A Fixed Point Theorem for Mean Non-Expansive Mappings Semigroups in Uniformly Convex Banach Spaces

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Abstract. In this work we deal with the mean non-expansive mappings as a generalization of non-expansive mappings. A fixed point theorem for strong semigroups of mean non-expansive mappings are given.

Keywords: asymptotic center; asymptotic radius; mean nonexpansive mapping; fixed point; normal structure; uniformly convex
1 Introduction

Let $X$ be a Banach space on the complex field numbers $C$, and $C$ be a non empty convex subset of $X$. A map $T : C \to C$ is said to be an mean non-expansive mapping if there are two positive constants $a_T$ and $b_T$ such that $a_T + b_T \leq 1$, and

$$
\|T x - T y\| \leq a_T \|x - y\| + b_T \|x - Ty\|, \quad \forall x, y \in C.
$$

(1)

we also say that $T$ is mean non-expansive if $T$ is $(a_T, b_T)$ mean non-expansive for some nonnegative real number $a_T$ and $b_T$, such that $a_T + b_T \leq 1$.

This type of mappings is introduced by (Zhang.1975)[?], and extensively studied by (Wu and Zhang,2007)[?], and (Young and Cui,2008).

As we see, a non-expansive mapping is a mean non-expansive mapping, in fact it suffices to take $b_T = 0$ in equ1, but the converse is not true, as show it the following example:

Take the map $T : [0,5] \to [0,2]$ be a mapping given by

$$
T(x) = \begin{cases} 
2 & \text{if } x \in [0,4] \\
1 & \text{if } x \in ]4,5[ \\
0 & \text{if } x = 5 
\end{cases}
$$

It is not difficult to see that this map is never non-expansive for any chosen norm on $R$, but it is a mean non-expansive mapping. It suffices to take $a_T = b_T = \frac{1}{2}$.

Recall that a family of mappings $\{T(t) : t \in R^+\}$ on a subset $C$ of $X$ is said to be a strong semigroup or simply a $C_0$-semigroup if it satisfy the three following conditions :

(i) $T(0) = I$, where $I$ denotes the mapping identity on $X$,

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \in R^+$,

(iii) $\lim_{t \to 0^+} \|T(t)x - x\| = 0$, for all $x \in C$.

Sukanya Somprom and Satit Saejung [?] have shown the following theorem:

**Theorem 1.1**[?] Let $C$ be a closed convex subset of a Hilbert space $H$. Suppose that $T$ be an mean non-expansive mapping on $C$ into itself, with $a_T > 0, b_T \geq 0$ and $a_T + b_T \leq 1$. If $F(T) \neq \emptyset$, then, for each $x \in C$, the sequence given by

$$
S_n(x) = \frac{1}{n} \sum_{i=1}^{n-1} T^i x,
$$

converges weakly to a fixed point of $T$ in $C$. 

Also many results about this class of mappings have been obtained both in the point of view of the ergodic Theory and the fixed point Theory, see e.g[?], [?], [?].

It is of a great interesting to look for extending this results to the Banach spaces case, but some difficulties arise. This difficulties are due to the loss of many geometric properties like the parallelogram equality ..., so in order to overcome them we use the uniform convexity approach.

2 Preliminaries

Let $X$ be a real Banach space. Recall that $X$ is strictly convex if its unit sphere does not contain any line segments, that is, $X$ is strictly convex if and only if the following implication holds:

$$x, y \in X, \|x\| = \|y\| = 1 \text{ and } \|\frac{x+y}{2}\| = 1 \Rightarrow x = y$$

In order to measure the degree of convexity (rotundity) of $X$, we define its modulus of convexity $\delta_X : [0, 2] \to [0, 1]$ by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon \right\}.$$ 

The characteristic $\varepsilon_0$ of convexity of $X$ is defined as

$$\varepsilon_0 = \varepsilon_0(X) = \sup \{ \varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0 \}$$

The following properties of the modulus $\delta_X$ of convexity of $X$ are quite well known, see e.g[?].

(a) $\delta_X$ is increasing on $[0, 2]$ and moreover strictly increasing on $[\varepsilon_0, 2]$.

(b) $\delta_X$ is continuous on $[0, 2]$ (but not necessarily at $\varepsilon = 2$).

(c) $\delta_X(2) = 1$ if and only if $X$ is strictly convex.

(d) $\delta_X(0) = 0$ and

$$\lim_{\varepsilon \to 2^-} \delta_X(\varepsilon) = 1 - \frac{\varepsilon_0}{2}.$$ 

(e) $\|a - x\| \leq r, \|a - y\| \leq r \text{ and } \|x-y\| \geq \varepsilon \Rightarrow \|a - \left( \frac{x+y}{2} \right) \| \leq r(1 - \delta_X(\frac{\varepsilon}{2}))$.

