Q-Bi Fuzzy Sub Semi Groups of Bi-Ideals

S. Subramanian*, R. Nagarajan** and B. Chellappa***

(*) Department of Mathematics,
Saranathan College of Engineering,
Tiruchirappalli -12. Tamil Nadu, India
mathsspmmanian@gmail.com

(**) Department of Mathematics
J.J College of Engineering & Technology
Tiruchirappalli-09.TamilNadu, India
nagalogesh@yahoo.co.in

(***) Department of Mathematics,
Alagappa Govt.Arts College,
Karaikudi-03,Tamil Nadu, India
chellappa58@gmail.com

Abstract. We consider the Bi Q-fuzzification of the concept of several ideals in a semi-group S, and investigate some properties of such ideals.

Mathematics Subject Classification: 20M12, 04A72

1. Introduction

After the introduction of fuzzy sets by L.A. Zadeh [9], several researchers explored on the generalization of the notion of fuzzy set, the concept of intuitionistic fuzzy set was introduced by K.T. Atanassav [2] as a generalization of the notion of fuzzy set. In [4], N. Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideals in a semi-groups then concept (1,2)- ideals in a semi-group was introduced by S. Lajos [6]. In this paper we consider the Bi Q-fuzzification of the concept of several ideals in a semi-group S and investigate some properties of such ideals.
2. Preliminaries

Let ‘S’ be a semi-group. By a sub semi-groups of S we mean a non-empty subset A of S such that \( A^2 \subseteq A \) and by a left (right) ideal of S we mean a non-empty subset A of S such that \( SA \subseteq A \) (\( AS \subseteq A \)). By two sided ideal or simply ideal, we mean a non-empty subset of S which is both left and right ideal of S. A sub semi-group ‘A’ of a semi-group S is called a bi-ideal of S if as \( A^2 \subseteq A \). A sub semi-group A of S is called a (1,2)- ideal of S if \( ASA^2 \subseteq A \). A semi-group S is said to be (2, 2) – regular if \( x \in x^2Sx^2 \) for \( x \in S \). A semi-group ‘S’ is said to be regular if, for each \( x \in S \), there exists \( y \in S \) such that \( x = xyx \). A semi-group ‘S’ is said to be completely regular if for each \( x \in S \), there exists \( y \in S \) such that \( x = xyx \) and \( xy = yx \). For a semi-group ‘S’, note that S is completely regular iff S is a union of groups iff S is (2,2)- regular. A semi-group ‘S’ is said to be left (resp. right) ideal if every left (resp. right) ideal of S is a two sided ideal of S.

A bi fuzzy set (briefly IFS) ‘A’ is a non-empty set \( X \) is an object having the form \( A = \{(x, t_A(x), f_A(x) / x \in X\} \) where the functions \( t_A : X \rightarrow [0,1] \) and \( f_A : X \rightarrow [0,1] \) denote the truth degree of membership and false degree of membership respectively and as \( t_A(x) + f_A(x) \leq 1 \), for all \( x \in X \).

In what follows, let S denote a semi-group unless otherwise specified.

**Definition 2.1:** An IFS \( A = (t_A, f_A) \) in S is called an bi Q-fuzzy sub semi-group of S

if, (i) \( t_A(xy,q) \geq \min \{t_A(x,q), t_A(y,q)\} \)

(ii) \( f_A(xy,q) \leq \max \{f_A(x,q), f_A(y,q)\} \) for all \( x, y \in S \).

**Definition 2.2:** An IFS \( A = (t_A, f_A) \) in S is called an bi Q-fuzzy left ideal of S if \( t_A(xy,q) \geq t_A(y,q) \) and \( f_A(xy,q) \leq f_A(y,q) \), for \( x, y \in S \). An bi Q-fuzzy right ideal of S define in an analogous way. An IFS \( A = (t_A, f_A) \) in S is called an bi Q-fuzzy ideal of S if it is both an bi Q-fuzzy right bi Q-fuzzy left (right) ideal of S is an biQ-fuzzy subgroup of S.

**Definition 2.3:** An bi Q-fuzzy sub semi-group \( A = (t_A, f_A) \) of S is called an bi Q-fuzzy bi-ideal of S if,

(i) \( t_A(xwy,q) \geq \min \{t_A(x,q), t_A(y,q)\} \)

(ii) \( f_A(xwy,q) \leq \max \{f_A(x,q), f_A(y,q)\} \) for all \( w, x, y \in S \).

