$n$-Tuples and Chaoticity

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Abstract

In this paper we characterize the Condition for Chaoticity of Tuples of operators on a Frechet space.

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1 Introduction

Let $T_1, T_2, ..., T_n$ be commutative bounded linear operators on a Banach space $\mathcal{X}$. For $n$-Tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$, put

$$\Gamma = \{T_1^{m_1}T_2^{m_2}...T_n^{m_n} : m_1, m_2, ..., m_n \geq 0\}$$

the semigroup generated by $\mathcal{T}$. For $x \in \mathcal{X}$, the orbit of $x$ under $\mathcal{T}$ is the set $\text{Orb}(\mathcal{T}, x) = \{S(x) : S \in \Gamma\}$, that is

$$\text{Orb}(\mathcal{T}, x) = \{T_1^{m_1}T_2^{m_2}...T_n^{m_n}(x) : m_1, m_2, ..., m_n \geq 0\}$$

The vector $x$ is called Hypercyclic vector for $\mathcal{T}$ and $n$-Tuple $\mathcal{T}$ is called Hypercyclic $n$-Tuple, if the set $\text{Orb}(\mathcal{T}, x)$ is dense in $\mathcal{X}$, that is

$$\text{Orb}(\mathcal{T}, x) = \{T_1^{m_1}T_2^{m_2}...T_n^{m_n}(x) : m_1, m_2, ..., m_n \geq 0\} = \mathcal{X}$$

The vector $x$ in $\mathcal{X}$ is called a Periodic vector for the $n$-Tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$, if there exist some numbers $\mu_1, \mu_2, ..., \mu_n \in \mathbb{N}$ such that

$$T_1^{\mu_1}T_2^{\mu_2}...T_n^{\mu_n}(x) = x.$$
Also the $n$-Tuple $T = (T_1, T_2, ..., T_n)$, is called chaotic tuple, if we have tree below conditions together,

1. It is topologically transitive, that is, if for any given open sets $U$ and $V$, there exist positive integer numbers $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{N}$ such that

   $T_{\alpha_1}^1 T_{\alpha_2}^2 ... T_{\alpha_n}^n (U) \cap V \neq \emptyset$

2. It has a dense set of periodic points, in other word, there is a set $X$ such that for each $x \in X$, there exist some numbers $\beta_1, \beta_2, ..., \beta_n \in \mathbb{N}$ such that

   $T_{\beta_1}^1 T_{\beta_2}^2 ... T_{\beta_n}^n (x) = x$

3. It has a certain property called sensitive dependence on initial conditions.

Notice that, all operators in this paper are commutative. For some topics see [1–12].

### 2 Main Results

**Theorem 2.1.** [The Hypercyclicity Criterion] Let $X$ be a separable Banach space and $T = (T_1, T_2, ..., T_n)$ is an $n$-tuple of continuous linear mappings on $X$. If there exist two dense subsets $Y$ and $Z$ in $X$, and $n$ strictly increasing sequences $\{m_j, 1 \}$, $\{m_j, 2 \}$, ..., $\{m_j, n \}$ such that:

1. $T_{m_j, 1}^1 T_{m_j, 2}^2 ... T_{m_j, n}^n \to 0$ on $Y$ as $j \to \infty$,
2. There exist function $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_{m_j, 1}^1 T_{m_j, 2}^2 ... T_{m_j, n}^n S_j z \to z$,

then $T$ is a Hypercyclic $n$-tuple.

If the tuple $T$ satisfying the hypothesis of previous theorem then we say that $T$ satisfying the Hypercyclicity criterion.