For a subset $A$ of $X$, let us set:

$$r_a(A) = \sup \{ \|u-v\| : v \in A \} \quad (u \in X);$$

For a subset $K$ of $X$, we denote by $\text{diam}(K)$ the number

$$\sup \{ \|x-y\| : x, y \in K \}.$$
Then it is obvious that for any \( u \in A \), we have
\[
    r(A) \leq r_u(A) \leq \text{diam}(A).
\]

We should note here that \( r_u(A) \) taken as a function of \( u \) defined on a bounded, closed and convex subset \( A \) of \( X \), then \( r_u : A \to \mathbb{R} \) is both continuous and convex, and it follows that \( C(A) \) is closed and convex. A point \( u \in A \) is said to be **diametral** if \( r_u(A) = \text{diam}(A) \), else \( u \) is said to be **nondiametral**. It is possible that a bounded, closed and convex set \( A \) may consist entirely of diametral points, such sets are called diametral. A bounded convex subset \( K \) of a Banach space \( X \) is said to have normal structure if every convex subset \( H \) of \( K \) such that \( \text{diam}(H) > 0 \), contains at least a nondiametral point.

A Banach space \( X \) is said to have normal structure if every bounded, convex subset of \( X \) has normal structure. The normal structure coefficient of a Banach space \( X \) is the number:
\[
    N(X) = \inf \left\{ \frac{\text{diam}(K)}{r(K)} : K \subset X \text{ is bounded and convex } \text{ diam}(K) > 0 \right\}.
\]

Where \( r_K(K) \) is the Chebyshev radius of \( K \) relative to itself, i.e:
\[
    r_K(K) = \inf_{x \in K} \sup_{y \in K} \| x - y \| \text{ and } \text{diam}(K) = \sup_{x,y \in K} \| x - y \|.
\]

Obviously \( N(X) \geq 1 \), and \( N(X) > 1 \) if and only if \( X \) has uniform normal structure.

It is already known that uniformly convex Banach spaces have uniform normal structure.

A general formula for \( N(X) \) is not known, this coefficient is fully known only in a few special cases.

One connection between normal structure and the modulus of convexity is that \( X \) has normal structure as soon as \( \delta_X(1) > 0 \).

Also, we should note that the equation
\[
    \lambda^2 \delta_X^{-1}(1 - \frac{1}{\lambda}) N(X)^{-1} = 1.
\]
possesses a unique solution \( \lambda > 1 \).

Let us write \( G \) to denote the set of positive reals \( \mathbb{R}^+ \) or the set of integers \( \mathbb{N} \), now let \( x \in X \) and \( \{ x_u \}_{u \in G} \) be a family in \( X \), the **asymptotic radius** of \( \{ x_u \}_{u \in G} \) at \( x \) is the number
\[
    r(x, \{ x_u \}_{u \in G}) = \lim_{u \to +\infty} \sup \| x_u - x \|.
\]

For \( \{ x_u \}_{u \in G} \) fixed, equ3 defines a nonnegative continuous and convex function of \( x \) from \( X \) into \( \mathbb{R}^+ \).
Now let $K$ be a nonempty closed subset, the **asymptotic radius** of $\{x_u\}_{u \in G}$ in $K$ is the number

$$r(K, \{x_u\}_{u \in G}) = \inf \{r(x, \{x_u\}_{u \in G}) : x \in K\}.$$ 

The **asymptotic center** of $\{x_u\}$ in $K$ is the set

$$A(K, \{x_u\}_{u \in G}) = \left\{ x \in K : r(x, \{x_u\}_{u \in G}) = r(K, \{x_u\}_{u \in G}) \right\}.$$ 

It is known that if $X$ is reflexive then $A(K, \{x_u\}_{u \in G})$ is a nonempty bounded closed and convex set, and if $X$ is uniformly convex then $A(K, \{x_u\}_{u \in G})$ is a singleton.

### 3 Main results

In what follows $X$ is assumed to be a uniformly Banach space. For a subset $K$ of $X$, we write $\text{co}(K)$ to denote the convex hull of $K$, that is $\text{co}(K) = \cap \{H : H$ convex and $K \subset H\}$.

Now, before to give our theorem, we will need the following lemma.

**Lemma 3.1**\cite{?} If $C$ is a nonempty closed convex subset of a reflexive Banach space $X$, then for every bounded net $\{x_t\}_{t \in G}$ of elements of $X$, $A(C, \{x_t\})$ is a nonempty bounded closed convex subset of $C$.

In particular, if $X$ is uniformly convex Banach space, then $A(C, \{x_t\})$ consists of a single point.

