On Certain New Gronwall-Bellman Type Integral Inequalities of Two Independent Variables

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Abstract. The aim of the present paper is to establish some new integral inequalities of Gronwall type involving functions of two independent variables which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in the qualitative theory of certain partial differential equations.

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1. Introduction

Closely related to the foregoing first-order ordinary differential operators is the following result of Bellman [10]: If the functions $g(t)$ and $u(t)$ are nonnegative for $t \geq 0$, and if $c \geq 0$, then the inequality

$$u(t) \leq c + \int_0^t g(s)u(s)\,ds, \quad t \geq 0$$

Implies that

$$u(t) \leq c \exp\left(\int_0^t g(s)\,ds\right), \quad t \geq 0$$
This result may be established either directly or by means of the technique of first-order linear differential equations (please, see Gronwall [13] and Giuliano [8]). Various applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [9]. Numerous applications to existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [14], Bihari [7], and Langenhop [3]. Several authors generalized inequalities of Bellman type (sometimes, inequalities of this type were called “Gronwall-Bellman inequalities” or “Inequalities of Gronwall type”) to the case of functions of two or more variables [1-14]. Of course, such results have application in the theory of partial differential equations and Volterra integral equations. In the book by Beckenbach and Bellman [6] the following unpublished Wendroff result was given: If

\[
 u(x, y) \leq a(x) + b(y) + \int_0^y \int_0^x v(r, s) u(r, s) dr ds,
\]

where \(a(x), b(y) > 0\), \(a'(x), b'(y) \geq 0\) and \(u(x, y), v(x, y) \geq 0\), then

\[
 u(x, y) \leq E(x, y) \exp \left\{ \int_0^x \int_0^y v(r, s) dr ds \right\},
\]

where

\[
 E(x, y) \leq \frac{(a(0) + b(y))(a(x) + b(0))}{a(0) + b(0)},
\]

2. Main Results

**Theorem 2.1:** Let \(\phi(x, y), A(x, y), B(x, y)\) and \(H(x, y)\) be realvalued nonnegative ,nondecreasing continuous functions defined for \((x, y) \in \mathbb{R}_+, c > 0\) and suppose

\[
 \phi(x, y) \leq c + \int_0^x A(s, y) \phi(s, y) ds + \int_0^y B(x, t) \phi(x, t) dt + \int_0^y H(s, t) \phi(s, t) ds dt, \quad (x, y) \in \mathbb{R}_+
\]

Then

\[
 \phi(x, y) \leq cQ(x, y)E(x, y) \exp \left[ \int_0^y H(s, t) Q(s, t) E(s, t) ds dt \right]
\]

where

\[
 Q(x, y) = \exp \left[ \int_0^y B(x, t) E(x, t) dt \right]
\]

and

\[
 E(x, y) = \exp \left[ \int_0^x A(s, y) ds \right]
\]
Proof: Define

\[ m(x, y) = c + \int_0^x \int_0^y H(s, t) \phi(s, t) ds dt \]  
(2.4)

\[ m(x, 0) = m(0, y) = c, \quad m_x(x, 0) = m_y(0, y) = 0 \]  
(2.5)

By substituting (2.4) in (2.1), we get

\[ \phi(x, y) \leq m(x, y) + \int_0^x A(s, y) \phi(s, y) ds + \int_0^y B(x, t) \phi(x, t) dt \]  
(2.6)

Since \( m(x, y) \) is positive, nondecreasing continuous function, therefore

\[ \frac{\phi(x, y)}{m(x, y)} \leq 1 + \int_0^x A(s, y) \frac{\phi(s, y)}{m(s, y)} ds + \int_0^y B(x, t) \frac{\phi(x, t)}{m(x, t)} dt \]  
(2.7)

Let

\[ z(x, y) = 1 + \int_0^x A(s, y) \frac{\phi(s, y)}{m(s, y)} ds + \int_0^y B(x, t) \frac{\phi(x, t)}{m(x, t)} dt \]  
(2.8)

\[ z(0, 0) = 1 \]  
(2.9)

From (2.7) and (2.8), we observe that

\[ \frac{\phi(x, y)}{m(x, y)} \leq z(x, y) \]  
(2.10)

