Beurling’s Theorem for Vector-Valued Hardy Spaces

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1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space $H^2 := H^2(\mathbb{D})$ is the space of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$
\|f\|_{H^2} := \lim_{r \to 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}|^2 \, d\theta \right)^{1/2} < +\infty
$$

A closed linear subspace $M$ of $H^2(\mathbb{D})$ is said to be shift invariant if $M$ is invariant under the multiplication operator by $z$ (shift operator) on $H^2(\mathbb{D})$. Beurling’s theorem states that every shift invariant subspace of $H^2(\mathbb{D})$ is either zero subspace or of the form $\varphi H^2(\mathbb{D})$, where $\varphi$ is an inner function in $H^2(\mathbb{D})$, a bounded analytic function on $\mathbb{D}$ with non-tangential boundary values of modulus 1 almost everywhere with respect to the Lebesgue measure on the unit circle $\partial \mathbb{D}$ [1]. Beurling’s theorem is viewed as one of the most celebrated theorems in operator theory and it has been extended to many directions [1,
In this paper, we study the invariant space problem for the shift operator on vector-valued Hardy space $H^2(\mathbb{D}, E)$. Recall that a holomorphic function $f : \mathbb{D} \to E$ belongs to the vector-valued Hardy space $H^2(E) := H^2(\mathbb{D}, E)$, if

$$
\|f\|_{H^2(E)} := \lim_{r \to 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_E^2 d\theta \right)^{1/2} < +\infty.
$$

In fact, $H^2(E)$ becomes a Hilbert space with the following inner product

$$
< f, g >_{H^2(E)} := \lim_{r \to 1} \left( \frac{1}{2\pi} \int_0^{2\pi} < f(re^{i\theta}), g(re^{i\theta}) >_E d\theta \right).
$$

A more detailed discussion of vector-valued analytic functions and Hardy spaces can be found in Hille and Philips [5], Rosenblum and Rovnyak [7], Hensgen [4] and a convenient reference for classical Hardy spaces is Duren [2]. It is not too hard to see the shift operator $S$ on $H^2(E)$ is well defined and bounded. The main result of this paper gives a characterization of shift invariant subspaces of vector-valued Hardy space $H^2(E)$. It involves with the well known Hadamard product on Hilbert space. Recall that for given Hilbert space $E$ with an orthonormal basis $\{e_n : n \in I\}$ the Hadamard product of two vectors $x$ and $y$ in $E$ is defined by

$$
x \ast y = \sum_{n \in I} < x, e_n > < e_n, y >
$$

An application of Cauchy-Schwarz inequality with the Parseval identity implies that the Hadamard product is well defined and $\|x \ast y\| \leq \|x\| \|y\|$. Moreover, the Hadamard product of two analytic function $f$ and $g$ is determined by

$$(f \ast g)(z) := f(z) \ast g(z) = \sum_{n=1}^{+\infty} < f(z), e_n > < g(z), e_n > e_n.
$$

For a separable Hilbert space $E$ and a nonzero shift invariant subspace $\mathcal{M}$ of $H^2(E)$ we show there exist a vector-valued function $\Phi : \mathbb{D} \to E$ such that $\|\Phi(z)\| = 1$ almost every where on $\partial \mathbb{D}$ and

$$
\mathcal{M} = \Phi \ast H^2(E) = \{ \Phi \ast F : F \in H^2(E) \},
$$

provided that $E \ast \mathcal{M} \subseteq \mathcal{M}$. Here $E \ast \mathcal{M}$ is the collection of all functions $a \ast f$ mapping $z \to f(z)a$ for $a \in E$ and $f \in \mathcal{M}$. In particular, when $E$ is of finite dimensional the condition $E \ast \mathcal{M} \subseteq \mathcal{M}$ is hold and the shift invariant subspaces of $H^2(E)$ are represented as above.
2 vector-valued hardy space

