Implicit Function Theorem to Sequentially Strongly Differentiable Mappings in Topological Vector Spaces

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Abstract

In the present paper, we shall need to establish the implicit function theorem to sequentially strongly differentiable mapping $f : U \times V \subset E_\xi \times F_\eta \to E_\xi$, where $\xi \equiv \{p_\alpha\}$ is a collection of quasi-norms on a normed space $E$, $\eta \equiv \{q_\beta\}$ is a collection of semi-norms on a normed space $F$.

The strategy followed can be described by the following scheme:
(1) Give an acceptable definition of sequentially strongly differentiable mappings with respect to $(E_\xi, F_\eta)$ that combines the basic ideas of differentiable mappings in normed spaces.
(2) Give an acceptable definition of sequentially strongly Lipschitzian mappings with respect to $(E_\xi, F_\eta)$.
(3) Prove the implicit function theorem for such mappings.

In general, various technical conditions are needed in order for the Theorem to hold for such maps. However, in the cases of interest to us, we use some properties of sequentially continuous linear mappings from $E_\xi$ to $F_\eta$ (see [2]) to establish the result.

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1 Preliminaries

Let $N$ be the set of all nonnegative integer and $R$ be the set of all real number.

Let $E$ be a vector space, $(\ell_1, \prec)$ be a directed set and let $\{p_\alpha\}_{\alpha \in \ell_1}$ be a family of positive function on $E$.

**Definition 1.1** The family $\{p_\alpha\}_{\alpha \in \ell_1}$ is said to be a family of quasi-norms on $E$ iff the following conditions hold:

1. $\forall \alpha \in \ell_1, p_\alpha(0) = 0$;  
2. $\forall \lambda \in \ell_1 \exists \mu \in \ell_1$ such that $\lambda \prec \mu$ and $\forall x, y \in E, p_\lambda(x + y) \leq p_\mu(x) + p_\mu(y)$;  
3. $\lambda \prec \mu \Rightarrow \forall x \in E, \forall r \in R, p_\lambda(rx) \leq |r|p_\mu(x)$.

If in addition the condition

$(p_\lambda(x) = 0 \quad \forall \lambda \in \ell_1) \Rightarrow x = 0$

is satisfied, then the family of quasi-norms $\{p_\alpha\}_{\alpha \in \ell_1}$ is said to be separating.

Let us now recall the following result due to Hyers (see [3]) which gives us a characterization of topological vector spaces via families of quasi-norms:

**Theorem 1.2** Assume now that $(E, \xi)$ is a separated topological vector space. Then the topology $\xi$ can be generated by a separating family $\{p_\alpha\}_{\alpha \in \ell_1}$ of quasi-norms.

1.1 Basic notions and results relating to the differentiability for maps between two topological vector spaces

Let $(E, \xi, \|\cdot\|_1)$ be a bitopological vector space such that $(E, \xi)$ is a topological vector space and $(E, \|\cdot\|_1)$ is a normed space.

Assume that the topology $\xi$ is generated by the family $\{p_\alpha\}$ of quasi-norms.

Let also $(F, \eta, \|\cdot\|_2)$ be a bitopological vector space such that $(F, \eta)$ is a locally convex space and $(F, \|\cdot\|_2)$ is a normed space.

Let $\{q_\beta\}$ be a family of semi-norms generating the topology $\eta$.

Let us recall first some basic notations (see [3]) which will be used later:

- $B(E)$ – System of all bounded set in $(E, \|\cdot\|_1)$,
- $B(E_\xi)$ – System of all bounded set in $E_\xi$,
\( C_0(E) \) – System of all sequence which converge to 0 in \((E, \|\cdot\|_1)\),

\( C_0(E_\xi) \) – System of all sequence which converge to 0 in \(E_\xi\),

\( C(E) \) – System of all relatively compact set in \((E, \|\cdot\|_1)\),

\( C(E_\xi) \) – System of all sequentially relatively compact set in \(E_\xi\),

\( l(E, F) \) – Space of all continuous linear mapping from \((E, \|\cdot\|_1)\) to \((F, \|\cdot\|_2)\),

\( l(E_\xi, F_\eta) \) – Space of all sequentially continuous linear mapping from \(E_\xi\) to \(F_\eta\),

\( (F_\eta)' \) – Space of all continuous linear mapping from \(F_\eta\) to \(R\);

\( \mathcal{L}(E_\xi, F_\eta) \) – Space of all continuous linear mapping from \(E_\xi\) to \(F_\eta\).