**Theorem 2.2.** Suppose $X$ be an F-sequence space whith the unconditional basis $\{e_\kappa\}_{\kappa \in \mathcal{N}}$. Let $T_1, T_2, ..., T_n$ are unilateral weighted backward shifts with weight sequence $\{a_{i, 1} : i \in \mathcal{N}\}$, $\{a_{i, 2} : i \in \mathcal{N}\}$, ..., $\{a_{i, n} : i \in \mathcal{N}\}$ and $T = (T_1, T_2, ..., T_n)$ be an $n$-tuple of operators $T_1, T_2, ..., T_n$. Then the following assertions are equivalent:

1. $T$ is chaotic,
2. $T$ is Hypercyclic and has a non-trivial periodic point,
3. $T$ has a non-trivial periodic point,
4. the series $\sum_{m=1}^{\infty} \prod_{k=1}^{m} (a_{k, i})^{-1} e_m$ convergence in $X$ for $i = 1, 2, ..., n$.

**Proof.** Proof of the cases (1) $\to$ (2) and (2) $\to$ (3) are trivial, so we just proof (3) $\to$ (4) and (4) $\to$ (1). First we proof (3) $\to$ (4), for this, Suppose
that \( \mathcal{T} \) has a non-trivial periodic point, and \( x = \{x_n\} \in \mathcal{X} \) be a non-trivial periodic point for \( \mathcal{T} \), that is there are \( \mu_1, \mu_2, \ldots, \mu_n \in \mathbb{N} \) such that,

\[
T_1^{\mu_1}T_2^{\mu_2} \ldots T_n^{\mu_n}(x) = x.
\]

Comparing the entries at positions \( j + kN \), and \( j + kN, \ k \in \mathbb{N} \cup \{0\} \), of \( x \) and \( T_1^{M}T_2^{N}(x) \) we find that

\[
x_{j+kM_1} = \left( \prod_{t=1}^{M} (a_{j+kN+t}) \right) x_{j+(k+1)}
\]

\[
x_{j+kM_2} = \left( \prod_{t=1}^{N} (b_{j+kN+t}) \right) x_{j+(k+1)}
\]

\[
\ldots
\]

\[
x_{j+kM_n} = \left( \prod_{t=1}^{M} (b_{j+kN+t}) \right) x_{j+(k+1)}
\]

so that we have,

\[
x_{j+kM_1} = \left( \prod_{t=j+1}^{j+kM_1} (a_t)^{-1} \right) x_j = c_1 \left( \prod_{t=1}^{j+kM_1} (a_t)^{-1} \right), \ k \in \mathbb{N} \cup \{0\}
\]

\[
x_{j+kM_2} = \left( \prod_{t=j+1}^{j+kM_2} (a_t)^{-1} \right) x_j = c_2 \left( \prod_{t=1}^{j+kM_2} (a_t)^{-1} \right), \ k \in \mathbb{N} \cup \{0\}
\]

\[
\ldots
\]

\[
x_{j+kM_n} = \left( \prod_{t=j+1}^{j+kM_n} (a_t)^{-1} \right) x_j = c_n \left( \prod_{t=1}^{j+kM_n} (a_t)^{-1} \right), \ k \in \mathbb{N} \cup \{0\}
\]

with

\[
c_1 = \left( \prod_{t=1}^{j} (m_{j,1}) \right) x_j
\]

\[
c_2 = \left( \prod_{t=1}^{j} (m_{j,2}) \right) x_j
\]

\[
\ldots
\]

\[
c_n = \left( \prod_{t=1}^{j} (m_{j,n}) \right) x_j.
\]

Since \( \{e_\kappa\} \) is an unconditional basis and \( x \in \mathcal{X} \) it follows from [..] that

\[
\sum_{k=0}^{\infty} \left( \prod_{t=1}^{j+kM_1} (m_{j,1}) \right) e_{j+kM_1} = \frac{1}{c_1} \sum_{k=0}^{\infty} x_{j+kM_1} e_{j+kM_1}
\]
we deduce that convergence in $\mathcal{X}$. Without loss of generality we may assume that $j \geq N$. Applying the operators $T$, $T^2$, $T^3$, ..., $T^{R-1}$, with $R = \text{Min}\{M_i : i = 1, 2, ..., n\}$, to this series and note that $T_1(e_n) = a_ne_{n-1}$ and $T_2(e_n) = b_ne_{n-1}$ for $n \geq 2$, we deduce that