Also we notice from (2.8) and (2.10) that

\[ z(x, y) \leq 1 + \int_0^x A(s, y) z(s, y) ds + \int_0^y B(x, t) z(x, t) dt \]  
(2.11)

Now define

\[ b(x, y) = 1 + \int_0^y B(x, t) z(x, t) dt \]  
(2.12)

\[ b(x, 0) = 1 \]  
(2.13)

By substituting (2.12) in (2.11), we get

\[ z(x, y) \leq b(x, y) + \int_0^x A(s, y) z(s, y) ds \]  
(2.14)

Since \( b(x, y) \) is positive, monotonic nondecreasing continuous function, therefore

\[ \frac{z(x, y)}{b(x, y)} \leq 1 + \int_0^x A(s, y) \frac{z(s, y)}{b(s, y)} ds \]  
(2.15)

Let

\[ v(x, y) = 1 + \int_0^x A(s, y) \frac{z(s, y)}{b(s, y)} ds \]  
(2.16)

\[ v(0, y) = 1 \]  
(2.17)
From (2.15) and (2.16), we observe that
\[
\frac{z(x,y)}{b(x,y)} \leq v(x,y)
\] (2.18)

Differentiating (2.16) w.r.t. \( x \) and using (2.18), we have
\[
\frac{v_x(x,y)}{v(x,y)} \leq A(x,y)
\] (2.19)

Keeping \( y \) fixed, set \( x = s \) and integrate from 0 to \( x \), we get
\[
v(x,y) \leq \exp \left[ \int_0^x A(s,y)ds \right] = E(x,y)
\] (2.20)

From (2.18) and (2.20), we have
\[
z(x,y) \leq b(x,y)E(x,y)
\] (2.21)

Differentiating (2.12) w.r.t. \( y \), we have
\[
b_y(x,y) = B(x,y)z(x,y)
\] (2.22)

By substituting (2.21) in (2.22) and keeping \( x \) fixed, set \( y = t \) and integrate from 0 to \( y \), we observe that
\[
b(x,y) \leq \exp \left[ \int_0^y B(x,t)E(x,t)dt \right] = Q(x,y)
\] (2.23)

From (2.22) and (2.23), we observe that
\[
z(x,y) \leq E(x,y)Q(x,y)
\] (2.24)

By substituting from (2.24) in (2.10), we get
\[
\phi(x,y) \leq Q(x,y)E(x,y)m(x,y)
\] (2.25)

Differentiating (2.4) w.r.t. \( x \) and \( y \) and using (2.25), we have
\[
\frac{m_y(x,y)}{m(x,y)} \leq H(x,y)Q(x,y)E(x,y)
\]

\[
\frac{m_y(x,y)m(x,y) - m(x,y)m_y(x,y)}{m^2(x,y)} \leq H(x,y)Q(x,y)E(x,y)
\]

\[
\frac{\partial}{\partial y} \left[ \frac{m(x,y)}{m(x,y)} \right] \leq H(x,y)Q(x,y)E(x,y)
\] (2.26)

By keeping \( y \) fixed, set \( x = s \), integrate from 0 to \( x \) in (2.26) and using (2.5) and again keeping \( x \) fixed, set \( y = t \), integrate from 0 to \( y \) in the resulting inequality with using (2.5), we obtain
\[
m(x,y) \leq c \exp \left[ \int_0^x \int_0^y H(s,t)Q(s,t)E(s,t)dsdt \right]
\]
Substituting the above bound in (2.25), we have

$$\phi(x, y) \leq cQ(x, y)E(x, y)\exp\left[\int_0^x \int_0^y H(s, t)Q(s, t)E(s, t)dsdt\right]$$

Theorem 2.2: Let $\phi(x, y)$, $A(x, y)$, $B(x, y)$ and $H(x, y)$ be defined as in Theorem 2.1, $c > 0$ and $0 < p < 1$ are constants and suppose

$$\phi(x, y) \leq c + \int_0^x A(s, y)\phi(s, y)ds + \int_0^y B(x, t)\phi(x, t)dt + \int_0^x \int_0^y H(s, t)\phi^p(s, t)dsdt, \quad (x, y) \in R_+$$

then

$$\phi(x, y) \leq Q(x, y)E(x, y)\left[c^{1-p} + (1 - p)\int_0^x \int_0^y H(s, t)Q^p(s, t)E^p(s, t)dsdt\right]^{\frac{1}{1-p}}, \quad (x, y) \in R_+$$

(2.28)

where $Q(x, y)$ and $E(x, y)$ are defined as in (2.2) and (2.3) respectively.