**Lemma 3.2**\cite{?} Assume that $X$ has a uniformly normal structure. Then for every bounded family $\{x_t\}_{t \in G}$ of elements of $X$, there exists some $y \in \text{co}(\{x_t : t \in G\})$ such that

$$\limsup_{t \to +\infty} \|x_t - y\| \leq N(X)^{-1}A(\{x_t\}), \quad (4)$$

and

$$A(\{x_t\}_{t \in G}) = \lim_{t \to +\infty} \left( \sup \{\|x_u - x_v\| : t \leq u, v \in G\} \right) \quad (5)$$

is the asymptotic diameter of $\{x_t\}_{u \in G}$.

Now, we are able to give our main theorem.

**Theorem 3.1** Let $X$ be a uniformly convex Banach space and $C$ be a nonempty closed and convex subset of $X$, and let $\Gamma = \{T(t) : t \in G\}$ be a $C_0$-semigroup of mean non-expansive mappings on $C$. Assume that the constants $a_T$ and $b_T$ are independent of the parameter $t$ with $a_T \geq 0$, $b_T < 1$ and $\frac{a_T + 2b_T}{1-b_T} < \lambda$, where $\lambda$ is the unique solution of equ2, and assume that there exists some $x_0 \in C$ such that $\{T(t)x_0 : t \in G\}$ is bounded.

Then there exists some $z \in C$ such that $T(t)z = z$ for all $t \geq 0$. 
The idea of the proof is to construct by induction a sequence \( \{x_n\} \) which will converge to the wanted fixed point. This sequence will be constructed by mean of the sets \( A(C, \{T(t)x_n\}) \), since the space \( X \) is uniformly convex these sets are singletons and the constructed sequence is well defined.

Let us choose arbitrarily a fixed element \( x_0 \in C \), and define the sequence \( \{x_n\} \) as follows

\[
x_{n+1} = A(C, \{T(t)x_n\}) \quad \forall n \in \mathbb{N}.
\]

Then, for all \( n \in \mathbb{N} \), \( x_n \) is the unique element of \( C \) which satisfy

\[
\limsup_{t \to +\infty} \|T(t)x_{n-1} - x_n\| = \inf_{u \in C} \limsup_{t \to +\infty} \|T(t)x_{n-1} - u\|.
\]

Now, let us set \( r_n = r_K(\{T(t)x_n\}_t) \), so in light of Lemma ?? we get

\[
r_n = \limsup_{t \to +\infty} \|T(t)x_n - x_{n+1}\|
\leq N(X)^{-1} A(\{T(t)x_n\}_t)
= N(X)^{-1} \lim_{t \to +\infty} \left(\sup\{\|T(u)x_n - T(v)x_n\| : t \leq u, v \in G\}\right)
\leq N(X)^{-1} \left(\frac{aT + 2bT}{1 - bT}\right) d(x_n),
\]

where \( d(x_n) = \sup\{\|T(t)x_n - x_n\| : t \in G\} \). If there is \( n \in \mathbb{N} \) such that \( d(x_n) = 0 \) then \( x_n = z \) is a fixed point and there is nothing to prove. So we can assume that \( d(x_n) > 0 \) for all \( n \geq 0 \).

Now, let \( n \geq 0 \) fixed and let \( \varepsilon > 0 \) be small enough, and let us choose \( s \in G \) in such a way that

\[
\|T(s)x_{n+1} - x_{n+1}\| > d(x_{n+1}) - \varepsilon,
\]

in other hand choose \( t_0 \geq 0 \) large enough such that

\[
\|T(t)x_n - x_{n+1}\| < r_n + \varepsilon, \quad \forall t \geq t_0.
\]

So, for all \( t \geq t_0 + s \), we have

\[
\|T(t)x_n - T(s)x_{n+1}\|
\leq aT\|T(t-s)x_n - x_{n+1}\| + bT\|T(t-s)x_n - T(s)x_{n+1}\|
\leq aT\|T(t-s)x_n - x_{n+1}\| + bT\|T(t-s)x_n - T(t)x_n\| + bT\|T(t)x_n - T(s)x_{n+1}\|
\leq (aT + 2bT)\|T(t)x_n - x_{n+1}\| + bT\|T(t)x_n - T(s)x_{n+1}\|
\]

This implies that:

\[
\|T(t)x_n - T(s)x_{n+1}\| \leq \left(\frac{a + 2b}{1 - b}\right)(r_n + \varepsilon),
\]