$$\sum_{k=0}^{\infty} \left( \prod_{l=1}^{k} \frac{1}{(m_{j,2})} \right) e_{j+kM_2} = \frac{1}{c_2} \sum_{k=0}^{\infty} x_{j+kM_2} e_{j+kM_2}$$

$$\sum_{k=0}^{\infty} \left( \prod_{l=1}^{k} \frac{1}{(m_{j,n})} \right) e_{j+kM_n} = \frac{1}{c_n} \sum_{k=0}^{\infty} x_{j+kM_n} e_{j+kM_n}$$

convergence in $\mathcal{X}$ for $\gamma = 0, 1, 2, ..., N - 1$. By adding these series, we see that condition (4) holds. Proof of (4) $\Rightarrow$ (1). It follows from theorem (2.1), that under condition (4) the operator $T$ is Hypercyclic. Hence it remains to show that $\mathcal{T}_t$ is Hypercyclic. Hence it remains to show that $\mathcal{T}_t$ has a dense set of periodic points. Since $\{e_\kappa\}$ is an unconditional basis, condition (4) with proposition 2.3 implies that for each $j \in \mathcal{N}$ and $M, N \in \mathcal{N}$ the series

$$\psi_1(j, M_1) = \sum_{k=0}^{\infty} \left( \prod_{l=1}^{j+kM_1} \frac{1}{(m_{k,1})} \right) e_{j+kM_1} = \left( \prod_{l=1}^{j} m_{k,1} \right) \left( \sum_{k=0}^{\infty} \frac{1}{\prod_{l=1}^{j+kM_1} m_{k,1}} e_{j+kM_1} \right)$$

$$\psi_2(j, M_2) = \sum_{k=0}^{\infty} \left( \prod_{l=1}^{j+kM_2} \frac{1}{(m_{k,2})} \right) e_{j+kM_2} = \left( \prod_{l=1}^{j} m_{k,2} \right) \left( \sum_{k=0}^{\infty} \frac{1}{\prod_{l=1}^{j+kM_2} m_{k,2}} e_{j+kM_2} \right)$$

$$\psi_n(j, M_n) = \sum_{k=0}^{\infty} \left( \prod_{l=1}^{j+kM_n} \frac{1}{(m_{k,n})} \right) e_{j+kM_n} = \left( \prod_{l=1}^{j} m_{k,n} \right) \left( \sum_{k=0}^{\infty} \frac{1}{\prod_{l=1}^{j+kM_n} m_{k,n}} e_{j+kM_n} \right)$$

converges and define $n$ elements in $\mathcal{X}$. Moreover, if $M \geq i$ then

$$T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} = \psi_1(j, M_1)$$

$$T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} = \psi_2(j, M_2)$$

$$...$$

$$T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} = \psi_n(j, M_n)$$

$$T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} = \psi_1(j, M_1) T_1^{M_1} T_2^{M_2} ... T_n^{M_n} \psi_1(j, M_1) = \omega(j, M_1)$$

(1)
\[ T_{1}^{m_{1}} T_{2}^{m_{2}} \ldots T_{n}^{m_{n}} = \psi_{2}(j, M_{2}) T_{1}^{M_{1}} T_{2}^{M_{2}} \ldots T_{n}^{M_{n}} \psi_{2}(j, M_{2}) = \omega(j, M_{2}) \]

\[ T_{1}^{m_{1}} T_{2}^{m_{2}} \ldots T_{n}^{m_{n}} = \psi_{1}(j, M_{1}) T_{1}^{M_{1}} T_{2}^{M_{2}} \ldots T_{n}^{M_{n}} \psi_{1}(j, M_{1}) = \omega(j, M_{1}) \]