Denote by \( B_1(F) \) the unit ball in \((F, \|\cdot\|_2)\). Let \( \sigma \subset B(E_\xi)\), \( \sigma \neq \emptyset \). Suppose that \( \sigma \) satisfies the condition:

\[
\bigcup_{C \in \sigma} C = E; \quad (1.1.1)
\]

\[
\forall C_1 \subset E \ \forall C_2 \in \sigma \ \forall C_1 \subset C_2 \Rightarrow C_1 \in \sigma. \quad (1.1.2)
\]

Let also \( \sigma_1, \sigma_2 \subset B(E_\xi); \ \tau_1, \tau_2 \subset B(F_\eta) \). Assume that \( \sigma_2 \) and \( \tau_2 \) satisfy the condition (1.1.2).

**Definition 1.3** Let \( C_1 \in \sigma_1\), \( x_0 \in E\), \( \{x_n\} \subset E \). We say that \( x_n \) converge to \( x_0 \) along the set \( C_1 \) and we denote \( x_n \rightarrow_{C_1} x_0 \) if there exists a sequence \((\xi_n) \in C_0(R^+)\) such that \( x_n - x_0 \in \xi_n C_1 \) for \( n \) large enough.

Let \( U \subset E \).
**Definition 1.4** A mapping \( r : U \subset E_\xi \to F_\eta \) is said to be infinitesimally \( \sigma_2 \)-small at \( x_0 \in U \), if

\[
\forall \varepsilon > 0 \ \forall \beta \ \forall C \in \sigma_2 \ \exists \delta > 0 \ \forall h \in C \ \forall t \in R : 0 < |t| < \delta
\]

\[
x_0 + th \in U \Rightarrow q_\beta(t^{-1}r(x_0 + th)) \leq \varepsilon.
\]

In particular, if \( \sigma_2 \) is the smallest system in \( \mathcal{B}(E_\xi) \) for which the conditions (1.1.1), (1.1.2) hold, then we say that \( r \) is Gâteaux infinitesimally-small.

If \( \sigma_2 = \mathcal{B}(E_\xi) \), then we say that \( r \) is boundedly small.

If \( \sigma_2 = \mathcal{C}(E_\xi) \), then we say that \( r \) is sequentially relatively compactly small.

**Definition 1.5** A mapping \( r : U \subset E_\xi \to F_\eta \) is said to be sequentially strongly \( \sigma_1 \sigma_2 \)-small at \( x_0 \in U \), if \( r(x_0) = 0 \) and

\[
\forall \beta \ \forall C_1 \in \sigma_1 \ \forall C_2 \in \sigma_2 \ \forall \{x_n\} \subset U : x_n \to C_1 x_0 \ \forall t_p \to 0 : t_p \neq 0
\]

\[
\sup_{h \in C_2, x_n + t_p h \in U} q_\beta(t_p^{-1}[r(x_n + t_p h) - r(x_n)]) \to 0, \text{ as } n, p \to \infty.
\]

**Definition 1.6** A mapping \( f : U \subset E_\xi \to F_\eta \) is said to be sequentially strongly \( \sigma_1 \sigma_2 \)-equivalent to an operator \( A \in l(E_\xi, F_\eta) \) at \( x_0 \in U \), if the mapping \( h \mapsto f(x_0 + h) - f(x_0) - Ah \) is sequentially strongly \( \sigma_1 \sigma_2 \)-small at 0.

If such a \( A \) is unique, then we say that \( f \) is sequentially strongly \( \sigma_1 \sigma_2 \) differentiable at \( x_0 \). In this case, \( A \) is called the sequential strong \( \sigma_1 \sigma_2 \) differential of \( f \) at \( x_0 \) and we denote it by \( f'(x_0) \).

**Proposition 1.7** Let \( U \) a convex set in \( E \), \( x_0 \in U \). Suppose that \( \tilde{E} \equiv \text{Lin}[(U - x_0) \cap ( \bigcup_{C \in \sigma_2} C)] \) is sequentially dense in \( E_\xi \) and assume also that the topology \( \eta \) is separated, \( A \in l(E_\xi, F_\eta) \), \( f : U \subset E_\xi \to F_\eta \) is \( \sigma_2 \)-equivalent to \( A \) at \( x_0 \). Then \( f \) is \( \sigma_2 \) differentiable at \( x_0 \).

**Proof.**

Let \( \tilde{A} \in l(E_\xi, F_\eta) \) such that \( f \) is \( \sigma_2 \) equivalent to \( \tilde{A} \) at \( x_0 \). Assume that \( Ah \neq \tilde{A}h \) for some \( h \in E \). Then \( \exists \beta \exists \varepsilon > 0 \ q_\beta(\tilde{A}h - Ah) > \varepsilon \). Moreover, \( \exists k_n \ \exists \lambda_i h_n \exists x_i \in (U - x_0) \cap ( \bigcup_{C \in \sigma_2} C) : h_n \equiv \sum_{i=1}^{k_n} \lambda_i x_i^{-} \) converges to \( h \) with respect to \( \xi \).