Proof: Define

$$m(x, y) = c + \int_0^x \int_0^y H(s, t)\phi^p(s, t)dsdt \quad (2.29)$$

$$m(x, 0) = m(0, y) = c, \quad m_y(x, 0) = m_y(0, y) = 0 \quad (2.30)$$

By substituting (2.29) in (2.27), we get

$$\phi(x, y) \leq m(x, y) + \int_0^x A(s, y)\phi(s, y)ds + \int_0^y B(x, t)\phi(x, t)dt$$

By following the same steps from (2.6)-(2.25) from theorem 2.1, we have

$$\phi(x, y) \leq Q(x, y)E(x, y)m(x, y) \quad (2.31)$$

Differentiating (2.29) w.r.t. $x$ and $y$ and using (2.31), we obtain

$$\frac{m_y(x, y)}{m^p(x, y)} \leq H(x, y)Q^p(x, y)E^p(x, y)$$

(2.32)
By keeping \( y \) fixed, set \( x = s \), integrate from 0 to \( x \) in (2.32) and using (2.30) and again keeping \( x \) fixed, set \( y = t \), integrate from 0 to \( y \) in the resulting inequality with using (2.30), we obtain

\[
m(x, y) \leq \left[ c^{1-p} + (1 - p) \int_0^x \int_0^y H(s, t)Q^p(s, t)E^p(s, t)dsdt \right]^{\frac{1}{1-p}}
\]

Substituting the above bound in (2.31) we get the desired inequality (2.28).

**Theorem 2.3:** Let \( \phi(x, y), A(x, y), B(x, y) \) and \( H(x, y) \) be defined as in Theorem 2.1, \( c > 0 \) and \( 0 < p < 1 \) are constants and suppose

\[
\phi(x, y) \leq c + \int_0^x A(s, y)\phi(s, y)ds + \int_0^y B(x, t)\phi^p(x, t)dt + \int_0^x \int_0^y H(s, t)\phi^p(s, t)dsdt, \quad (x, y) \in \mathbb{R}_+
\]

(2.33)

then

\[
\phi(x, y) \leq Q_1(x, y)E(x, y)\left[ c^{1-p} + (1 - p) \int_0^x \int_0^y H(s, t)Q^p(s, t)E^p(s, t)dsdt \right]^{\frac{1}{1-p}}, \quad (x, y) \in \mathbb{R}_+
\]

(2.34)

where \( E(x, y) \) is defined as in (2.3) and

\[
Q_1(x, y) = \left[ 1 + (1 - p) \int_0^y B(x, t)E^p(x, t)m^{p-1}(x, t)dt \right]^{\frac{1}{1-p}}
\]

(2.35)

**Proof:** Define

\[
m(x, y) = c + \int_0^x \int_0^y H(s, t)\phi^p(s, t)dsdt
\]

(2.36)

\[
m(x, 0) = m(0, y) = c, \quad m_x(0, 0) = m_y(0, y) = 0
\]

(2.37)

By substituting (2.36) in (2.33), we get

\[
\phi(x, y) \leq m(x, y) + \int_0^x A(s, y)\phi(s, y)ds + \int_0^y B(x, t)\phi^p(x, t)dt
\]

(2.38)

Since \( m(x, y) \) is positive, nondecreasing continuous function, therefore

\[
\frac{\phi(x, y)}{m(x, y)} \leq 1 + \int_0^x A(s, y)\frac{\phi(s, y)}{m(s, y)}ds + \int_0^y B(x, t)\frac{\phi^p(x, t)}{m(x, t)}dt
\]

(2.39)
Let 
\[ z(x, y) = 1 + \int_0^x A(s, y) \frac{\phi(s, y)}{m(s, y)} \, ds + \int_0^y B(x, t) \frac{\phi^p(x, t)}{m(x, t)} \, dt \]  
(2.40)
\[ z(0, 0) = 1 \]  
(2.41)