**Example 2.4:** Let \( S = \{a, b, c, d, e\} \) be a semi-group with the following Cayley table.
Define an IFS $A = (t_A, f_A)$ in $S$ by $t_A(a, q) = 0.4$, $t_A(b, q) = 0.6$, $t_A(c, q) = 0.3$, $t_A(d, q) = 0.7$, $t_A(e, q) = 0.6$. By routine calculation, we can check that ‘$A$’ is a fuzzy bi-ideal of $S$.

Let ‘$X$’ be a non-empty set. A mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy set in $X$. The complement of a fuzzy set $\mu$ in $X$, denoted by $\mu^c$ is the fuzzy set in $X$ given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. In what follows, let $Q$ and $S$ denote a set and a semi-group, respectively unless otherwise specified. A mapping $\mu : S \times Q \rightarrow [0,1]$ is called a $Q$ fuzzy set in $X$.

### 3. Characteristic of $Q$-fuzzy bi-ideals

**Proposition 3.1:** Every bi $Q$-fuzzy bi-ideal is an bi $Q$-fuzzy $(1,2)$-ideal.

**Proof:** Let $A = (t_A, f_A)$ be an bi $Q$-fuzzy bi-ideal of $S$ and let $w, x, y, z \in S$ and $q \in Q$ then

\[
\begin{align*}
t_A(xw(yz), q) &= t_A((xwy)z, q) \\
&\geq \min \{ t_A(xwy, q), t_A(z, q) \} \\
&\geq \min \{ \min \{ t_A(x,q), t_A(y,q) \} t_A(z, q) \} \\
&= \min \{ t_A(x,q), t_A(y,q), t_A(z,q) \}
\end{align*}
\]
and
\[ f_A(xw(yz), q) = f_A((xwy)z, q) \]
\[ \leq \text{Max} \{ f_A(xwy, q), f_A(z, q) \} \]
\[ \leq \text{Max} \{ \text{Max} \{ f_A(x, q), f_A(y, q) \}, f_A(z, q) \} \]
\[ = \text{Max} \{ f_A(x, q), f_A(y, q), f_A(z, q) \} \]
Hence \( A = (t_A, f_A) \) be an biQ-fuzzy \((1, 2)\)-ideal of \( S \).

To consider the converse of proposition 3.1, we need to strengthen the condition of a sub semi-group \( S \).

**Proposition 3.2:** If \( S \) is a regular semi-group, then every bi Q-fuzzy \((1,2)\)-ideal of \( S \) is an biQ-fuzzy bi-ideal of \( S \).

**Proof:** Assume that a sub semi-group \( S \) is regular and let \( A = (t_A, f_A) \) be an bi Q-fuzzy \((1,2)\)-ideal of \( S \). Let \( w, x, y \in S \) and \( q \in Q \). Since \( S \) is regular, we have \( xw \in (xsx)s \subseteq xsx \) which implies that \( xw = xsx \) for some \( s \in S \) thus,
\[ t_A(xwy, q) = t_A((xsx)y, q) = t_A(xs(xy), q) \]
\[ \geq \text{Min} \{ t_A(x, q), t_A(x, q), t_A(y, q) \} \]
\[ = \text{Min} \{ t_A(x, q), t_A(y, q) \} \]
and
\[ f_A(xwy, q) = f_A((xsx)y, q) = f_A(xs(xy), q) \]
\[ \leq \text{Max} \{ f_A(x, q), f_A(x, q), f_A(y, q) \} \]
Therefore \( A = (t_A, f_A) \) is an bi Q-fuzzy bi-ideal of \( S \).

**Proposition 3.3:** Let ‘\( A \)’ be an bi Q-fuzzy bi-ideal of \( S \). If \( S \) is a completely regular, then \( A(a, q) = A(a^2, q) \) for all \( a \in S \) and \( q \in Q \).