In order to state our main result we have to introduce some notation. Unless otherwise stated we assume that $E$ is a Hilbert space with the orthonormal basis \{e_i : i \in I\} for some index set $I$. For a vector $x \in E$ and a vector-valued function $f : \mathbb{D} \rightarrow E$, the scalar function $x \otimes f : \mathbb{D} \rightarrow \mathbb{C}$ is defined by

$$(x \otimes f)(z) = \langle f(z), x \rangle.$$ 

It is easy to see, $f$ is holomorphic if and only if $x \otimes f$ is holomorphic for every $x \in E$. By this, we can present any holomorphic vector-valued function $f$ by

$$f = \sum_{i \in I} (e_i \otimes f)e_i.$$ 

Also for a subspace $M$ of vector-valued holomorphic functions, the subspace $x \otimes M$, is defined by

$$x \otimes M = \{x \otimes f : f \in M\}.$$ 

in a natural way. If \{H_i : i \in I\} is a collection of Hilbert spaces, then $H := (\bigoplus_{i \in I} H_i)_{\ell^2}$ is defined as the collection of square summable nets; that is, nets of the form $h = \{h_i\}_{i \in I}$ such that $h_i \in H_i$ for all $i$ and $\sum_{i \in I} \|h_i\|^2 < +\infty$. Moreover, $H$ is a Hilbert space by the following inner product:

$$<f, g>_H = \sum_{i \in I} <f_i, g_i>_{H_i}$$

for all $f = \{f_i\}_{i \in I}, g = \{g_i\}_{i \in I} \in H$. Of course, if a net of operators $T_i$ acting on $H_i$ is such that $\{|T_i|\}_{i \in I}$ is bounded, then $T := \bigoplus_{i \in I} T_i$ defined in a natural way, is a bounded linear operator on $(\bigoplus_{i \in I} H_i)_{\ell^2}$.

In the following the structure of vector-valued Hardy spaces has been stated as a direct sum of scalar-valued copies:

**Theorem 2.1.** Let $E$ be a Hilbert space then $H^2(E)$ is isometrically isomorphic to $(\bigoplus_{i \in I} H^2)_{\ell^2}$, for some index set $I$.

**Proof.** Let $\{e_i : i \in I\}$ be an orthonormal basis for Hilbert space $E$, for some index set $I$. Put $\mathcal{H} = (\bigoplus_{i \in I} H^2)_{\ell^2}$ and define $\Lambda : H^2(E) \rightarrow \mathcal{H}$ by $\Lambda(f) = \{e_i \otimes f\}_{i \in I}$, then

$$<\Lambda(f), \Lambda(g)>_{\mathcal{H}} = <\{e_i \otimes f\}_{i \in I}, \{e_i \otimes g\}_{i \in I}>_{\mathcal{H}}$$

$$= \sum_{i \in I} <e_i \otimes f, e_i \otimes g>_H$$

$$= \sum_{i \in I} \lim_{r \rightarrow 1} \left( \frac{1}{2\pi} \int_0^{2\pi} (e_i \otimes f)(re^{i\theta})(e_i \otimes g)(re^{i\theta}) \, dm(t) \right)$$
\[
= \lim_{r \to 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{i \in I} \langle f(re^{i\theta}), e_i \rangle < g(re^{i\theta}), e_i \rangle \right) dm(t) \right)
= \lim_{r \to 1} \left( \frac{1}{2\pi} \int_0^{2\pi} < f(re^{i\theta}), g(re^{i\theta}) > dm(t) \right) = < f, g >_{H^2(E)} .
\]

It remains to prove \( \Lambda \) is a surjection. Let \( g = \{ f_i \}_i \in \mathcal{H} \), then by the Parseval identity,

\[
f(t) = \sum_{i \in I} f_i(t) e_i
\]
defines a holomorphic function in \( H^2(E) \) and \( \Lambda(f) = g \).

