Generators of Diagram Groups over Graphical Presentations of Integers with Four Initial Generators Using Lifting Method

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Abstract
In this study, we investigate to determine the generators of diagram groups over the graphical presentations with four initial generators by using lifting method. We consider the graphs $\Gamma_n, n \in \mathbb{N}$ obtained from these kinds of graphical presentation. The determinations were made by systematically applying a lifting method to a graph according to length of words. We determined the general formula for the total number of lifts of generators and the lifts of generators at any given word in general for the 2-complex of diagram groups of the graphical presentation with four initial generators.

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1 Introduction

In our previous researches we described the component of graphs, the spanning tree, and the generators for diagram group over the graphical presentation \[ S = \langle x_1, x_2, x_3 \mid x_1 = x_2, x_1 = x_3, x_2 = x_3 \rangle \] using lifting method. We also discussed construction of graphical presentations and class of diagram groups (refer to [8], [9], [10], [11], and [12]).

In this study we discuss the determination of lift of generators for diagram groups, particularly of the graphical with the presentation of integers with four initial generators \[ S = \{ x_i, x_2, x_3, x_4 \mid x_j = x_j; \ 1 \leq i < j \leq 4 \}. \] Let \( S = \langle X \mid R \rangle \) be a graphical presentation. Then we may obtain the diagram group \( D(S, \mathcal{W}) \) which \( w \) is a word on \( X \) as defined by Guba and Sapir (please see [5], [6], and [7]).

The 2-complex, associated with presentation \( S \) is denoted by \( K(S) \). As the complex we have fundamental group and we denoted by \( \pi_1(K(S), \mathcal{W}) \). Kilibarda has shown that the fundamental group is isomorphic to diagram group \( D(S, \mathcal{W}) \) (refer to [3], and [4]).

We will consider the fundamental group \( \pi_1(K(S), \mathcal{W}) \) constructed from graphical presentation \[ S = \langle x_1, x_2, x_3 \mid x_1 = x_2, x_1 = x_3, x_2 = x_3 \rangle. \]

For our presentation, the 2-complex consists of infinitely connected component \( \Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_n \). Note that all vertices in \( \Gamma_i \) are words of length \( i \). Thus if length \( (u) = \text{length} (v) \) then \( \pi_1(K(S), u) = \pi_1(K(S), v) \) (see [1]).

As a group, it is sufficient to determine its generators and relations. The generators of this group can be determined from the 2-complex \( K(S) \) by identifying the spanning tree \( T \). Fix a vertex \( v \in K(S) \) and the edge \( e \not\in T \). Then \( \gamma_{t(e)}^e \gamma_{t(e)}^{-1} \) is the generator, where \( \gamma_{t(e)}, \gamma_{t(e)} \) are paths in \( T \) from \( v \in K(S), \) to the initial and terminal of \( e \) respectively.

Let \( u_i \) be a word of length \( i \). We will show that the generator for \( \pi_1(K(S), u_{i+1}) \) can be obtained from generator of \( \pi_1(K(S), u_i) \). This is a lifting method. Hence it is sufficient to determine the generator for \( \pi_1(K(S), x_1) \). Lifting method can determine all generators for the whole fundamental groups \( \pi_1(K(S), u_i) \).
In section 2 we briefly explain diagram groups for presentations of graphical and in section 3, we give some definitions about graphs, paths and lifting of paths. Then the determination of lifts of generators will be shown in section 4. This section includes determining the lift of the generator $g$ at a word $W$, the general formula for the number of lifts of generators and also the total number of lifts of generators of fundamental groups in the specific graphs $\Gamma_n, n \in N$.

2 Determining the graphs $\Gamma_n, n \in \mathbb{N}$ and Preliminaries

Now we want to determine the graphs $\Gamma_n, n \in \mathbb{N}$ and the generators of diagram groups in these graphs.

Let $\langle x_1, x_2, x_3, x_4 \mid x_i = x_j, 1 \leq i < j \leq 4 \rangle$ be a graphical presentation. Note that the graph obtained from $S$ is collections of sub graphs. The graph $\Gamma$ obtained from $S$ is a union of $\Gamma_n$ connecting all vertices of length $n$ and respective edges. For example in Figure 1, the graph of $\Gamma_1$ obtained for graphical presentation with four initial generators (we called $x_1, x_2, x_3$ and $x_4$ initial generators) is in Figure 1.

