Common Fixed Point Theorems of Compatible Mappings of Type (P) in Fuzzy Metric Spaces

M. Koireng and Yumnam Rohen

National Institute of Technology Manipur, Department of Mathematics
Takyelpat, Pin-795004, Manipur, India
ymnehor2008@yahoo.com

Abstract

In this paper we prove some common fixed point theorems for compatible mappings of type (P) in fuzzy metric spaces. Our results modify the results of Seong Hoon Cho [14].

Mathematics subject classification: 54H25, 54E50.

Keywords: Compatible mappings of type (P), Common fixed point, Fuzzy metric space

1. Introduction

The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck [7] as a generalization of commuting mappings. Pathak, Chang and Cho [11] introduced the concept of compatible mappings of type(P).

Zadeh [15] introduced the concept of fuzzy sets. The idea of fuzzy metric space was introduced by Kramosil and Michalek [10] which was modified by George and Veeramani [3,4]. Bijendra Singh and M. S. Chauhan [13] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point
Theorems in fuzzy metric spaces in the sense of George and Veeramani with continuous $t$-norm $*$ defined by $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

Our aim in this paper is to prove some common fixed point theorems of compatible mappings of type (P) by modifying the results of S.H. Cho [14].

2. Preliminaries

Following definitions and lemma were discussed by S.H. Cho [14].

**Definition 2.1** [12] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous $t$-norm if it satisfies the following conditions

(i) $*$ is associative and commutative.
(ii) $*$ is continuous.
(iii) $a*1 = a$ for all $a \in [0,1]$.  
(iv) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

**Definition 2.2** [3] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if $X$ is an arbitrary (non-empty) set, $*$ is continuous $t$-norm, and $M$ is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$,
5. $M(x, y, ) : (0, \infty) \to [0,1]$ is continuous

for all $x, y, z \in X$ and $t, s > 0$.

Let $(X, d)$ be a metric space, and let $a*b = ab$ or $a*b = \min\{a, b\}$. Let $M(x, y, t) = \frac{1}{\mu_{\tau d(x,y)}}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric $M$ induced by $d$ is called the standard fuzzy metric space [3].

**Definition 2.3** [5] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called convergent to a point $x \in X$ (denoted by $\lim_{n \to \infty} x_n = x$), if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1-\varepsilon$ for all $n \geq n_0$.

The completeness and non completeness of fuzzy metric space was discussed in George and Veeramani [3] and M. Grabiec [5].

**Definition 2.4** [3] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called Cauchy sequence if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $n, m \geq n_0$.
Definition 2.5 [13] Self mappings $A$ and $S$ of a fuzzy metric space $(X, M, \ast)$ is said to be compatible if \( \lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1 \) for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z$ for some $z \in X$.

Definition 2.6 [11] Self mappings $A$ and $S$ of a fuzzy metric space $(X, M, \ast)$ is said to be compatible of type (P) if \( \lim_{n \to \infty} M(AAx_n, SSx_n, t) = 1 \) for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z$ for some $z \in X$.

Lemma 2.7 [5] Let $(X, M, \ast)$ be a fuzzy metric space. Then for all $x, y$ in $X$, $M(x, y, .)$ is non-decreasing.

Lemma 2.8 [14] Let $(X, M, \ast)$ be a fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(x, y, qt) \geq M(x, y, t/q^n)$ for positive integer $n$. Taking limit as $n \to \infty$, $M(x, y, t) \geq 1$ and hence $x = y$.

Lemma 2.9 [9] The only $t$-norm $\ast$ satisfying $r \ast r \geq r$ for all $r \in [0,1]$ is the minimum $t$-norm, that is, $a \ast b = \min \{a, b\}$ for all $a, b \in [0,1]$.


Proposition 2.10 [11] Let $(X, M, \ast)$ be a fuzzy metric space and let $A$ and $S$ be continuous mappings of $X$ then $A$ and $S$ are compatible if and only if they are compatible of type (P).

Proposition 2.11 [11] Let $(X, M, \ast)$ be a fuzzy metric space and let $A$ and $S$ be compatible mappings of type(P) and $Az = Sz$ for some $z \in X$, then $AAz = ASz = SAz = SSz$.