**Proof:** Let \( a \in S \) and \( q \in Q \), then there exists \( x \in S \) such that \( a = a^2xa^2 \).
Hence,
\[ t_A(a,q) = t_A(a^2x^a^2,q) \]
\[ \geq \min \{ t_A(a^2,q), t_A(a^2,q) \} \]
\[ = t_A(a^2,q) \]
\[ \geq \min \{ t_A(a,q), t_A(a,q) \} \]
\[ = t_A(a,q) \]

and
\[ f_A(a,q) = f_A(a^2x^a^2,q) \]
\[ \leq \max \{ f_A(a^2,q), f_A(a^2,q) \} \]
\[ = f_A(a^2,q) \]
\[ \leq \max \{ f_A(a,q), f_A(a,q) \} \]
\[ = f_A(a,q) \]

It follows that \( t_A(a,q) = t_A(a^2,q) \) and \( f_A(a,q) = f_A(a^2,q) \) so that \( A(a,q) = A(a^2,q) \).

**Proposition 3.4:** Let \( A \) be bi Q-fuzzy ideal of \( S \). If \( \text{‘} S \text{’} \) is an intra-regular then \( A(a,q) = A(a^2,q) \) for all \( a \in S \) and \( q \in Q \).

**Proof:** Let \( a \in S \) then \( S \) is intra-regular there exists \( x \) and \( y \) in \( S \) such that \( a = xa^2y \). Hence since \( A \) is bi Q-fuzzy bi-ideal.

\[ t_A(a,q) = t_A(xa^2y,q) \]
\[ \geq t_A(xa^2,q) \]
\[ \geq t_A(a^2,q) \]
\[ \geq \min \{ t_A(a,q), t_A(a,q) \} \]
\[ = t_A(a,q) \]

and
\[ f_A(a,q) = f_A(xa^2y,q) \]
\[ \leq f_A(xa^2,q) \]
\[ \leq f_A(a^2, q) \]
\[ \leq \text{Max} \{ f_A(a, q), f_A(a, q) \} \]
\[ = f_A(a, q) \]

Hence we have \( t_A(a, q) = t_A(a^2, q) \) for all \( x, y \in S \) and \( q \in Q \).

**Proposition 3.5:** Let \( 'A' \) be bi Q-fuzzy bi-ideal of \( S \). If \( S \) is an intra-regular then \( A(ab, q) = A(ba, q) \) for all \( a, b, \in S \) and \( q \in Q \).

**Proof:** Let \( a, b, \in S \) and \( q \in Q \) then by proposition (3.3), we have
\[
\begin{align*}
t_A(ab, q) &= t_A((ab)^2, q) \\
&\geq t_A(a(ba)b, q) \\
&\geq t_A(ba, q) = t_A((ba)^2, q) \\
&\geq t_A((b(ab)a, q) \\
&= t_A(ab, q)
\end{align*}
\]
and
\[
\begin{align*}
f_A(ab, q) &= f_A((ab)^2, q) \\
&\leq f_A(a(ba)b, q) \\
&\leq f_A(ba, q) = f_A((ba)^2, q) \\
&\leq f_A((b(ab)a, q)) \\
&= f_A(ab, q)
\end{align*}
\]
So we have \( t_A(ab, q) = t_A(ba, q) \) and \( f_A(ab, q) = f_A(ba, q) \). Therefore \( A(ab, q) = A(ba, q) \).

**Proposition 3.6:** An IFS \( 'A' \) is bi Q-fuzzy bi-ideal of \( S \) if and only if the Q-fuzzy sets \( t_A \) and \( f_A \) are Q-fuzzy bi-ideals of \( S \).

**Proof:** Let \( 'A' \) be bi Q-fuzzy bi-ideal of \( S \), then clearly \( t_A \) is a Q-fuzzy bi-ideal of \( S \). Let \( x, a, y \in S, q \in Q \) then
\[
\bar{f}_A(xy, q) = 1 - f_A(xy, q)
\]
\[
\begin{align*}
\geq & \quad 1 - \text{Max} \{f_A(x,q), f_A(y,q)\} \\
= & \quad \text{Min} \{1 - f_A(x,q), 1 - f_A(y,q)\} \\
= & \quad \text{Min} \{\overline{f_A}(x,q), \overline{f_A}(y,q)\} \\
\end{align*}
\]
and
\[
\overline{f_A}(xay, q) = 1 - f_A(xay, q)
\]
\[
\geq 1 - \text{Max}\{f_A(x,q), f_A(y,q)\}
\]
\[
= \quad \text{Min} \{1 - f_A(x,q), 1 - f_A(y,q)\}
\]
\[
= \quad \text{Min} \{\overline{f_A}(x,q), \overline{f_A}(y,q)\}
\]

