An Analog of Titchmarsh’s Theorem of Jacobi Transform

Radouan Daher
Faculty of Sciences Ain Ckock, Casablanca, Morocco
radaher@yahoo.fr

Mohamed El Hamma
Faculty of Sciences Ain Ckock, Casablanca, Morocco
m_elhamma@yahoo.fr

Abstract

In this paper, we prove an analog of Titchmarsh’s theorem for the Jacobi transform for functions satisfying the Jacobi-Lipschitz condition in $L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t)dt)$, using a generalized translation operator.

Mathematics Subject Classification: 33C45; 43A90; 42C15

Keywords: Jacobi operator, Jacobi transform, generalized translation operator

1 Introduction and notations

In the present paper, we study a the set of functions in $L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t)dt)$ satisfying the Cauchy Lipschitz condition, we use translation operator $\tau_h$ associated of Jacobi operators, the operator has played a decisive role in the development of Euclidean harmonic analysis, as evidenced, for example, by landmark paper [3] by Hörmander.

Titchmarsh’s [7, Theorem 85] characterized the set of functions $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of asymptotic estimate growth of the norm of their Fourier transform, namely we have

**Theorem 1.1** [7] Let $\alpha \in (0, 1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents:

1. $\|f(t + h) - f(t)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ as $h \to 0$

2. $\int_{|\lambda| \geq r} |\hat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ as $r \to \infty$

where $\hat{f}$ stands for the Fourier transform of $f$. 
The main aim of this paper is to establish of Theorem 1.1 in Jacobi operators setting by means the generalized translation operator $\tau_h$ defined in Section 2. We point out that similar results have been established in the context of noncompact rank 1 Riemannian symmetric spaces [6].

In this section, we briefly collect the pertinent definitions and facts relevant for Jacobi analysis, can be found in [4].

Let $(a)_0 = 1$ and $(a)_k = a(a+1)......(a+k-1)$. The hypergeometric function

$$F(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, \quad |z| < 1$$

the function $z \rightarrow F(a, b, c, z)$ is the unique solution of the differential equation

$$z(1-z)u''(z) + (c-(a+b+1)z)u'(z) - abu(z) = 0$$

which is regular in 0 and equals 1 there.

The Jacobi function with parameters $(\alpha, \beta)$ is defined by the formula

$$\varphi^{(\alpha,\beta)}_\lambda(t) = F\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha + 1, -\sinh^2 t\right)$$

For $\alpha \geq-\frac{1}{2}$, $\alpha > \beta \geq-\frac{1}{2}$, $\rho = \alpha + \beta + 1$, the system $\{\varphi^{(\alpha,\beta)}_\lambda\}_{\lambda \geq 0}$ is a continuous orthonormal system in $\mathbb{R}^+$, with respect to the weight

$$\Delta_{\alpha,\beta}(t) = (2 \sinh t)^{2\alpha+1}(2 \cosh t)^{2\beta+1}$$

The Jacobi operator

$$L = L_{\alpha,\beta} = \frac{d^2}{dt^2} + ((2\alpha + 1) \coth t + (2\beta + 1) \tanh t) \frac{d}{dt}$$

By means of which the Jacobi function $\varphi^{(\alpha,\beta)}_\lambda$ may alternatively be characterized as the unique solution to

$$L\varphi + (\lambda^2 + \rho^2)\varphi = 0 \quad (1)$$

on $\mathbb{R}^+$ satisfying $\varphi^{(\alpha,\beta)}_\lambda(0) = 1$, $\varphi^{(\alpha,\beta)}_\lambda'(0) = 0$, and $\lambda \rightarrow \varphi^{(\alpha,\beta)}_\lambda(t)$ is analytic for all $t \geq 0$.

We adhere to the conventions and normalization used [2], the c-function

$$c(\lambda) = \frac{2^\rho \Gamma(\rho)(1+i\lambda)}{\Gamma(1/2(\rho+i\lambda))\Gamma(1/2(\rho+i\lambda)-\beta)}$$

where $\alpha > 0$, $\alpha > \beta \geq-\frac{1}{2}$.

The Jacobi transform of a function $f \in L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t)dt)$ is defined by
\[ \hat{f}(\lambda) = \int_0^\infty f(t) \varphi^{(\alpha,\beta)}_\lambda(t) \Delta_{\alpha,\beta}(t) \, dt \]  
(2)

and the inversion formula is statement that (cf. [4])

\[ f(t) = \frac{1}{2\pi} \int_0^\infty \hat{f}(\lambda) \varphi^{(\alpha,\beta)}_\lambda(t) d\mu(\lambda) \]  
(3)

where \( d\mu(\lambda) = |c(\lambda)|^{-2} d\lambda \).