Proposition 2.12 [11] Let $(X, M, \ast)$ be a fuzzy metric space and let $A$ and $S$ be compatible mappings of type(P) and let $Ax_n, Sx_n \to z$ as $n \to \infty$ for some $z \in X$. Then
(i) $\lim_{n \to \infty} SSx_n = Az$ if $A$ is continuous at $z$,
(ii) $\lim_{n \to \infty} AAx_n = Sz$ if $S$ is continuous at $z$,
(iii) $ASz = Saz$ and $Az = Sz$ if $A$ and $S$ are continuous at $z$.

3. Common Fixed Point Theorems

We prove the following theorem.
Theorem 3.1 Let \((X, M, *)\) be a complete fuzzy metric space and let \(A, B, S\) and \(T\) be a self mappings of \(X\) satisfying the following conditions:

(i) \(AX \subseteq TX, BX \subseteq SX\),
(ii) \(S\) and \(T\) are continuous
(iii) the pairs \(\{A, S\}\) and \(\{B, T\}\) are compatible mapping of type \((P)\) on \(X\).
(iv) there exists \(q \in (0, 1)\) such that for every \(x, y \in X\) and \(t > 0,\)
\[ M(AX, By, qt) \geq M(Sx, Ty, t)^*M(AX, Sx, t)^*M(Bx, Ty, t)^*M(AX, Ty, t) \]

Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Proof: Since \(AX \subseteq TX\) and \(BX \subseteq SX\), for any \(x_0 \in X\), there exists \(x_1 \in X\) such that \(AX_0 = TX_1\) and for this \(x_1 \in X\), there exists \(x_2 \in X\) such that \(BX_1 = SX_2\). Inductively, we define a sequence \(\{x_n\}\) in \(X\) such that
\[ y_{2n-1} = TX_{2n-1} = AX_{2n-2} \quad \text{and} \quad y_{2n} = SX_{2n} = BX_{2n-1}\]
for all \(n = 0, 1, 2, \ldots\).

From (iv),
\[ M(y_{2n+1}, y_{2n+2}, qt) = M(AX_{2n}, BX_{2n+1}, qt) \geq M(Sx_{2n}, Tx_{2n+1}, t)^*M(AX_{2n}, Sx_{2n}, t)^*M(Bx_{2n+1}, Tx_{2n+1}, t)^*M(AX_{2n}, Tx_{2n+1}, t) \]
\[ = M(y_{2n+1}, y_{2n+2}, t)^*M(y_{2n+1}, y_{2n+1}, t)^*M(y_{2n+2}, y_{2n+1}, t)^*M(y_{2n+1}, y_{2n+1}, t) \]
\[ \geq M(y_{2n+1}, y_{2n+2}, t)^*M(y_{2n+2}, y_{2n+1}, t) \]

From lemma 2.7 and 2.10, we have
\[ M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t) \]  \hspace{1cm} (3.1.1)

Similarly, we have
\[ M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t) \]  \hspace{1cm} (3.1.2)

From (3.1.1) and (3.1.2), we have
\[ M(y_{n+1}, y_{n+2}, qt) \geq M(y_{n}, y_{n+1}, t) \]  \hspace{1cm} (3.1.3)

From (3.1.3), we have
\[ M(y_{n+1}, y_{n+1}, t) \geq M(y_{n}, y_{n+1}, \frac{t}{q}) \]
\[ \geq M(y_{n}, y_{n+1}, \frac{t}{q^2}) \]
\[ \geq \ldots \ldots \geq M(y_{1}, y_{2}, \frac{t}{q^{n-1}}) \to 1 \text{ as } n \to \infty. \]

So, \(M(y_{n}, y_{n+1}, t) \to 1\) as \(n \to \infty\) for any \(t > 0\). For each \(\varepsilon > 0\) and each \(t > 0\), we can choose \(n_0 \in \mathbb{N}\) such that
\[ M(y_{n}, y_{n+1}, t) > 1 - \varepsilon \text{ for all } n > n_0. \]

For \(m, n \in \mathbb{N}\), we suppose \(m \geq n\). Then we have that
\[ M(y_{m}, y_{m}, t) \geq M(y_{m}, y_{n+1}, \frac{t}{q^{m-n}})^*M(y_{n+1}, y_{n+2}, \frac{t}{q^{m-n}})^* \ldots *M(y_{m-1}, y_{m}, \frac{t}{q^{m-n}}) \]
\[ \geq (1 - \varepsilon)^*(1 - \varepsilon)^* \ldots (m - n) \times (m - n) \text{ times.} \]
\[ \geq (1 - \varepsilon) \]
and hence \(\{y_n\}\) is a Cauchy sequence in \(X\).