and, if \( N \geq j \) then

\[ T_{1}^{m_{1},1} T_{2}^{m_{2},2} \ldots T_{n}^{m_{n},n} \omega((j, i), N) = \omega((j, i), N) \quad (2) \]

for \( m_{j,i} \geq N \) and \( i = 1, 2, \ldots, n \). So that each \( \psi(j, N) \) for \( j \leq N \) is a periodic point for \( T \). We shall show that \( T \) has a dense set of periodic points. Since \( \{e_{\kappa}\} \) is a basis, it suffices to show that for every element \( x \in \text{span}\{e_{\kappa}: \kappa \in \mathcal{N}\} \) there is a periodic point \( y \) arbitrarily close to it. For this, let \( x = \sum_{j=1}^{m} x_{j} e_{j} \) and \( \varepsilon > 0 \). We can assume without lost of generality that

\[
| x_{i} \prod_{t=1}^{i} a_{t,1} | \leq 1 \quad , \quad i = 1, 2, 3, \ldots, m_{1}
\]

\[
| x_{j} \prod_{t=1}^{i} a_{t,2} | \leq 1 \quad , \quad i = 1, 2, 3, \ldots, m_{2}
\]

\[
| x_{i} \prod_{t=1}^{i} a_{t,n} | \leq 1 \quad , \quad i = 1, 2, 3, \ldots, m_{n}
\]

Since \( \{e_{n}\} \) is an unconditional basis, then condition (4) implies that there are an \( M, N \geq m \) such that

\[
\left\| \sum_{n=M_{1}+1}^{\infty} \varepsilon_{\kappa,1} \frac{1}{\prod_{t=1}^{i} a_{t,1}} e_{\kappa} \right\| < \frac{\varepsilon}{m_{1}}
\]

\[
\left\| \sum_{n=M_{2}+1}^{\infty} \varepsilon_{\kappa,2} \frac{1}{\prod_{t=1}^{i} a_{t,2}} e_{\kappa} \right\| < \frac{\varepsilon}{m_{2}}
\]

\[
\left\| \sum_{n=M_{n}+1}^{\infty} \varepsilon_{\kappa,n} \frac{1}{\prod_{t=1}^{i} a_{t,n}} e_{\kappa} \right\| < \frac{\varepsilon}{m_{n}}
\]

for every sequences \( \{\varepsilon_{\kappa,i}\}, i = 1, 2, \ldots, n \) taking values 0 or 1. By (1) and (2) the elements

\[
y_{1} = \sum_{i=1}^{m_{1}} x_{i} \psi(i, M_{1})
\]

\[
y_{2} = \sum_{i=1}^{m_{2}} x_{i} \psi(i, M_{2})
\]

...
\[ y_n = \sum_{i=1}^{m_n} x_i \psi(i, M_n) \]

of \( X \) is a periodic point for \( T \), and we have
\[
\| y_\lambda - x \| = \| \sum_{i=1}^{m_\lambda} x_i (\psi(i, M_\lambda) - e_i) \|
\]
\[
= \| \sum_{i=1}^{m_\lambda} x_i \prod_{t=1}^{\frac{i}{M_\lambda}} d_{t,M_\lambda} \left( \sum_{k=1}^{\infty} \frac{1}{\prod_{t=1}^{\frac{1+kM_\lambda}{M_\lambda}} a_{t,M_\lambda}} e_{i+kM_\lambda} \right) \|
\]
\[
\leq \sum_{i=1}^{m_\lambda} \left\| \sum_{k=1}^{\infty} \frac{1}{\prod_{t=1}^{\frac{1+kM_\lambda}{M_\lambda}} a_{t,M_\lambda}} e_{i+kM_\lambda} \right\|
\leq \epsilon
\]
as \( \lambda = 1, 2, \ldots, n \) and by this, the proof is complete.

References


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