From (2.39) and (2.40), we observe that 
\[ \frac{\phi(x, y)}{m(x, y)} \leq z(x, y) \]  
(2.42)

Also we notice from (2.40) and (2.42) that 
\[ z(x, y) \leq 1 + \int_0^x A(s, y)z(s, y)\, ds + \int_0^y B(x, t)m^{p-1}(x, t)z^p(x, t)\, dt \]  
(2.43)

Now define 
\[ b(x, y) = 1 + \int_0^y B(x, t)m^{p-1}(x, t)z^p(x, t)\, dt \]  
(2.44)
\[ b(x, 0) = 1 \]  
(2.45)

By substituting (2.44) in (2.43), we get 
\[ z(x, y) \leq b(x, y) + \int_0^x A(s, y)z(s, y)\, ds \]  
(2.46)

By following the same steps from (2.14)-(2.21) in (2.46), we get 
\[ z(x, y) \leq b(x, y)E(x, y) \]  
(2.47)

By differentiating (2.44) w.r.t \( y \), we have 
\[ b_y(x, y) = B(x, y)m^{p-1}(x, y)z^p(x, y) \]  
(2.48)

By substituting (2.47) in (2.48) and keeping \( x \) fixed, set \( y = t \) and integrate from 0 to \( y \) and using (2.45), we observe that
\[ b(x, y) \leq \left[ 1 + (1-p) \int_0^y B(x, t)E^p(x, t)m^{p-1}(x, t)\, dt \right]^{\frac{1}{1-p}} = Q_1(x, y) \]  
(2.49)

From (2.47) and (2.49), we observe that 
\[ z(x, y) \leq E(x, y)Q_1(x, y) \]  
(2.50)

By substituting from (2.50) in (2.42), we get 
\[ \phi(x, y) \leq Q_1(x, y)E(x, y)m(x, y) \]  
(2.51)

Differentiating (2.36) w.r.t. \( x \) and \( y \) and using (2.50), we obtain
\[ \frac{m_{xy}(x, y)}{m^p(x, y)} \leq H(x, y)Q^p(x, y)E^p(x, y) \]
\[ \frac{m_{xy}(x, y)m(x, y)}{m^{p+1}(x, y)} - \frac{m_{x}(x, y)m_{y}(x, y)}{m^{p+1}(x, y)} \leq H(x, y)Q^p(x, y)E^p(x, y) \]
\[
\frac{\partial}{\partial y} \left[ \frac{m_y(x,y)}{m(x,y)} \right] \leq H(x,y)Q^p(s,t)E^p(s,t)
\]  
(2.52)

By keeping \( y \) fixed, set \( x = s \), integrate from 0 to \( x \) in (2.52) and using (2.37) and again keeping \( x \) fixed, set \( y = t \), integrate from 0 to \( y \) in the resulting inequality with using (2.37), we obtain

\[
m(x,y) \leq \left[ c^{1-p} + (1-p) \int_0^x H(s,t)Q^p(s,t)E^p(s,t)dsdt \right]^{1-1-p}
\]

Substituting the above bound in (2.51) we get the desired inequality (2.34).

**Theorem 2.4:** Let \( \phi(x,y), A(x,y), B(x,y) \) and \( H(x,y) \) be defined as in Theorem 2.1, \( c > 0 \) and \( 0 < p < 1 \) are constants and suppose

\[
\phi(x,y) \leq c + \int_0^x A(s,y)\phi^p(s,y)ds + \int_0^y B(x,t)\phi(x,t)dt + \int_0^x \int_0^y H(s,t)\phi^p(s,t)dsdt, \quad (x,y) \in \mathbb{R}_+
\]

(2.53)

then

\[
\phi(x,y) \leq Q_1(x,y)E_1(x,y) \left[ c^{1-p} + (1-p) \int_0^x H(s,t)Q^p(s,t)E_1^p(s,t)dsdt \right]^{1-1-p}, \quad (x,y) \in \mathbb{R}_+
\]

(2.54)

where

\[
Q_1(x,y) = \exp \left[ \int_0^y B(x,t)dt \right]
\]

