.3 There exist the following natural isomorphism between the dihedral and quaternion homology

\[ {^1HQ_n}(A) \cong {^1HD_n}(A), \]
\[ {^{-1}HQ_n}(A) \cong {^{-1}HD_n}(A), \]

Similarly, one can define a quaternion cohomology for an unital Banach algebra with an involution.

References


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