![Figure 1: Graph of $\Gamma_1$](image)

While in Figure 2, graph of $\Gamma_2$ is in Figure 2.
Note that $\Gamma_2$ is four copies of $\Gamma_1$ and each vertex in each copy are joined together respectively. Similarly with four copies of $\Gamma_2$, we may obtain $\Gamma_3$. Repeat similar procedures for $\Gamma_4$ and so on.

For finding the generators of fundamental groups in graphs, the edges which are not to the spanning tree will be generators. For example we have three generators of fundamental groups in $\Gamma_1$ and there are:

- $g_{1_{l_3}} = (1, x_1 \to x_2, l)(1, x_2 \to x_3, l)(1, x_1 \to x_3, l)^{-1}$
- $g_{1_{l_4}} = (1, x_1 \to x_2, l)(1, x_2 \to x_3, l)(1, x_3 \to x_4, l)(1, x_1 \to x_4, l)^{-1}$
- $g_{1_{l_4}} = (1, x_1 \to x_2, l)(1, x_2 \to x_3, l)(1, x_3 \to x_4, l)(1, x_2 \to x_4, l)^{-1}(1, x_1 \to x_2, l)^{-1}$
Similarly in $\Gamma_2$ we may obtain its generators.

**Definition 2.1:** A path on a graph $\Gamma$ is either a vertex or a non-empty sequence of edges $e_1e_2\ldots e_n$ such that $\tau(e_i) = \iota(e_{i+1})$ for any $i = 1, 2, 3, \ldots, n-1$ ($e_i \in E$). If $\alpha = e_1e_2\ldots e_n$ is a path then the inverse path $\alpha^{-1}$ is the path $e_n^{-1}\ldots e_2^{-1}e_1^{-1}$. A path consisting of one vertex is called an empty path. An empty path coincides with its inverse. We define $\alpha$ to be a closed path if $\tau(\alpha) = \iota(\alpha)$. For example in the graph of $\Gamma_1$ in Figure 1, $(l, x_1 \rightarrow x_2, l)(l, x_2 \rightarrow x_3, l)(l, x_1 \rightarrow x_3, l)^{-1}$ is a path and $(l, x_1 \rightarrow x_2, l)(l, x_2 \rightarrow x_3, l)(l, x_3 \rightarrow x_4, l)(l, x_1 \rightarrow x_4, l)^{-1}$ is a closed path.

**Definition 2.2:** Let $\phi : \Gamma^* \rightarrow \Gamma$ be a mapping of graphs. If $\nu$ and $\nu^*$ are the vertices of $\Gamma$ and $\Gamma^*$ respectively such that $\phi(\nu^*) = \nu$ then $\nu^*$ is said to lie over $\nu$. Let $\alpha$ be a path in $\Gamma$ with $\iota(\alpha) = \nu$ and suppose $\nu^*$ lies over $\nu$. A path $\alpha^*$ in $\Gamma^*$ is said to be a lift of $\alpha$ at $\nu^*$ if $\phi(\alpha^*) = \alpha$, (see [2]).

Let $\alpha = (l, x_1 \rightarrow x_2, l)(l, x_2 \rightarrow x_3, l)(l, x_3 \rightarrow x_4, l)$ be a path, $\iota(\alpha) = x_1, \tau(\alpha) = x_4$ and suppose $x_1x_2$ lies over $x_1$. The lift of $\alpha$ at $x_1x_2$ is $\alpha^* = (l, x_1 \rightarrow x_2, x_2)(l, x_2 \rightarrow x_3, x_2)(l, x_3 \rightarrow x_4, x_2)$

**Example 2.3:** We have three generators in $\pi_1(4\Gamma_1)$ and $g_{l_0}$ is

$(l, x_1 \rightarrow x_2, l)(l, x_2 \rightarrow x_3, l)(l, x_1 \rightarrow x_3, l)^{-1}$.

Then lifts of $g_{l_0}$ at $x_1^2$ are $(x_1, x_1 \rightarrow x_2, l)(x_1, x_2 \rightarrow x_3, l)(x_1, x_1 \rightarrow x_3, l)^{-1}$, and

$(l, x_1 \rightarrow x_2, x_1)(l, x_2 \rightarrow x_3, x_1)(l, x_1 \rightarrow x_3, x_1)^{-1}$.