The Jacobi transform is a unitary isomorphism from \( L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t) \, dt) \) onto \( L^2(\mathbb{R}^+, \frac{1}{2\pi} d\mu(\lambda)) \), i.e.

\[ \|f\| = \|f\|_{L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t) \, dt)} = \|\hat{f}\|_{L^2(\mathbb{R}^+, \frac{1}{2\pi} d\mu(\lambda))} \]  
(4)

The limiting case \( \alpha = \beta = -\frac{1}{2} \) is the Fourier-cosine transform, which we will not study.

We have

\[ \hat{\Delta}(\lambda) = -\lambda^2 \hat{f}(\lambda) \]  
(5)

The generalized translation operator was defined by Flensted-Jensen and Koornwinder [2, Formula (5.1)] given by

\[ \tau_y f(x) = \int_0^\infty f(z) K(x, y, z) \Delta_{\alpha,\beta}(z) \, dz \]

with kernel

\[ K(x, y, z) = \frac{2^{-2\rho} \Gamma(\alpha + 1)(\cosh x \cosh y \cosh z)^{-\alpha-\beta-1}}{\Gamma(\frac{1}{2}) \Gamma(\alpha + \frac{1}{2}) (\sinh x \sinh y \sinh z)^{2\alpha}} (1 - B^2)^{\alpha - \frac{1}{2}} \times F((\alpha + \beta, \alpha - \beta, \alpha + \frac{1}{2}, \frac{1}{2}, 1 - B)) \]

for \( |x - y| < z < x + y \) and \( K(x, y, z) = 0 \) elsewhere and

\[ B = \frac{\cosh^2 x + \cosh^2 y + \cosh^2 z - 1}{2 \cosh x \cosh y \cosh z} \]

in [1], we have

\[ \tau_h f(\lambda) = \varphi^{(\alpha,\beta)}_\lambda(h) \hat{f}(\lambda) \]  
(6)

For \( \alpha \geq -\frac{1}{2} \), we introduce the Bessel normalized function of the first kind \( j_\alpha \) defined by
\[ j_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(n + \alpha + 1)}, \quad z \in \mathbb{C} \] (7)

Moreover, from (7) we see that

\[ \lim_{z \to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0 \] (8)

by consequence, there exist \( c > 0 \) and \( \eta > 0 \) satisfying

\[ |z| \leq \eta \implies |j_\alpha(z) - 1| \geq c|z|^2 \] (9)

**Lemma 1.2** The following inequalities are valid for Jacobi functions \( \varphi^{(\alpha,\beta)}(t) \)

1. \( |\varphi^{(\alpha,\beta)}(t)| \leq 1 \) (10)
2. \( 1 - \varphi^{(\alpha,\beta)}(t) \leq t^2(\lambda^2 + \rho^2) \) (11)

**Proof:** analog (see [lemmas 3.1-3.2, 3])

**Lemma 1.3** Let \( \alpha > \frac{1}{2}, \quad \alpha \geq \beta \geq -\frac{1}{2} \). Then for \( |\eta| \leq \rho \), there exists a positive constant \( c_1 \) such that

\[ |1 - \varphi^{(\alpha,\beta)}(t)| \geq c_1 |1 - j_\alpha(\mu t)| \] (12)

**Proof:** (see[Lemma 9, 1])

## 2 An analog of Titchmarsh’s Theorem

In this section we give the main resultat of this paper, we need first to define Jacobi-Lipschitz class.

**Definition 2.1** Let \( \delta \in (0, 1) \). A function \( f \in L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t)dt) \) is said to be in the Jacobi-Lipschitz class, denote by \( \text{Lip}(\delta, 2) \), if

\[ \|\tau_h f(x) - f(x)\| = O(h^\delta), \quad \text{as } h \to 0 \]

**Theorem 2.2** Let \( f \in L^2(\mathbb{R}^+, \Delta_{\alpha,\beta}(t)dt) \). Then the following are equivalents

1. \( f \in \text{Lip}(\delta, 2) \).
2. \( \int_r^\infty |\hat{f}(\lambda)|^2 d\mu(\lambda) = O(r^{-2\delta}) \) as \( r \to +\infty \).
**Proof:** \(1 \implies 2\): Assume that \(f \in \text{Lip}(\delta, 2)\). Then we have

\[ \|\tau_h f(x) - f(x)\| = O(h^\delta), \quad \text{as } h \to 0 \]

Formulas (4) and (6) gives

\[ \|\tau_h f(x) - f(x)\|^2 = \int_0^\infty |1 - \varphi_X^{(\alpha,\beta)}(h)|^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) \]