Since \( A, \tilde{A} \in l(E_\xi, F_\eta) \), then \( \exists n \in N \ q_\beta((\tilde{A} - A)h_n) > \varepsilon \).

\[
(1.1.5)
\]
On the other hand,

\[
q_\beta((\tilde{A} - A)h_n) \leq \sum_{i=1}^{k_n} |\lambda_i^n| q_\beta((\tilde{A} - A)x_i^n)
\]

\[
\leq \sum_{i=1}^{k_n} |\lambda_i^n| \left[q_\beta(t^{-1}(f(x_0 + tx_i^n) - f(x_0)) - Ax_i^n) + q_\beta(t^{-1}(f(x_0 + tx_i^n) - f(x_0)) - \tilde{A}x_i^n)\right]
\]

\[
< \varepsilon \text{ for } t \text{ small enough.}
\]

But this contradicts (1.1.5).

**Theorem 1.8** Let \( U \) a convex set in \( E \), \( f : U \subset E \rightarrow F \) a Gâteaux differentiable mapping. Then

\[
\forall \lambda, \mu \in U \forall \beta \ q_\beta(f(x_1) - f(x_2)) \leq \sup_{0 \leq \theta \leq 1} q_\beta(f((1 - \theta)x_2 + \theta x_1)(x_1 - x_2)).
\]

**Proof.**
Let \( x_1, x_2 \in U \). For all \( \varphi \in (F_\eta)' \), set \( g(t) = \varphi(f(x_2 + t(x_1 - x_2))) \forall t \in [0,1] \). In this case \( g'(t) = \varphi(f'(x_2 + t(x_1 - x_2))(x_1 - x_2)) \). By the mean value theorem, we have then \( g(1) = g(0) + g'(\theta) \) for some \( \theta \in [0,1] \). This implies that

\[
\forall \varphi \in (F_\eta)' \ \varphi(f(x_1) - f(x_2)) = \varphi(f'(x_2 + \theta(x_1 - x_2))(x_1 - x_2)),
\]

where \( \theta \) depends on \( \varphi \). Pick any \( \beta \) and put

\[
U^0 = \{ \varphi \in (F_\eta)' \mid \forall x \in F \ |\varphi(x)| \leq q_\beta(x) \}.
\]

Thus, \( \forall \varphi \in U^0 \)

\[
\varphi(f(x_1) - f(x_2)) = \varphi(f'(x_2 + \theta(x_1 - x_2))(x_1 - x_2))
\]

\[
\leq q_\beta[f'(x_2 + \theta(x_1 - x_2))(x_1 - x_2)]
\]

\[
\leq \sup_{0 \leq \theta \leq 1} q_\beta[f'((1 - \theta)x_2 + \theta x_1)(x_1 - x_2)].
\]

On the other hand, \( q_\beta(f(x_1) - f(x_2)) \leq \sup_{\varphi \in U^0} \varphi(f(x_1) - f(x_2)) \). Hence,

\[
q_\beta(f(x_1) - f(x_2)) \leq \sup_{0 \leq \theta \leq 1} q_\beta[f'((1 - \theta)x_2 + \theta x_1)(x_1 - x_2)].
\]

As an immediate consequence of theorem 8, we get the following result:
Theorem 1.9 Let $U$ a convex set in $E$, $x_0 \in U$, $f : U \subset E_\xi \to F_\eta$ a Gâteaux differentiable mapping. Then

$$\forall x_1, x_2 \in U \, \forall \beta \, \quad q_\beta[f(x_1)-f(x_2)-f'(x_0)(x_1-x_2)] \leq \sup_{0 \leq \theta \leq 1} q_\beta[(f'((1-\theta)x_2+\theta x_1)-f'(x_0))(x_1-x_2)].$$

Proposition 1.10 Let $x_0 \in U$. Suppose that $B(E) \subset B(E_\xi)$, $\sigma_1, \sigma_2 \subset B(E)$, $C(E) \subset \sigma_1$. Assume also that $U$ is a convex set in $E$, $f : U \subset E_\xi \to F_\eta$ is Gâteaux differentiable, i.e. $f'(U) \subset l(E_\xi, F_\eta)$. Then the following conditions are equivalent:

1) $f' : U \to l_{\sigma_2}(E_\xi, F_\eta)$ is continuous at $x_0$;

2) $f$ is sequentially strongly $\sigma_1 \sigma_2$ differentiable at $x_0$.