Hence \(\overline{f_A}\) is a Q-fuzzy bi-ideal of \(S\). Conversely, suppose that \(t_A\) and \(f_A\) are Q-fuzzy bi-ideals of \(S\). Let \(a, x, y \in S\).

\[
1 - f_A(xy, q) = \overline{f_A}(xy, q)
\]
\[
\geq \quad \text{Min} \{\overline{f_A}(x,q), \overline{f_A}(y,q)\}
\]
\[
= \quad \text{Min} \{1 - f_A(x,q), 1 - f_A(y,q)\}
\]
\[
= \quad \text{Max}\{f_A(x,q), f_A(y,q)\}
\]
\[
1 - f_A(xay, q) = \overline{f_A}(xay, q)
\]
\[
= \quad \text{Min} \{\overline{f_A}(x,q), \overline{f_A}(y,q)\}
\]
\[
= \quad \text{Min} \{1 - f_A(x,q), 1 - f_A(y,q)\}
\]
\[
= \quad \text{Max}\{f_A(x,q), f_A(y,q)\}
\]

which imply that \(f_A(xy, q) \leq \text{Max}\{f_A(x,q), f_A(y,q)\}\)

and \(f_A(xay, q) \leq \text{Max}\{f_A(x,q), f_A(y,q)\}\)

This completes the proof.

**Proposition 3.7:** An IFS \(A = (t_A, f_A)\) is bi Q-fuzzy bi-ideal of \(S\) if and only if

\[
\square A = (t_A, \overline{f_A})\quad \text{and} \quad \diamondsuit A = (\overline{f_A}, f_A)\quad \text{are bi Q-fuzzy bi-ideals of} \ S.
\]
Proof: It is sufficient to show that $\overline{t}_A$ satisfies the condition (i) in definition 2.1. and (ii) in definition of 2.3. 

For any $a, x, y \in S$, we have

$$\overline{t}_A (xy, q) = 1 - t_A (xy, q)$$

$$\leq 1 - \min \{t_A(x, q), t_A(y, q)\}$$

$$= \max \{1 - t_A(x, q), 1 - t_A(y, q)\}$$

$$= \max \{\overline{t}_A (x, q), \overline{t}_A (y, q)\}$$

and

$$\overline{t}_A (xay, q) = 1 - t_A (xay, q)$$

$$\leq 1 - \min \{t_A(x, q), t_A(y, q)\}$$

$$= \max \{1 - t_A(x, q), 1 - t_A(y, q)\}$$

$$= \max \{\overline{t}_A (x, q), \overline{t}_A (y, q)\}$$

Therefore $A$ is bi Q-fuzzy bi-ideal of $S$.

Similarly, we can show $A$ is intuitionistic Q-fuzzy bi-ideal of $S$.

Proposition 3.8: Let $f : S \rightarrow T$ be a homomorphism of semi-groups. If $B = (t_B, f_B)$ is bi Q-fuzzy bi-ideal of $T$, then the pre image $f^{-1}(B)$ of $B$ under $f$ is bi Q-fuzzy bi-ideal of $S$.

Proof: Assume that $B = (t_B, f_B)$ is bi Q-fuzzy bi-ideal of $T$ and let $x, y \in S$ then

$$f^{-1}(t_B)(xy, q) = t_B(f(xy, q))$$

$$= t_B(f(x, q), f(y, q))$$

$$\geq \min \{t_B(f(x, q), t_B(f(y, q))\}$$

$$= \min \{f^{-1}(t_B)(x, q), f^{-1}(t_B)(y, q)\}$$

Also

$$f^{-1}(t_B)(xy, q) = f_B(f(xy, q))$$

$$= f_B(f(x, q), f(y, q))$$
\[ \leq \text{Max } \{ f_B(f(x,q), f_B(f(y,q)) \} \]
\[ = \text{Max } \{ f^{-1}(f_B(x,q)), f^{-1}(f_B(y,q)) \} \]