Since (12) and \(\lambda \in \mathbb{R}^+\), we have

\[ \int_{\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} |1 - \varphi_X^{(\alpha,\beta)}(h)|^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) \geq c_1 \int_{\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} |1 - j_\alpha(\lambda h)|^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) \]

From (9), we obtain

\[ \int_{\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} |1 - \varphi_X^{(\alpha,\beta)}(h)|^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) \geq \frac{c_1 c^2 \eta^4}{16} \int_{\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} |\hat{f}(\lambda)|^2 d\mu(\lambda) \]

There exists then a positive constant \(K\) such that

\[ \int_{\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} |\hat{f}(\lambda)|^2 d\mu(\lambda) \leq K \int_0^\infty |1 - \varphi_X^{(\alpha,\beta)}(h)|^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) \]

\[ \leq K h^{2\delta} \]

For all \(h > 0\). Then

\[ \int_{r}^{2r} |\hat{f}(\lambda)|^2 d\mu(\lambda) \leq K r^{-2\delta} \]

for all \(r > 0\). Furthermore, we have

\[ \int_r^\infty |\hat{f}(\lambda)|^2 d\mu(\lambda) = \sum_{i=0}^{\infty} \int_{2^i r}^{2^{i+1} r} |\hat{f}(\lambda)|^2 d\mu(\lambda) \]

\[ \leq K \sum_{i=0}^{\infty} (2^i r)^{-2\delta} \]

\[ \leq K r^{-2\delta} \]

This proves

\[ \int_r^\infty |\hat{f}(\lambda)|^2 d\mu(\lambda) = O(r^{-2\delta}) \quad \text{as } r \to +\infty \]

\(2 \implies 1\): Suppose now that
\[ \int_r^\infty |\hat{f}(\lambda)|^2 d\mu(\lambda) = O(r^{-2\delta}) \quad \text{as } r \to +\infty \]

we write
\[ \|\tau_h f(x) - f(x)\|^2 = I_1 + I_2 \]

where
\[ I_1 = \int_0^1 |1 - \varphi^{(\alpha,\beta)}(h)|^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) \]

and
\[ I_2 = \int_1^\infty |1 - \varphi^{(\alpha,\beta)}(h)|^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) \]

Estimate the summands \( I_1 \) and \( I_2 \).
we have from (10)
\[ I_2 \leq 4 \int_{\frac{1}{h}}^\infty |\hat{f}(\lambda)|^2 d\mu(\lambda) = O(h^{2\delta}) \]

To estimate \( I_1 \), we use the inequality (10).
\[ I_1 = \int_0^1 |1 - \varphi^{(\alpha,\beta)}(h)|^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) \leq 2 \int_0^1 |1 - \varphi^{(\alpha,\beta)}(h)||\hat{f}(\lambda)|^2 d\mu(\lambda) \]

From the inequality (11), we have
\[ I_1 \leq 2h^2 \int_0^\infty (\lambda^2 + \rho^2) |\hat{f}(\lambda)|^2 d\mu(\lambda) \]
\[ = 2\rho^2 h^2 \int_0^\infty |\hat{f}(\lambda)|^2 d\mu(\lambda) + 2h^2 \int_0^\frac{1}{h} \lambda^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) \]

Note that
\[ 2\rho^2 h^2 \int_0^\frac{1}{h} |\hat{f}(\lambda)|^2 d\mu(\lambda) \leq 2\rho^2 h^2 \int_0^\infty |\hat{f}(\lambda)|^2 d\mu(\lambda) \]
\[ = 2\rho^2 \|f\|^2 \]
\[ = O(h^{2\delta}) \]

since \( 2\delta < 2 \)
We put
An analog of Titchmarsh’s theorem

\[ \psi(r) = \int_r^\infty |\hat{f}(\lambda)|^2 d\mu(\lambda) \]

Integrating by parts, we obtain

\[ 2h^2 \int_0^{\frac{1}{h}} \lambda^2 |\hat{f}(\lambda)|^2 d\mu(\lambda) = 2h^2 \int_0^{\frac{1}{h}} (-r^2 \psi'(r)) dr \]

\[ = 2h^2 (-\frac{1}{h^2} \psi(1) + 2 \int_0^{\frac{1}{h}} r \psi(r) dr) \]

\[ = -2 \psi(1) + 4h^2 \int_0^{\frac{1}{h}} r \psi(r) dr \]

\[ \leq 4Ch^2 \int_0^{\frac{1}{h}} r^{1-2\delta} dr \]

\[ = O(h^{2\delta}) \]

Finally, then

\[ \| \tau_h f(x) - f(x) \| = O(h^\delta), \quad \text{as } h \to 0 \]

which completes the proof of Theorem.

References


Received: September, 2011