Proof.

Let us prove that 1) implies 2). Suppose that $f' : U \to l_{\sigma_2}(E_\xi, F_\eta)$ is continuous at $x_0$. Pick any $\beta$ and let $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that

$$\forall x \in U : \|x-x_0\| < 2\delta_1 \quad \forall h \in C_2 \quad \text{the following inequality holds:}$$

$$q_\beta([f'(x)-f'(x_0)]h) \leq \varepsilon. \quad (1.1.7)$$

On the other hand,

$$\forall (\xi_n) \in C_0(R^+) \quad \xi_n C_1 \subset \delta_1 B_1(E) \quad \text{and} \quad \xi_n C_2 \subset \delta_1 B_1(E) \quad \text{for} \quad n \text{ large enough.}$$

Consequently, If $\{x_n\} \subset U$, $x_n \to x_0$, $t_p \to 0$, $t_p \neq 0$ and $x_n + t_p h \in U$, then it follows from theorem 8 and (1.1.7) that for $n, p$ large enough

$$\sup_{h \in C_2} q_\beta \left[ t_p^{-1}[f(x_n+t_p h)-f(x_n)]-f'(x_0)h \right] \leq \sup_{h \in C_2} \sup_{0 \leq \rho \leq 1} q_\beta \left[ t_p^{-1}[f'(x_n+\rho t_p h)-f'(x_0)]t_p h \right] \leq \varepsilon,$$

i.e. the assumption 2) holds. Let us prove now that 2) implies 1). Suppose the contrary, then the assumption 2) holds and

$$\exists \varepsilon > 0 \exists \beta \exists C_2 \subset \sigma_2 \quad \forall n \in N \exists x_n \in U$$

$$\exists h_n \in C_2 \quad \|x_n-x_0\| \leq n^{-2} \quad \text{and} \quad q_\beta([f'(x_n)-f'(x_0)]h_n) > \varepsilon. \quad (1.1.8)$$

Since $\|n(x_n-x_0)\| \to 0$ and $C(E) \subset \sigma_1$, then $C_1 \equiv \{n(x_n-x_0) | n \in N\} \in \sigma_1$. Let $t_p \neq 0$ such that $t_p \to 0$. Since $x_n-x_0 \in \frac{1}{n} C_1$, then it follows that

$$q_\beta(t_p^{-1}[f(x_n+t_p h_n)-f(x_n)]-f'(x_0)h_n) < \varepsilon. \quad \text{Tending} \quad p \to \infty, \quad \text{we obtain}$$

$$q_\beta([f'(x_n)-f'(x_0)]h_n) \leq \varepsilon, \quad (1.1.9)$$

a contradiction with the assertion (1.1.8).
1.2 Lipschitzian Mappings and Implicit Function Theorem in Topological Vector Spaces

Suppose that all conditions of the above paragraph are satisfied.

**Definition 1.11** A mapping \( f : U \subset E_\xi \to F_\eta \) is said to be sequentially locally \((\sigma_2, \tau_2)\) lipschitzian at \( x_0 \in U \), if \( \forall C_2 \in \sigma_2 \forall \{h_n\} \subset C_2 \forall (t_n) \in C_0(R) : t_n \neq 0 \) and \( x_0 + t_n h_n \in U \quad \{t_n^{-1}[f(x_0 + t_n h_n) - f(x_0)]\} \in \tau_2 \).

**Definition 1.12** A mapping \( f : U \subset E_\xi \to F_\eta \) is said to be strongly sequentially locally \((\sigma_1 \sigma_2, \tau_2)\) lipschitzian at \( x_0 \in U \) if

\[
\forall (\delta_n) \in C_0(R) : \delta_n > 0 \forall C_1 \in \sigma_1 \forall C_2 \in \sigma_2 \forall \{x_n\} \subset U : x_n - x_0 \in \delta_n C_1 \forall \{h_n\} \subset C_2 \forall \{t_n\} : |t_n| < \delta_n, t_n \neq 0, x_n + t_n h_n \in U \quad \{t_n^{-1}[f(x_n + t_n h_n) - f(x_n)]\} \in \tau_2. (1.2.1)
\]