**Corollary 2.2.** The vector-valued function \( f \) belongs to \( H^2(E) \) if and only if

\[
\sum_{i \in I} \| e_i \otimes f \|_{H^2}^2 < +\infty
\]

Conversely, if \( \{ f_i \} \) is a net in \( H^2 \), such that \( \sum_{i \in I} \| f_i \|_{H^2}^2 < +\infty \) then

\[
f(z) := \sum_{i \in I} f_i(z) e_i
\]
defines a holomorphic function in \( H^2(E) \) and \( \| f \|_{H^2(E)} = \sum_{i \in I} \| f_i \|_{H^2}^2 .
\]

### 3 Beurling's Theorem

In what follows \( E \) is a separable Hilbert space with an orthonormal basis \( \{ e_n : n \in \mathbb{N} \} \). Recall that the shift operator \( S : H^2(E) \to H^2(E) \) defined by \( S(f)(z) = zf(z) \) is well defined and bounded. From now on, we call the invariant subspaces \( H^2(E) \) under \( S \) as a shift invariant subspace. By an scalar-valued inner function we mean a bounded analytic function \( \varphi \) such that \( |\varphi(e^{i\theta})| = 1 \) almost everywhere.

**Theorem 3.1** (Beurling). Let \( M \) be a nonzero shift invariant subspace of \( H^2 \) then there is an inner function \( \varphi \) such that \( M = \varphi H^2 \).

**Definition 3.2.** For any two \( E \)-valued function \( f, g \) in \( H^2(E) \), the Hadamard product \( f \) and \( g \) is defined by the following:

\[
(f \ast g)(z) = \sum_{n=1}^{+\infty} < f(z), e_n > < g(z), e_n > e_n.
\]
In fact,

\[ f \ast g = \sum_{n=1}^{+\infty} (e_n \otimes f)(e_n \otimes g)e_n. \]

By Cauchy-Schwarz inequality, \( f \ast g \) is a holomorphic \( E \)-valued function on \( \mathbb{D} \).

In particular, for any \( a \in E \) the Hadamard product \( a \ast g \) means the Hadamard product of the constant vector-valued function \( f \equiv a \) with \( g \). More generally, for a nonzero subspace \( \mathcal{M} \) of \( H^2(E) \) the Hadamard product \( E \ast \mathcal{M} \) is denoted by \( E \ast \mathcal{M} \) and defined by

\[ E \ast \mathcal{M} := \{ a \ast f : a \in E \text{ and } f \in \mathcal{M} \} \]

**Theorem 3.3.** Let \( \mathcal{M} \) be a nonzero shift invariant subspace of \( H^2(E) \) then \( E \ast \mathcal{M} \subseteq \mathcal{M} \) if and only if there is a vector-valued bounded analytic \( \Phi : \mathbb{D} \rightarrow E \) such that \( \| \Phi(e^{i\theta}) \|_E = 1 \) almost everywhere on \( \partial \mathbb{D} \) and \( \mathcal{M} = \Phi \ast H^2(E) \).

**Proof.** Assume that \( \mathcal{M} = \Phi \ast H^2(E) \) then it is easy to see that \( E \ast \mathcal{M} \subseteq \mathcal{M} \).

For the converse let \( \mathcal{M} \) be a shift invariant subspace such that \( E \ast \mathcal{M} \subseteq \mathcal{M} \). Since \( \mathcal{M} \) is shift invariant subspace of \( H^2(E) \), every close subspace \( e_n \otimes \mathcal{M} \) in \( H^2 \) is also shift invariant for any integer \( n \geq 1 \). Let \( A := \{ n \in \mathbb{N} : e_n \otimes \mathcal{M} \neq \{0\} \} \). Consider the scalar sequence \( \{\beta_n\} \) defined in terms of the cardinality of \( A \):

\[ \beta_n = \begin{cases} |A|^{1/2} & \text{if } n \in A \text{ is finite} \\ 2^{n/2} & \text{if } n \in A \text{ is infinite} \end{cases} \]

where \( |A| \) means the cardinality of \( A \). In fact, the sequence \( \{\beta_n\} \) is chosen in such a way that

\[ \sum_{n \in A} \beta_n^2 = 1. \]

By using the Beurling’s Theorem in scalar-valued case we obtain an inner function \( \varphi_n \) such that

\[ \beta_n^{-1}e_n \otimes \mathcal{M} = \varphi_n H^2 \]

for any \( n \in A \). If the complement of \( A \) is nonempty we set \( \varphi_n = 0 \) for every \( n \notin A \). Put

\[ \Phi = \sum_{n=1}^{+\infty} \beta_n \varphi_n e_n = \sum_{n \in A} \beta_n \varphi_n e_n. \]

Then \( \Phi : \mathbb{D} \rightarrow E \) is a holomorphic function and we claim \( \mathcal{M} = \Phi \ast H^2(E) \).