Hence \( f^{-1}(B) = (f^{-1}(t_B), f^{-1}(f_B)) \) is bi-Q-fuzzy sub semi-group of S. For any \( x, a, y \in S \) we have

\[ f^{-1}(t_B)(xay,q) = t_B(f(xay,q)) \]
\[ = t_B(f(x,a), f(a,q), f(y,q)) \]
\[ \geq \text{Min } \{ t_B(f(x,q), t_B(y,q)) \} \]
\[ = \text{Min } \{ f^{-1}(t_B(x,q), f^{-1}(t_B(y,q)) \} \]

and

\[ f^{-1}(f_B)(xay,q) = f_B(f(xay,q)) \]
\[ = f_B(f(x,a), f(a,q), f(y,q)) \]
\[ \leq \text{Max } \{ f_B(f(x,q), f_B(f(y,q)) \} \]
\[ = \text{Max } \{ f^{-1}(f_B(x,q), f^{-1}(f_B(y,q)) \} \]

Therefore \( f^{-1}(B) \) is bi-Q-fuzzy bi-ideal of S.

**Proposition 3.9:** If \( \{A_i\}_{i \in A} \) is a family of bi-Q-fuzzy bi-ideals of S then \( \cap A_i \) is biQ-fuzzy bi-ideal of S, where

\[ \cap A_i = \{ \land t_{A_i}, \lor f_{A_i} \} \] and

\[ \land t_{A_i}(x,q) = \text{Min } \{ t_{A_i}(x,q) / i \in A, x \in S \} \]
\[ \lor f_{A_i}(x,q) = \text{Max } \{ f_{A_i}(x,q) / i \in A, x \in S \} \]

**Proof:** Let \( x, y \in S \) then we have

\[ \land t_{A_i}(x,q) = \land \{ \text{Min } \{ t_{A_i}(x,q), t_{A_i}(y,q) \} \} \]
\[ = \text{Min } \{ \text{Min } \{ t_{A_i}(x,q), t_{A_i}(y,q) \} \} \]
\[ = \text{Min } \{ \text{Min } t_{A_i}(x,q), \text{Min } t_{A_i}(y,q) \} \]
\[ = \text{Min } \{ \land t_{A_i}(x,q), \land t_{A_i}(y,q) \} \]
\( Vf_{A_1}(xy,q) \leq \bigvee \{ \max \{ f_{A_1}(x,q), f_{A_1}(y,q)\} \}
\]
\[
= \max \{ \max \{ f_{A_1}(x,q), f_{A_1}(y,q)\} \}
\]
\[
= \max \{ \max f_{A_1}(x,q), \min f_{A_1}(y,q)\} 
\]
\[
= \max \{ \max f_{A_1}(x,q), \max f_{A_1}(y,q)\} 
\]
Hence \( \cap A_i \) is bi Q-fuzzy sub semi-group of \( S \). Next for \( x, y, a \in S \) we obtain
\[
\wedge t_{A_1}(xay,q) \geq \wedge \{ \min \{ t_{A_1}(x,q), t_{A_1}(y,q)\} \}
\]
\[
= \min \{ \min \{ t_{A_1}(x,q), t_{A_1}(y,q)\} \}
\]
\[
= \min \{ \min t_{A_1}(x,q), \min t_{A_1}(y,q) \}
\]
\[
= \min \{ \min t_{A_1}(x,q), \min t_{A_1}(y,q) \}
\]
\[
Vf_{A_1}(xay,q) \leq \bigvee \{ \max \{ f_{A_1}(x,q), f_{A_1}(y,q)\} \}
\]
\[
= \max \{ \max \{ f_{A_1}(x,q), f_{A_1}(y,q)\} \}
\]
\[
= \max \{ \max f_{A_1}(x,q), \min f_{A_1}(y,q)\} 
\]
\[
= \max \{ \max f_{A_1}(x,q), \max f_{A_1}(y,q)\} 
\]
Hence \( \cap A_i \) is bi Q-fuzzy bi-ideal of \( S \). This completes the proof.

**Conclusion:** Kuroki. N [4] introduced the concept of fuzzy ideals and bi-ideals in a semi group and Lajos.S [6] investigate the concept of (1,2)-ideals of union of groups. [3] discussed the concept of Q-Vague groups and vague normal subgroups with respect to (T,S) norms. In this paper, we investigate the concept of bifuzzy membership functions in several ideals of semi group and investigate some properties of such ideal

**References**

Q-bi fuzzy sub semi groups of bi-ideals


Received: June, 2011