**Definition 1.13** A mapping \( f : U \subset E_\xi \to F_\eta \) is said to be uniformly strongly sequentially locally \((\sigma_1 \sigma_2, \tau_2)\) lipschitzian at \( x_0 \in U \) if

\[
\forall C_1 \in \sigma_1 \forall C_2 \in \sigma_2 \forall \{x_n\} \subset U : x_n - x_0 \to C_1 x_0 \forall h \in C_2 \forall \{t_p\} : t_p \to 0, t_p \neq 0, x_n + t_p h \in U \quad \{t_p^{-1}[f(x_n + t_p h) - f(x_n)]\} \in \tau_2.
\]

It’s clear that if \( r : U \subset E_\xi \to F_\eta \) is strongly \( \sigma_1 \sigma_2 \)-small at \( x_0 \in U \), then \( r : U \subset E_\xi \to F_\eta \) is strongly sequentially locally \((\sigma_1 \sigma_2, C(F_\eta))\) lipschitzian at \( x_0 \).

**Proposition 1.14** Suppose that \( C(E_\xi) \subset \sigma_1, C(E_\xi) \subset \sigma_2, f : U \subset E_\xi \to F_\eta \) is strongly sequentially locally \((\sigma_1 \sigma_2, B(F_\eta))\) lipschitzian at \( x_0 \in U \). Then \( f : U \subset E_\xi \to F_\eta \) is strongly sequentially locally \((B(E_\xi), \eta, B(E_\xi), \eta))\) lipschitzian at \( x_0 \).

**Proof.**
Let \( (\delta_n) \in C_0(R) : \delta_n > 0, \theta_1, \theta_2 \in B(E_\xi), \{x_n\} \subset U : x_n - x_0 \in \delta_n \theta_1, \{h_n\} \subset \theta_2, \{t_n\} : |t_n| < \delta_n, t_n \neq 0, x_n + t_n h_n \in U, \chi = \{z_n \equiv t_n^{-1}[f(x_n + t_n h_n) - f(x_n)]\} \). Assume by contradiction that \( \chi \notin B(F_\eta) \). We can suppose without loss of generality that \( \exists \beta q_\beta(z_n) > n^2 \). Set \( \varepsilon_n = \max(\sqrt{\delta_n}, n^{-1}), t'_n = \varepsilon_n^{-1} t_n, h'_n = \varepsilon_n h_n \).

Then \( \varepsilon_n \to 0, t'_n \to 0, \varepsilon_n^{-1}(x_n - x_0) \in \varepsilon_n \theta_1 \). Consequently, \( \varepsilon_n^{-1}(x_n - x_0) \to 0, h'_n \to 0 \) with respect to \( \xi \). This implies that \( C_1 \equiv \{\varepsilon_n^{-1}(x_n - x_0)\} \in C(E_\xi) \subset \sigma_1 \) and \( C_2 \equiv \{h'_n\} \in C(E_\xi) \subset \sigma_2 \). In this case, \( x_n - x_0 \in \varepsilon_n C_1, h'_n \in C_2 \). Thus \( q_\beta(z_n) = \varepsilon_n^{-1} q_\beta(\varepsilon_n^{-1}[f(x_n + t'_n h'_n) - f(x_n)]) \leq n M \) for some \( M > 0 \), but this contradicts the fact that \( q_\beta(z_n) > n^2 \).
Theorem 1.15 Let \( G \) a vector space, \( \eta_1 = \{ r_\gamma \} \) a collection of semi-norms on \( G, A_1 \in l(E_\xi, F_\eta), A_2 \in l(F_\eta, G_\eta) \). Suppose that \( f : U \subset E_\xi \to F_\eta \) is sequentially strongly \( \sigma_1 \sigma_2 \) equivalent to \( A_1 \) at \( x_0 \in U \), \( g : f(U) \subset F_\eta \to G_\eta \) is sequentially strongly \( \tau_1 \tau_2 \) equivalent to \( A_2 \) at \( f(x_0) \in f(U) \). Assume also that \( A_2 \in L(F_\eta, G_\eta) \), \( f \) is locally sequentially \( (\sigma_1, \tau_1) \) Lipschitzian at \( x_0 \), \( f \) is uniformly locally strongly sequentially \( (\sigma_1 \sigma_2, \tau_2) \) Lipschitzian at \( x_0 \). Then \( g \circ f : U \subset E_\xi \to G_\eta \) is sequentially strongly \( \sigma_1 \sigma_2 \) equivalent to \( A_2 \circ A_1 \) at \( x_0 \).

Proof.
Set \( R(x, h) = f(x + h) - f(x) - A_1 h \) and \( S(y, k) = g(y + k) - g(y) - A_2 k \). Then
\[
g(f(x + h)) - g(f(x)) = A_2 [f(x + h) - f(x)] + S(f(x), f(x + h) - f(x)) = A_2 A_1 h + A_2 R(x, h) + S(f(x), f(x + h) - f(x)).
\]

Let \( C_1 \in \sigma_1, C_2 \in \sigma_2, \{ x_n \} \subset U : x_n \to C_1, x_0 \in C_2, \{ t_p \} : t_p \to 0, t_p \neq 0, x_n + t_p h \in U \).