Since \((X, M, *)\) is complete, \(\{y_n\}\) converges to some point \(z \in X\), and so \(\{AX_{2n}\}, \{SX_{2n}\}, \{BX_{2n+1}\}\) and \(\{TX_{2n+1}\}\) also converges to \(z\).

From proposition 2.11 and (iii), we have
Common fixed point theorems

185

(3.1.4)

\[ AAx_{2n} \to Sz \text{ and } SSx_{2n} \to Az \]

(3.1.5)

\[ BBx_{2n-1} \to Tz \text{ and } TTx_{2n-1} \to Bz \]

From (iv), we get

\[
M(AAx_{2n}, BBx_{2n-1}, qt) \\
\geq M(SAx_{2n-2}, TTx_{2n-1}, t)\ast M(AAx_{2n-2}, SAx_{2n-2}, t)\ast M(BBx_{2n-1}, TTx_{2n-1}, t)\ast M(AAx_{2n-2}, TTx_{2n-1}, t)
\]

Taking limit as \( n \to \infty \) and using (3.1.4) and (3.1.5), we have

\[
M(Sz, Tz, qt) \\
\geq M(Sz, Sz, t)\ast M(Sz, Tz, t)\ast M(Tz, Tz, t)\ast M(Sz, Tz, t)
\]

It follows that \( Sz = Tz \).  \hspace{1cm} (3.1.6)

Now, from (iv),

\[
M(Az, Bz, qt) \\
\geq M(Az, Az, t)\ast M(Az, Bz, t)\ast M(Az, Tz, t)
\]

and hence \( Az = Tz \).  \hspace{1cm} (3.1.7)

From (iv), (3.1.6) and (3.1.7),

\[
M(Az, Bz, qt) \\
\geq M(Az, Az, t)\ast M(Az, Bz, t)\ast M(Az, Tz, t)
\]

and hence \( Az = Bz \). \hspace{1cm} (3.1.8)

From (3.1.6), (3.1.7) and (3.1.8), we have

\[ Az = Bz = Tz = Sz \]  \hspace{1cm} (3.1.9)

Now, we show that \( Bz = z \).

From (iv),

\[
M(Ax_{2n}, Bz, qt) \\
\geq M(Sx_{2n}, Tz, t)\ast M(Ax_{2n}, Sx_{2n}, t)\ast M(Bz, Tz, t)\ast M(Ax_{2n}, Tz, t)
\]

And, taking limit as \( n \to \infty \) and using (3.1.6) and (3.1.7), we have

\[
M(z, Bz, qt) \\
\geq M(z, Sz, t)\ast M(z, Sz, t)\ast M(Bz, Tz, t)\ast M(z, Sz, t)
\]

And hence \( Bz = z \). Thus from (3.1.9), \( z = Az = Bz = Tz = Sz \) and \( z \) is a common fixed point of \( A, B, S \) and \( T \).

In order to prove the uniqueness of fixed point, let \( w \) be another common fixed point of \( A, B, S \) and \( T \). Then

\[
M(z, w, qt) = M(Az, Bw, qt) \\
\geq M(Sz, Tw, t)\ast M(Az, Sz, t)\ast M(Bw, Tw, t)\ast M(Az, Tw, t)
\]