It Then follows from property (e) that

\[
\left\|T(t)x_n - \frac{1}{2}(x_{n+1} + T(s)x_{n+1})\right\|
\leq \left(\frac{aT + 2bT}{1 - bT}\right)(r_n + \varepsilon)\left(1 - \delta_X\left(\frac{d(x_{n+1}) - \varepsilon}{(aT + 2bT)(r_n + \varepsilon)}\right)^2\right).
\]
for \( t \geq t_0 + s \), and hence
\[
  r_n \leq \limsup_{t \to \infty} \left\| T(t)x_n - \frac{1}{2} (x_{n+1} + T(s)x_{n+1}) \right\| 
  \leq \left( \frac{a_T + 2b_T}{1 - b_T} \right) (r_n + \varepsilon) \left( 1 - \delta_X \left( \frac{d(x_{n+1}) - \varepsilon}{(a_T + 2b_T)(1 - b_T)r_n} \right) \right) 
\]
(12)

Now, let us tending \( \varepsilon \) to zero, we get that
\[
  r_n \leq \left( \frac{a_T + 2b_T}{1 - b_T} \right) r_n \left( 1 - \delta_X \left( \frac{d(x_{n+1})}{(a_T + 2b_T)(1 - b_T)r_n} \right) \right). 
\]
(13)

This last equality implies that
\[
  \delta_X \left( \frac{d(x_{n+1})}{(a_T + 2b_T)(1 - b_T)r_n} \right) \leq 1 - \frac{1}{\left( \frac{a_T + 2b_T}{1 - b_T} \right)} 
\]
(14)

Since \( X \) is uniformly convex, then the characteristic of convexity of \( X \),
\[
  \varepsilon_0 = \varepsilon_0(X) = \sup \{ \varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0 \} = 0 
\]
It then follows that \( \delta_X \) is strictly increasing on \([0, 2] \to [0, 1]\), and
\[
  d(x_{n+1}) \leq \left( \frac{a_T + 2b_T}{1 - b_T} \right) r_n \delta_X^{-1} \left( 1 - \frac{1}{\left( \frac{a_T + 2b_T}{1 - b_T} \right)} \right) 
\]
(15)

Hence
\[
  d(x_{n+1}) \leq \left( \frac{a_T + 2b_T}{1 - b_T} \right)^2 N(X)^{-1} \delta_X^{-1} \left( 1 - \frac{1}{\left( \frac{a_T + 2b_T}{1 - b_T} \right)} \right) d(x_n). 
\]
(16)

And
\[
  d(x_n) \leq Ad(x_{n-1}) \leq A^n d(x_0), 
\]
(17)

where
\[
  A = \left( \frac{a_T + 2b_T}{1 - b_T} \right)^2 N(X)^{-1} \delta_X^{-1} \left( 1 - \frac{1}{\left( \frac{a_T + 2b_T}{1 - b_T} \right)} \right) < 1 \text{ by assumption.} 
\]

In other hand we have
\[
  \| x_{n+1} - x_n \| \leq \limsup_{t \to \infty} \| T(t)x_n - x_{n+1} \| + \limsup_{t \to \infty} \| T(t)x_n - x_n \| 
  \leq r_n + d(x_n) 
  \leq \left( 1 + N(X)^{-1} \left( \frac{a_T + 2b_T}{1 - b_T} \right) \right) d(x_n), 
\]
(18)
so in light of equ10, we conclude that \( \{x_n\} \) is a Cauchy sequence, hence it converges strongly to some \( z \in X \).

Now, for all \( t \in G \), we have

\[
\| z - T(t)z \| \leq \| z - x_n \| + \| x_n - T(t)x_n \| + \| T(t)x_n - T(t)z \| \\
\leq (1 + a_T)\| x_n - z \| + \| x_n - T(t)x_n \| + b_T\| x_n - T(t)z \| \\
\leq (1 + a_T + b_T)\| x_n - z \| + \| x_n - T(t)x_n \| + b_T\| z - T(t)z \|
\]

That is

\[
(1 - b_T)\| z - T(t)z \| \leq (1 + a_T + b_T)\| x_n - z \| + \| x_n - T(t)x_n \| 
\]

Hence

\[
\leq \left( \frac{1 + a_T + b_T}{1 - b_T} \right)\| x_n - z \| + \left( \frac{1}{1 - b_T} \right)d(x_n),
\]

But since the last term tends to zero as \( n \) tends to \( +\infty \), then we deduce the required result, and the proof is complete.

4 APPLICATIONS

**Corollary 4.1** let \( X \) be a Hilbert space and \( C \) a nonempty closed bounded and convex subset of \( X \), and \( T : C \to C \) is Mean non-expansive mapping. Assume that the constants \( a_T \) and \( b_T \) satisfy \( a_T + 2b_T \leq 1 < \lambda < a_T \geq 0 \). Where \( \lambda > 1 \) is the unique solution of the equation (2). Then \( T \) has a fixed point in \( C \).

**References**


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