Then there exists \( (\delta_n) \in C_0(R) \) such that \( \delta_n > 0 \) and \( x_n - x_0 \in \delta_n C_1 \). Therefore, \( D_1 \equiv \{ t_p^{-1} [f(x_n) - f(x_0)] \} \in \tau_1, D \equiv \{ t_p^{-1} [f(x_n + t_p h) - f(x_n)] \} \in \tau_2 \). Consequently, \( f(x_n) - f(x_0) \in \delta_n D_1 \). Thus
\[
\sup_{h \in C_2, x_n + t_p h \in U} r_{\gamma} \left[ t_p^{-1} A_2 R(x_n, t_p h) + t_p^{-1} S(f(x_n), t_p [t_p^{-1} (f(x_n + t_p h) - f(x_n))] \right] \to 0.
\]

Lemma 1.16 Let \( V \) a set, \( 0 < q < 1, f : U \times V \to E, \tilde{A} : E \to E \) a linear operator. Suppose that the map \( x \mapsto \tilde{A}f(x, y) \) \( x \) is \( q \)-lipschitzian on \( U \) for every \( y \in V \). Assume also that there exists \( g : V \to U \) such that
\[
f(g(y), y) = 0 \ \forall y \in V.
\]

Put \( \varphi(x, y) = \tilde{A}f(x, y) \). Then
1) \( \forall y' \in V \forall x \in U \forall h \in U - x \ | h |_1 \leq (1-q)^{-1} | \varphi(x+h, y') - \varphi(x, y') |_1 \); (1.2.3)
2) \( \forall y, \tilde{y} \in V : | g(y) - g(\tilde{y}) |_1 \leq (1-q)^{-1} | \varphi(\tilde{y}, y) |_1 \); (1.2.4)
3) such a \( g \) is unique. (1.2.5)

Proof.
Since \( \| \varphi(x+h, y) - \varphi(x, y) - h \|_1 \leq q \| h \|_1 \), then \( \| h \|_1 - \| \varphi(x+h, y) - \varphi(x, y) \|_1 \leq q \| h \|_1 \).

Consequently, \( \| h \|_1 \leq (1-q)^{-1} \| \varphi(x+h, y) - \varphi(x, y) \|_1 \).

2) Let us prove now the second assumption. Set \( x = g(\tilde{y}), y' = y \) and \( h = g(y) - g(\tilde{y}) \).
It follows from (1.2.3) that
\[ \|g(y) - g(\hat{y})\|_1 \leq (1 - q)^{-1}\|\varphi(g(y), y) - \varphi(g(\hat{y}), y)\|_1 = (1 - q)^{-1}\|\varphi(g(\hat{y}), y)\|_1. \]

3) Suppose that there exists \( g_1 : V \to U \) such that \( f(g_1(y), y) = 0 \quad \forall y \in V \). Put \( h = g_1(y) - g(y), y' = y \). Then \( \|g_1(y) - g(y)\|_1 \leq (1 - q)^{-1}\|\varphi(g_1(y), y) - \varphi(g(y), y)\|_1 = 0 \). Hence, \( g_1(y) = g(y) \). Thus we achieve the proof.

**Theorem 1.17** Assume that all the conditions of lemma 16 hold. Let \( y_0 \in V, x_0 = g(y_0) \). Suppose that \( \tilde{A} \in l(E, E) \). Then

1) if \( f(x_0, y_n) \to_{\| \|} f(x_0, y_0) \) for some \( \{y_n\} \subset V \), then \( g(y_n) \to_{\| \|} g(y_0) \);

2) if \( V \subset F \) and \( f \) is strongly sequentially \((\sigma_1 \times \tau_1 \{0\} \times \tau_2, \mathcal{B}(E))\) lipschitzian at \((x_0, y_0)\),

\[ \mathcal{C}(E) \subset \sigma_1 \text{ and } f \text{ is sequentially } (\{0\} \times \tau_1, \mathcal{B}(E)) \text{ lipschitzian at } (x_0, y_0), \]

then \( g \) is strongly sequentially \((\tau_1 \times \tau_2, \mathcal{B}(E))\) lipschitzian at \( y_0 \);

3) If all the hypothesis of 2) hold and if \( \psi(x, y) = x - \varphi(x, y) \) is strongly sequentially

\[ (\sigma_1 \times \tau_1 \mathcal{B}(E) \times \tau_2, \sigma_2) \text{ lipschitzian at } (x_0, y_0), \]

then \( g \) is strongly sequentially \((\tau_1 \times \tau_2, \sigma_2)\) lipschitzian at \( y_0 \).