From lemma 2.8, \( z = w \). This completes the proof of theorem.
Corollary 3.2. Let \((X, M, \ast)\) be a complete fuzzy metric space and let \(A, B, S\) and \(T\) be self mappings of \(X\) satisfying \((i) - (iii)\) of theorem 3.1 and there exists \(q \in (0,1)\) such that
\[
M(Ax, By, qt) \geq M(Sx, Ty, t) \ast M(Ax, Sx, t) \ast M(By, Ty, t) \ast M(By, Sx, 2t) \ast M(Ax, Ty, t)
\]
for every \(x, y \in X\) and \(t > 0\). Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Corollary 3.3. Let \((X, M, \ast)\) be a complete fuzzy metric space and let \(A, B, S\) and \(T\) be self mappings of \(X\) satisfying \((i) - (iii)\) of theorem 3.1 and there exists \(q \in (0,1)\) such that
\[
M(Ax, By, qt) \geq M(Sx, Ty, t)
\]
for every \(x, y \in X\) and \(t > 0\). Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Corollary 3.4. Let \((X, M, \ast)\) be a complete fuzzy metric space and let \(A, B, S\) and \(T\) be self mappings of \(X\) satisfying \((i) - (iii)\) of theorem 3.1 and there exists \(q \in (0,1)\) such that
\[
M(Ax, By, qt) \geq M(Sx, Ty, t) \ast M(Sx, Ax, t) \ast M(Ax, Ty, t)
\]
for every \(x, y \in X\) and \(t > 0\). Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Theorem 3.5 Let \((X, M, \ast)\) be a complete fuzzy metric space. Then continuous self mappings \(S\) and \(T\) of \(X\) have a common fixed point in \(X\) if and only if there exists a self mapping \(A\) of \(X\) such that the following conditions are satisfied:
\[
\begin{align*}
(i) & \quad A(X) \subseteq T(X) \cap S(X), \\
(ii) & \quad \text{the pairs } \{A, S\} \text{ and } \{A, T\} \text{ are compatible mapping of type } (P) \text{ on } X, \\
(iii) & \quad \text{there exists } q \in (0,1) \text{ such that for every } x, y \in X \text{ and } t > 0, \\
& \quad M(Ax, Ay, qt) \geq M(Sx, Ty, t) \ast M(Sx, Ax, t) \ast M(Ax, Ty, t) \ast M(Ax, Ty, t) \\
\end{align*}
\]
In fact \(A, S\) and \(T\) have a unique common fixed point in \(X\).

Proof: We shown that the necessity of the conditions (i)- (iii). Suppose that \(S\) and \(T\) have a common fixed point in \(X\), say \(z\). Then \(Sz = z = Tz\).

Let \(Ax = z\) for all \(x \in X\). Then we have \(A(X) \subseteq T(X) \cap S(X)\) and we know that \([A, S]\) and \([A, T]\) are compatible mapping of type (P), in fact \(A \circ S = S \circ A\) and \(A \circ T = T \circ A\), and hence the conditions (i) and (ii) are satisfied.

For some \(q \in (0,1)\), we get
\[
M(Ax, Ay, qt) = 1 \geq M(Sx, Ty, t) \ast M(Ax, Sx, t) \ast M(Ay, Ty, t) \ast M(Ax, Ty, t)
\]
for every \(x, y \in X\) and \(t > 0\) and hence the condition (iii) is satisfied.

Now, for the sufficiency of the conditions, let \(A = B\) in theorem 3.1. Then \(A, S\) and \(T\) have a unique common fixed point in \(X\).

Corollary 3.6 Let \((X, M, \ast)\) be a complete fuzzy metric space. Then continuous self mappings \(S\) and \(T\) of \(X\) have a common fixed point in \(X\) if and only if there exists a
self mapping $A$ of $X$ satisfying (i) – (ii) of theorem 3.5 and there exists $q \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$M(Ax, Ay, qt) \geq M(Sx, Ty, t) * M(A, S, x, t) * M(A, A, y, t)$$

In fact $A, S$ and $T$ have a unique common fixed point in $X$.

**Corollary 3.7** Let $(X, M, \ast)$ be a complete fuzzy metric space. Then continuous self mappings $S$ and $T$ of $X$ have a common fixed point in $X$ if and only if there exists a self mapping $A$ of $X$ satisfying (i) – (ii) of theorem 3.5 and there exists $q \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$M(Ax, Ay, qt) \geq M(Sx, Ty, t)$$

In fact $A, S$ and $T$ have a unique common fixed point in $X$.

**Corollary 3.8** Let $(X, M, \ast)$ be a complete fuzzy metric space. Then continuous self mappings $S$ and $T$ of $X$ have a common fixed point in $X$ if and only if there exists a self mapping $A$ of $X$ satisfying (i) – (ii) of theorem 3.5 and there exists $q \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$M(Ax, Ay, qt) \geq M(Sx, Ty, t) * M(Sx, A, x, t) * M(A, A, y, t)$$

In fact $A, S$ and $T$ have a unique common fixed point in $X$.

**Acknowledgement:** Authors are thankful to Prof. M.R. Singh for his valuable suggestions towards the improvement of this paper.

**References**


Received: July, 2011