Assume that \( f \in \mathcal{M} \), and \( n \in A \). Then \( e_n \otimes f = \beta_n^{-1} \varphi_n f_n \) for some \( f_n \in H^2 \).
Let \( F := \sum_{n \in A} \beta_n^{-1} f_n e_n \), where \( (f_n e_n)(z) = f_n(z) e_n \). Note that each \( \varphi_n \) is an inner function, hence
\[
\|f_n\|_{H^2} = \|\varphi_n f_n\|_{H^2} = \beta_n \|e_n \otimes f\|_{H^2}.
\]
Since \( f \in H^2 \), \( \sum_{n=1}^{+\infty} \|e_n \otimes f\|_{H^2}^2 \) is finite, so is \( \sum_{n=1}^{+\infty} \beta_n^{-1} \|f_n\|_{H^2}^2 \) and hence \( F \in H^2(E) \) by Corollary 2.3. Moreover,
\[
f = \sum_{n=1}^{+\infty} (e_n \otimes f)e_n = \sum_{n=1}^{+\infty} \beta_n^{-1} (\varphi_n f_n)e_n = \sum_{n=1}^{+\infty} \beta_n^{-1} (e_n \otimes \Phi)(e_n \otimes F)e_n = \Phi \ast F
\]
Now suppose \( F \in H^2(E) \) and \( n \in A \). Then
\[
m^{-1} \varphi_n (e_n \otimes F) \in \varphi_n H^2 = e_n \otimes \mathcal{M},
\]
whence \( m^{-1} \varphi_n (e_n \otimes F) = e_n \otimes f_n \) for some \( f_n \in A \). Hence,
\[
\Phi \ast F = \sum_{n=1}^{+\infty} [e_n \otimes (\Phi \ast F)]e_n = \sum_{n=1}^{+\infty} (e_n \otimes \Phi)(e_n \otimes F)e_n = \sum_{n \in A} m^{-1} \varphi_n (e_n \otimes F)e_n = \sum_{n \in A} (e_n \otimes f_n)e_n
\]
Note that \( (e_n \otimes f_n)e_n = e_n \ast f_n \in E \ast \mathcal{M} \subseteq \mathcal{M} \) for any \( n \in A \). Since \( \mathcal{M} \) is a close subspace,
\[
\Phi \ast F = \sum_{n \in A} (e_n \otimes f_n)e_n \\
\text{This implies that } \Phi \ast H^2(E) = \mathcal{M}. \text{ It remains to prove } \|\Phi(e^{i\theta})\| = 1 \text{ almost everywhere on } \partial \mathbb{D}. \text{ For this let } B_n = \{ e^{i\theta} \in \partial \mathbb{D} : |\varphi_n(z)| \neq 1 \} \text{ and } B = \cup_{n \in A} B_n. \text{ Since all } \varphi_n \text{ are inner functions, } \|\varphi_n\|_{H^2} = 1 \text{ and } B \text{ is of zero measure.}
\]
Now for any \( e^{i\theta} \notin B \),
\[
\|\Phi(e^{i\theta})\|_{H^2(E)}^2 = \sum_{n \in A} \beta_n^2 \|\varphi_n(e^{i\theta})\|_{H^2}^2 = \sum_{n \in A} \beta_n^2 = 1
\]
This ends the proof.

It is easy to see that if \( E \) is a finite dimensional Hilbert space then \( E \ast \mathcal{M} \subseteq \mathcal{M} \) and so the following corollary is deduced:

**Corollary 3.4.** Let \( E \) is a finite dimensional Hilbert space and \( \mathcal{M} \) be a nonzero shift invariant subspace of \( H^2(E) \) then there is a vector-valued bounded analytic \( \Phi : \mathbb{D} \to E \) such that \( \|\Phi(e^{i\theta})\|_E = 1 \) almost everywhere on \( \partial \mathbb{D} \) and \( \mathcal{M} = \Phi \ast H^2(E) \).
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