(2.55)

and

\[
E_1(x,y) = \left[ 1 + (1-p) \int_0^x A(s,y)Q^p(s,y)m^{p-1}(s,y)dt \right]^{1-1-p}
\]

(2.56)

**Proof:** Define

\[
m(x,y) = c + \int_0^x \int_0^y H(s,t)\phi^p(s,t)dsdt
\]

(2.57)

\[
m(0,y) = m(0,y) = c, \quad m_x(0,y) = m_y(0,y) = 0
\]

(2.58)

By substituting (2.57) in (2.53), we get

\[
\phi(x,y) \leq m(x,y) + \int_0^x A(s,y)\phi^p(s,y)ds + \int_0^y B(x,t)\phi(x,t)dt
\]

(2.59)
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Since \( m(x,y) \) is positive, nondecreasing continuous function, therefore

\[
\frac{\phi(x,y)}{m(x,y)} \leq 1 + \int_0^x A(s,y) \frac{\phi^p(s,y)}{m(s,y)} ds + \int_y^x B(x,t) \frac{\phi(x,t)}{m(x,t)} dt \tag{2.60}
\]

Let

\[
z(x,y) = 1 + \int_0^x A(s,y) \frac{\phi^p(s,y)}{m(s,y)} ds + \int_y^x B(x,t) \frac{\phi(x,t)}{m(x,t)} dt \tag{2.61}
\]

\[
z(0,0) = 1 \tag{2.62}
\]

From (2.60) and (2.61), we observe that

\[
\frac{\phi(x,y)}{m(x,y)} \leq z(x,y) \tag{2.63}
\]

Also we notice from (2.61) and (2.63) that

\[
z(x,y) \leq 1 + \int_0^x A(s,y)m^{p-1}(s,y)z^p(s,y)ds + \int_y^x B(x,t)z(x,t)dt \tag{2.64}
\]

Now define

\[
b(x,y) = 1 + \int_0^x A(s,y)m^{p-1}(s,y)z^p(s,y)dt \tag{2.65}
\]

\[
b(x,0) = 1 \tag{2.66}
\]

By substituting (2.44) in (2.43), we get

\[
z(x,y) \leq b(x,y) + \int_0^y B(x,t)z(x,t)ds \tag{2.67}
\]

Since \( b(x,y) \) is positive, monotonic nondecreasing continuous function, therefore

\[
\frac{z(x,y)}{b(x,y)} \leq 1 + \int_0^y B(x,t) \frac{z(x,t)}{b(x,t)} ds \tag{2.68}
\]

Let

\[
v(x,y) = 1 + \int_0^y B(x,t) \frac{z(x,t)}{b(x,t)} ds \tag{2.69}
\]

\[
v(0,0) = 1 \tag{2.70}
\]

From (2.68) and (2.69), we have

\[
\frac{z(x,y)}{b(x,y)} \leq v(x,y) \tag{2.71}
\]

Differentiating (2.69) w.r.t. \( y \) and using (2.71), we have

\[
\frac{v_y(x,y)}{v(x,y)} \leq B(x,y) \tag{2.72}
\]

Keeping \( x \) fixed, set \( y = t \) and integrate from 0 to \( y \), we get
\begin{equation}
\nu(x,y) \leq \exp \left[ \int_0^y B(x,t) \, ds \right] = Q_1(x,y)
\end{equation}

From (2.71) and (2.73), we have
\begin{equation}
z(x,y) \leq b(x,y)E_1(x,y)
\end{equation}

By differentiating (2.65) w.r.t. \(x\), we have
\begin{equation}
b_s(x,y) \leq A(x,y)m^{p-1}(x,y)z^p(x,y)
\end{equation}

By substituting (2.74) in (2.75) and keeping \(y\) fixed, set \(x = s\) and integrate from 0 to \(x\) and using (2.66), we observe that
\begin{equation}
b(x,y) = \left[ 1 + (1 - p) \int_0^x A(s,y))E^p(s,y)m^{p-1}(s,y) \, dt \right]^\frac{1}{1-p} = E_1(x,y)
\end{equation}

From (2.74) and (2.76), we observe that
\begin{equation}
z(x,y) \leq E_1(x,y)Q_1(x,y)
\end{equation}