**Proof.**

The assertion 1) follows from (1.2.4). In fact, we have

\[ \|g(y_n) - g(y_0)\|_1 \leq (1 - q)^{-1}\|\varphi(g(y_0), y_n) - \varphi(g(y_0), y_0)\|_1. \]

Consequently, tending \( n \to \infty \), we deduce the result.

2) Let \( (\delta_n) \in \mathcal{C}_0(R) : \delta_n > 0, D_1 \in \tau_1, D_2 \in \tau_2, \{y_n\} \subset V : y_n \to y_0 \in \delta_n D_1, \{k_n\} \subset D_2, \{t_n\} : |t_n| < \delta_n, t_n \neq 0, y_n + t_n k_n \in V \). It follows from (1.2.4) that

\[ \delta_n^{-1}\|g(y_n) - g(y_0)\|_1 \leq (1 - q)^{-1}\|\delta_n^{-1}[\varphi(g(y_0), y_0 + \delta_n(\delta_n^{-1}(y_n - y_0))] - \varphi(g(y_0), y_0)\|_1 \]

\[ \leq (1 - q)^{-1}M_1 \text{ for some } M_1 > 0. \]  

(1.2.6)

Set \( \varepsilon_n = \sqrt{\delta_n} \). Then, it follows from (1.2.6) that \( \|\varepsilon_n^{-1}[g(y_n) - g(y_0)]\|_1 \leq \varepsilon_n(1 - q)^{-1}M_1 \to 0 \). Therefore, \( C_1 \equiv \{\varepsilon_n^{-1}[g(y_n) - g(y_0)]\} \in \mathcal{C}(E) \subset \sigma_1 \). In this case,

\[ g(y_n) - g(y_0) \in \varepsilon_n C_1 \text{ and thus} \]

(1.2.7)

\[ \|t_n^{-1}[g(y_n + t_n k_n) - g(y_n)]\|_1 \leq (1 - q)^{-1}\|t_n^{-1}[\varphi(g(y_n), y_n + t_n k_n) - \varphi(g(y_n), y_n)]\|_1 \]

\[ \leq (1 - q)^{-1}M_2 \text{ for some } M_2 > 0. \]  

(1.2.8)
The assertion 3) follows from (1.2.7) and (1.2.8). In fact, we have
\[ \{t_n^{-1}[g(y_n + t_n k_n) - g(y_n)]\} = \{t_n^{-1}[g(y_n + t_n k_n) - g(y_n) - \varphi(g(y_n + t_n k_n), y_n + t_n k_n) + \varphi(g(y_n), y_n)]\} = \{t_n^{-1}[\psi(g(y_n + t_n k_n), y_n + t_n k_n) - \psi(g(y_n), y_n)]\} \in \sigma_2. \]

**Theorem 1.18** Let \( V \subset F, 0 < q < 1, f : U \times V \to E, \tilde{A} \in \mathcal{L}(E_\xi, E_\xi) \). Suppose that the map \( x \mapsto \tilde{A}f(x, y) - x \) is \( q \)-lipschitzian on \( U \) for every \( y \in V \). Assume also that there exists \( g : V \to U \) such that \( f(g(y), y) = 0 \) \( \forall y \in V \). Put \( \varphi(x, y) = \tilde{A}f(x, y) \). Let \( y_0 \in V, x_0 = g(y_0), A : E \times F \to E, A_1 = A|_{E \times \{0\}}, A_2 = A|_{\{0\} \times F} \). Suppose also that \( U \) is open in \((E, \|\cdot\|_1)\), \( A_1 \) is one to one, \( f \) is sequentially \((\{0\} \times \tau_1, \mathcal{B}(E)) \) lipschitzian at \((x_0, y_0)\) and \( \mathcal{C}(E) \subset \sigma_1 \). Then

1) If \( \mathcal{B}(E_\xi) \subset \mathcal{B}(E) \),

\[ A_1^{-1} \in \mathcal{L}(E_\xi, E_\xi), \] (1.2.9)

\[ A_1^{-1}A_2 \tau_2 \subset \mathcal{B}(E), \] (1.2.10)

and

\[ f : U \times V \subset E_\xi \times F_\eta \to E_\xi \] is sequentially strongly \( (\sigma_1 \times \tau_1)(\mathcal{B}(E) \times \tau_2) \) equivalent to \( A \) at \((x_0, y_0)\), then

\[ g : V \subset F_\eta \to E_\xi \] is sequentially strongly \( \tau_1 \tau_2 \) equivalent to \( -A_1^{-1} \circ A_2 \) at \( y_0 \).