Lifts of $g_{l_0}$ at $x_1, \lambda^* \in \{x_2, x_3, x_4\}$

$(x_1, x_1 \rightarrow \lambda^*, l)(l, x_1 \rightarrow x_2, \lambda^*)(l, x_2 \rightarrow x_3, \lambda^*)(l, x_1 \rightarrow x_3, \lambda^*)^{-1}(x_1, x_1 \rightarrow \lambda^*, l)^{-1}$.

Similarly we can show that lifts of all generators $g_{l_4}$ and $g_{l_2}$ at $x_1^2, x_1x_2, x_1x_3, x_1x_4$ are the same as the lift of $g_{l_3}$ at $x_1^2, x_1x_2, x_1x_3, x_1x_4$.

Note that the first lifts of $g_{l_4}$ and $g_{l_2}$ at $x_1^2$ has no conjugate, but the lifts of $g_{l_4}$ and $g_{l_2}$ at $x_1x_2, x_1x_3, \ldots, x_1x_n$ has a conjugate.
3 Lift of generators

Now we will show our technique for computing the general formula for the total number of lifts of generators in $\Gamma_n$ and the number lifts of generators in $\Gamma_n$.

**Lemma 3.1:** Let $^4S = \{x_1, x_2, x_3, x_4 \mid x_i = x_j; \ 1 \leq i < j \leq 4\}$ be a graphical presentation. The general formula for total number of lifts of generators of diagram groups in the graph $\Gamma_{n-1}$ is $a_n = 4a_{n-1} + 3$, where $a_i (i=1,2,3,...,n)$ the number of lift of generators in $\pi_1(\Gamma_i)$ and $a_1 = 3$.

**Proof:** By definition, $\Gamma_n$ is four copies of $\Gamma_{n-1}$, then the spanning tree $T_n$ in $\Gamma_n$ is four copies of the spanning $T_{n-1}$ in $\Gamma_{n-1}$. We know that the edges do not belong to spanning tree will be generators. Thus the generators of fundamental group $\pi_1(\Gamma_n)$ is four copies of generators in $\pi_1(\Gamma_{n-1})$. Thus the number of lifts of generators of $\pi_1(\Gamma_n)$ is four copies of the number of lift of generators of $\pi_1(\Gamma_{n-1})$ plus three generators, which obtained from between edges of three spanning trees. Hence the number of lifts of generators of $\pi_1(\Gamma_n)$ is $a_n = 4a_{n-1} + 3$.

**Lemma 3.2:** Let $^4S = \{x_1, x_2, x_3, x_4 \mid x_i = x_j; \ 1 \leq i < j \leq 4\}$ be a graphical presentation. The total number of lifts of generators of $\pi_1(\Gamma_{n-1})$ is $a_n = (4^n - 1), (n=2,3,4,...)$.

**Proof:** By induction, for $n = 2$, we have $a_2 = 15$. Now suppose $a_k = (4^k - 1)$ is the total number lift of generators of diagram groups in $\Gamma_{k-1}$, we will prove that $a_{k+1} = (4^{k+1} - 1)$ is the number lift of generators of diagram groups in $\Gamma_k$. By Lemma 1, we have $a_{k+1} = 4a_k + 3 = 4(4^k - 1) + 3 = (4^{k+1} - 1)$.

**Theorem 3.3:** Let $^4S = \{x_1, x_2, x_3, x_4 \mid x_i = x_j; \ 1 \leq i < j \leq 4\}$ be the graphical presentation and $\rho_{in} = (u, x_1 \rightarrow x_2, v)(u, x_2 \rightarrow x_3, v)(u, x_1 \rightarrow x_3, v)^{-1}$ where length (uv) = $n - 1$, then:

- Lifts of $\rho_{in}$ at $x_{n+1}$ is:
  
  $(x_{n+1}^{m+1}, x_1 \rightarrow x_2, x_1^{-m})(x_{n+1}^{m+1}, x_2 \rightarrow x_3, x_1^{-m-1}) (x_{n+1}^{m+1}, x_1 \rightarrow x_1, x_1^{-m-1})$, $m = 1, 2, 3, ..., n + 1$, $m \leq n + 1$
Generators of diagram groups