By substituting from (2.77) in (2.63), we get
\begin{equation}
\phi(x,y) \leq Q_1(x,y)E_1(x,y)m(x,y)
\end{equation}

Differentiating (2.57) w.r.t. \(x\) and \(y\) and using (2.78), we obtain
\begin{equation}
\frac{m_{xy}(x,y)}{m^p(x,y)} \leq H(x,y)Q^p(x,y)E^p(x,y)
\end{equation}

\begin{equation}
\frac{m_{xy}(x,y)m(x,y)}{m^{p+1}(x,y)} - \frac{m_s(x,y)m_y(x,y)}{m^{p+1}(x,y)} \leq H(x,y)Q^p(x,y)E^p(x,y)
\end{equation}

\begin{equation}
\frac{\partial}{\partial y} \left[ \frac{m_s(x,y)}{m^p(x,y)} \right] \leq H(x,y)Q^p(x,y)E^p(x,y)
\end{equation}

By keeping \(y\) fixed, set \(x = s\), integrate from 0 to \(x\) in (2.79) and using (2.58), and again keeping \(x\) fixed, set \(y = t\), integrate from 0 to \(y\) in the resulting inequality with using (2.58), we obtain
\begin{equation}
m(x,y) \leq \left[ c^{1-p} + \left( 1 - p \right) \int_0^x H(s,t)Q^p(s,t)E^p(s,t) \, ds \right]^\frac{1}{1-p}
\end{equation}

Substituting the above bound in (2.78) we get the desired inequality (2.54).

**Theorem 2.5:** Let \(\phi(x,y)\), \(A(x,y)\), \(B(x,y)\) and \(H(x,y)\) be defined as in Theorem 2.1, \(c > 0\) and \(0 < p < 1\) are constants and suppose
\begin{equation}
\phi(x,y) \leq c + \int_0^x A(s,y)\phi^p(s,y) \, ds + \int_0^x B^p(x,t)\phi(x,t) \, dt + \int_0^y H(s,t)\phi^p(s,t) \, ds dt, \quad (x,y) \in R_+
\end{equation}
then
\[
\phi(x,y) \leq Q_2(x,y)E_2(x,y) \left[ c^{1-p} + (1-p) \int_0^y H(s,t)Q_2^p(s,t)E_2^p(s,t)dsdt \right]^{\frac{1}{1-p}}, \quad (x,y) \in R_+
\] (2.81)

where
\[
Q_2(x,y) = \left[ 1 + (1-p) \int_0^y B(x,t)m_{p-1}(x,t)b_{p-1}(x,t)dt \right]^{\frac{1}{1-p}}
\] (2.82)

and
\[
E_2(x,y) = \left[ 1 + (1-p) \int_0^x A(s,y)Q_2^p(s,y)m_{p-1}(s,y)ds \right]^{\frac{1}{1-p}}
\] (2.83)

**Proof:** Define
\[
m(x,y) = c + \int_0^y H(s,t)\phi^p(s,t)dsdt
\] (2.84)

\[
m(x,0) = m(0,y) = c, \quad m_y(x,0) = m_y(0,y) = 0
\] (2.85)

By substituting (2.84) in (2.80), we get
\[
\phi(x,y) \leq m(x,y) + \int_0^x A(s,y)\phi^p(s,y)ds + \int_0^y B(x,t)\phi^p(x,t)dt
\] (2.86)

Since \( m(x,y) \) is positive, nondecreasing continuous function, therefore
\[
\frac{\phi(x,y)}{m(x,y)} \leq 1 + \int_0^x A(s,y)\frac{\phi^p(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\phi^p(x,t)}{m(x,t)}dt
\] (2.87)

Let
\[
z(x,y) = 1 + \int_0^x A(s,y)\frac{\phi^p(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\phi^p(x,t)}{m(x,t)}dt
\] (2.88)

\[
z(0,0) = 1
\] (2.89)

From (2.87) and (2.88), we observe that
\[
\frac{\phi(x,y)}{m(x,y)} \leq z(x,y)
\] (2.90)

Also we notice from (2.88) and (2.90) that
\[
z(x,y) \leq 1 + \int_0^x A(s,y)m_{p-1}(s,y)z^p(s,y)ds + \int_0^y B(x,t)m_{p-1}(x,t)z^p(x,t)dt
\] (2.91)