2) If \( A_1^{-1}A_2 \tau_2 \subset \sigma_2 \),

\[ f : U \times V \subset E_\xi \times F_\eta \to E \] is sequentially strongly \( (\sigma_1 \times \tau_1)(\sigma_2 \times \tau_2) \) equivalent to \( A \) at \((x_0, y_0)\), then \( g : V \subset F_\eta \to E \) is sequentially strongly \( \tau_1 \tau_2 \) equivalent to \( -A_1^{-1} \circ A_2 \) at \( y_0 \).

**Proof.**

Let

\[ D_1 \in \tau_1, D_2 \in \tau_2, \{y_n\} \subset V : y_n \to D_1 y_0, k \in D_2, \]

\[ \{t_p\} \subset R : 0 < |t_p|, t_p \to 0, y_n + t_p k \in V. \]
It follows by (proposition 5,[?]) that $A \in \mathcal{L}(E,E)$. Set
\begin{equation}
R(x, y, h, k) = f(x+h, y+k) - f(x, y) - A_1 h - A_2 k, \quad S(y, k) = g(y+k) - g(y) + A_1^{-1} A_2 k. \tag{1.2.13}
\end{equation}

Let us remark that from (1.2.2), it follows that
\[0 = f(g(y+k), y+k) - f(g(y), y) = A_1[g(y+k) - g(y)] + A_2 k + R(g(y), y, g(y+k) - g(y), k).\]

Consequently, $S(y, k) = -A_1^{-1} R(g(y), y, g(y+k) - g(y), k)$. \tag{1.2.14}

On the other hand, it follows from (1.2.3) that \(\forall y, k : y+k \in V\) and $g(y) - A_1^{-1} A_2 k \in U$
\begin{align*}
\|S(y, k)\|_1 & \leq (1-q)^{-1} \| \tilde{A} [f(g(y) - A_1^{-1} A_2 k, y + k) - f(g(y) + k, y + k)]\|_1 \\
& = (1-q)^{-1} \| \tilde{A} [f(g(y) - A_1^{-1} A_2 k, y + k) - f(g(y), y) - A_1 (-A_1^{-1} A_2 k) - A_2 k]\|_1 \\
& = (1-q)^{-1} \| \tilde{A} [R(g(y), y, -A_1^{-1} A_2 k, k)]\|_1. \tag{1.2.15}
\end{align*}

Therefore, using (1.2.6) and the fact that $U$ is an open set, we deduce that $g(y_n) - A_1^{-1} A_2 t_p k \in U$ for $n, p$ large enough. Hence we get
\begin{equation}
\|t_p^{-1} S(y_n, t_p k)\|_1 \leq \| \tilde{A} [t_p^{-1} R(g(y_n), y_n, t_p (-A_1^{-1} A_2 k), t_p k)]\|_1. \tag{1.2.16}
\end{equation}

To prove the assertion 1), it suffice to use (1.2.11),(1.2.13),(1.2.16),(1.2.7),(1.1.4),(1.2.9).

Indeed, we have
\begin{equation}
\|t_p^{-1} [g(y_n + t_p k) - g(y_n)]\|_1 \leq \|A_1^{-1} A_2 k\|_1 + \|t_p^{-1} S(y_n, t_p k)\|_1 \tag{1.2.17}
\end{equation}
\begin{equation*}
\leq M \text{ for some } M > 0.
\end{equation*}

On the other hand, it follows from (1.2.14) that
\begin{equation}
t_p^{-1} S(y_n, t_p k) = -A_1^{-1} t_p^{-1} R(g(y_n), y_n, t_p [t_p^{-1} (g(y_n + t_p k) - g(y_n))], t_p k). \tag{1.2.18}
\end{equation}

Consequently, by (1.2.7),(1.2.17),(1.1.4),(1.2.10), we conclude that
\[\forall \alpha \sup_{k \in D_2 ; y_n + t_p k \in V} p_\alpha [t_p^{-1} S(y_n, t_p k)] \to 0.\]

Hence we get 1).

For the assertion 2), using (1.2.16),(1.2.7),(1.2.12),(1.1.4), it follows then that
\[\sup_{k \in D_2 ; y_n + t_p k \in V} \|t_p^{-1} S(y_n, t_p k)\|_1 \to 0.\] Thus we achieve the proof.
References


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