- Lifts of $\rho_{n}$ at $a_{1}a_{2}a_{3}...a_{n+1}$, $a_{i} \in \{x_{1},x_{2},x_{3},x_{4}\}$, $\exists (i=1,2,3,...,m)$ $a_{i}=x_{i}$ from $x_{j}^{n+1}$ is:
  $S_{1}S_{2}...S_{n-1}\rho_{n+1}S_{n-m}^{-1}...S_{j}^{-1}S_{1}^{-1}$
  where, $\rho_{n+1}=(u,x_{1} \rightarrow x_{2},v)(u,x_{2} \rightarrow x_{3},v)(u,x_{1} \rightarrow x_{3},v)^{-1}$, length $(uv)=n$ and $s_{i}=(u^{*},x_{1} \rightarrow a,v^{*})$, length $(u^{*}v^{*})=n$ and $a \in \{x_{2},x_{3},x_{4}\}$.
  
  Note that lift of $\rho_{2n}=(1,x_{1} \rightarrow x_{2},1)(1,x_{2} \rightarrow x_{3},1)(1,x_{1} \rightarrow x_{3},1)^{-1}$ and $\rho_{n}=(1,x_{1} \rightarrow x_{2},1)(1,x_{2} \rightarrow x_{3},1)(1,x_{3} \rightarrow x_{4},1)(1,x_{1} \rightarrow x_{4},1)^{-1}$, where length $(uv)=n-1$, at $w$ the same as this theorem.

**Proof:** By induction on length of words.
  
  Note that the lift of generator has no conjugate if it is the lift of $\rho_{n}$ at $x_{i}^{n+1}$, but the lift of $\rho_{n}$ at $wx_{i}$ or $x_{i}w$ has conjugates and the number of conjugate is $(n+1)-\exp_{x_{i}}(w)$.
  
  Here $\exp_{x_{i}}(w)$ is the exponent sum of $x_{i}$ in a word $w$ defined to be the number of $x_{i}$ appeared in $w$ minus the number of $x_{i}^{-1}$ appeared in $w$.

**Example 3.4:** Lift of $\rho_{1}=(u,x_{1} \rightarrow x_{2},v)(u,x_{2} \rightarrow x_{3},v)$ $(u,x_{1} \rightarrow x_{3},v)^{-1}$, length $(uv)=2$ at $x_{1}x_{2}x_{3}^{2}$ from $x_{1}^{4}$ is:
  
  $(x_{3},x_{1} \rightarrow x_{3},1)(x_{3},x_{1} \rightarrow x_{3},x_{1}^{-1}x_{3})(x_{3},x_{1} \rightarrow x_{3},x_{1}^{-1})(1,x_{1} \rightarrow x_{2},x_{2}x_{3})(1,x_{2} \rightarrow x_{3},x_{2}x_{3})$
  
  $(1,x_{1} \rightarrow x_{3},x_{2}x_{3})^{-1}(x_{3},x_{1} \rightarrow x_{3},x_{1})^{-1}(x_{3},x_{1} \rightarrow x_{3},x_{1})^{-1}$(min $x_{i},x_{1} \rightarrow x_{3},x_{1}^{-1}$)

**Example 3.5:** If $\gamma_{6}=(u,x \rightarrow y,v)(u,y \rightarrow z,v)(u,x \rightarrow z,v)^{-1}$, length $(uv)=5$, then the lifts of $\gamma_{6}$ at $x^{7}y^{2}zyx$ from $x^{7}$ is:
  
  $(x^{7},x \rightarrow y,x)(x^{4},x \rightarrow z,yx)(x^{7},x \rightarrow y,zyx)(x^{7},x \rightarrow y,zyx)$
  
  $(x,y \rightarrow y,zyx)(x,y \rightarrow z,y^{2}zyx)(x,x \rightarrow z,y^{2}zyx)^{-1}$
  
  $(x^{2},x \rightarrow y,zyx)(x^{3},x \rightarrow y,zyx)^{-1}(x^{4},x \rightarrow z,yx)^{-1}(x^{5},x \rightarrow y,x)^{-1}$

  Because $x^{7} \rightarrow x^{5}yx \rightarrow x^{4}zyx \rightarrow x^{3}zyyx \rightarrow x^{2}y^{2}zyx$ and by using Theorem 3.3, we have four conjugates.
References


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