Now define
\[
b(x,y) = 1 + \int_0^x A(s,y)m_{p-1}(s,y)z^p(s,y)dt
\] (2.92)

\[
b(x,0) = 1
\] (2.93)

By substituting (2.92) in (2.91), we get
\[ z(x,y) \leq b(x,y) + \int_0^y B(x,t)m^{p-1}(x,t)z^p(x,t)\,ds \]  

(2.94)

Since \( b(x,y) \) is positive, monotonic nondecreasing continuous function, therefore

\[ \frac{z(x,y)}{b(x,y)} \leq 1 + \int_0^y B(x,t)m^{p-1}(x,t)\frac{z^p(x,t)}{b(x,t)}\,ds \]  

(2.95)

Let

\[ v(x,y) = 1 + \int_0^y B(x,t)m^{p-1}(x,t)\frac{z^p(x,t)}{b(x,t)}\,ds \]  

(2.96)

\[ v(0,y) = 1, \]  

(2.97)

From (2.95) and (2.96), we have

\[ \frac{z(x,y)}{b(x,y)} \leq v(x,y) \]  

(2.98)

Differentiating (2.96) w.r.t. \( y \) and using (2.97), we have

\[ \frac{v_y(x,y)}{v^p(x,y)} \leq B(x,y)m^{p-1}(x,y)b^{p-1}(x,y) \]

Keeping \( x \) fixed, set \( y = t \) and integrate from 0 to \( y \), we get from above inequality

\[ v(x,y) \leq \left[ 1 + (1-p)\int_0^y B(x,t)m^{p-1}(x,t)b^{p-1}(x,t)\,dt \right]^{1/p} = Q_2(x,y) \]  

(2.99)

From (2.97) and (2.99), we have

\[ z(x,y) \leq b(x,y)Q_2(x,y) \]  

(2.100)

By differentiating (2.92) w.r.t \( x \), we have

\[ b_x(x,y) \leq A(x,y)m^{p-1}(x,y)z^p(x,y) \]  

(2.101)

By substituting (2.100) in (2.101) and keeping \( y \) fixed, set \( x = s \) and integrate from 0 to \( x \) and using (2.93), we observe that

\[ b(x,y) = \left[ 1 + (1-p)\int_0^x A(s,y))Q_2^p(s,y)m^{p-1}(s,y)\,ds \right]^{1/p} = E_2(x,y) \]  

(2.102)

From (2.100) and (2.102), we observe that

\[ z(x,y) \leq E_2(x,y)Q_2(x,y) \]  

(2.103)

By substituting from (2.102) in (2.90), we get

\[ \phi(x,y) \leq Q_2(x,y)E_2(x,y)m(x,y) \]  

(2.104)

Differentiating (2.84) w.r.t \( x \) and \( y \) and using (2.104), we obtain
Gronwall-Bellman type integral inequalities

\[
\frac{m_{xy}(x, y)}{m^p(x, y)} \leq H(x, y)Q_2^p(x, y)E_2^p(x, y)
\]

\[
\frac{m_{xy}(x, y)m(x, y)}{m_{p+1}(x, y)} - \frac{m_x(x, y)m_y(x, y)}{m_{p+1}(x, y)} \leq H(x, y)Q_2^p(x, y)E_2^p(x, y)
\]

\[
\frac{\partial}{\partial y} \left[ \frac{m_x(x, y)}{m^p(x, y)} \right] \leq H(x, y)Q_2^p(x, y)E_2^p(x, y) \tag{2.105}
\]

By keeping \( y \) fixed, set \( x = s \), integrate from 0 to \( x \) in (2.105) and using (2.85) and again keeping \( x \) fixed, set \( y = t \), integrate from 0 to \( y \) in the resulting inequality with using (2.85), we obtain

\[
m(x, y) \leq \left[ e^{1-p} + (1-p) \int_0^x \int_0^y H(s, t)Q_2^p(s, t)E_2^p(s, t) ds dt \right]^{\frac{1}{1-p}}
\]

Substituting the above bound in (2.104) we get the desired inequality (2.81).

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