A Complete Classification for Quenching Phenomena in Coupled Heat Equations

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Abstract

This paper deals with simultaneous and non-simultaneous quenching phenomena in coupled heat equations. We characterize completely the range of parameters for which non-simultaneous or simultaneous quenching occur. Moreover, all kinds of simultaneous and non-simultaneous quenching rates are obtained.

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Keywords: Quenching; Non-simultaneous quenching; Quenching rate

1 Introduction

In this paper, we consider the following heat equations coupled with inner absorption

\[
\begin{align*}
    u_t &= u_{xx} - u^{-p}v^{-m}, & v_t &= v_{xx} - u^{-n}v^{-q}, & 0 < x < 1, & t > 0, \\
    u_x(0, t) &= v_x(0, t) = 0, & u_x(1, t) &= v_x(1, t) = 0, & t > 0, \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & 0 \leq x \leq 1,
\end{align*}
\]

(1.1)

where \( p, q, m, n > 0 \), and the initial data \( u_0, v_0 \) are positive, smooth and compatible with the boundary data.

Problem (1.1) can be considered as a heat propagation model coupled with inner absorption. Because of the singular nonlinearity in the absorption terms of (1.1), the so-called finite-time quenching may occur for the model. The study of quenching phenomena was begun in 1975 by Kawarada [5]. Since then, a lot of works have been contributed to this subject (see, e.g., [1, 2, 4, 6, 7, 11]), and phenomena of non-simultaneous quenching for nonlinear parabolic systems have been observed and studied by many authors in recent years (see,
The solution \((u, v)\) of (1.1) is called quenching at time \(t = T < \infty\) if \(\liminf_{t \to T^-} \min_{0 \leq x \leq 1} \{u(x, t), v(x, t)\} = 0\). If this happens, \(T\) will be called as quenching time. Clearly at quenching time \(T\), a singularity develops in the absorption term, consequently the classical solution can no longer exists.

In [8], A. de Pablo etc. considered the following problem

\[
\begin{align*}
u_t &= u_{xx} - v^{-m}, & v_t &= v_{xx} - u^{-n}, & 0 < x < 1, \ t > 0, \\
u_x(0, t) &= v_x(0, t) = 0, & u_x(1, t) &= v_x(1, t) = 0, & t > 0, \\
u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & 0 \leq x \leq 1.
\end{align*}
\]

They pointed that quenching happens for system (1.2) for every initial data and proved that if \(m, n > 1\), then quenching is always simultaneous, if \(m < 1\) or \(n < 1\), then there exists a wide class of initial data with non-simultaneous quenching (i.e. one of the components of the system remains bounded away from zero, while the other vanishes at some point at time \(T\)), and if \(m < 1 \leq n\) or \(n < 1 \leq m\), then quenching is always non-simultaneous. Simultaneous quenching rate of (1.2) were given as \(u(0, t) \sim (T-t)^{\frac{m-2}{m-n-1}}, v(0, t) \sim (T-t)^{\frac{n-1}{m-n-1}}\) if \(m, n > 1\) or \(m, n < 1\); \(u(0, t) \sim (T-t)^{\frac{1}{2}}, v(0, t) \sim (T-t)^{\frac{1}{2}}\) if \(m = n = 1\);
\(u(0, t) \sim (T-t)|\log(T-t)|^{\frac{m-1}{m-n}}, v(0, t) \sim |\log(T-t)|^{\frac{n-1}{m-n}}\) if \(m > n = 1\) and \(v(0, t) \sim (T-t)\) for non-simultaneous quenching with \(v\) quenching only. Here and throughout our paper, the notation \(f \sim g\) means that there exist finite positive constants \(c_1, c_2\) such that \(c_1g \leq f \leq c_2g\) for \(t\) close to the quenching time \(T\).

It is easy to see that the solution of (1.1) quenches in finite time for any initial data since \((u, v)\) to (1.2) is a pair of supersolution of (1.1). In this paper, we propose the complete classification for the simultaneous and non-simultaneous quenching of solutions to (1.1). From now on, we always assume the initial data are nondecreasing and satisfy

\((H)\) \(u_0\)xx - \(u_0^{-p}v_0^{-m}\) \(\leq -\delta u_0^{-p}v_0^{-m}, v_0\)xx - \(u_0^{-n}v_0^{-q}\) \(\leq -\delta u_0^{-n}v_0^{-q}\), \(0 < \delta < 1\).

Now we state the main results of this paper.

**Theorem 1.1** If \(m \geq q + 1, n \geq p + 1\), then any quenching in (1.1) is simultaneous.

**Theorem 1.2** If \(m < q + 1, n \geq p + 1\), then any quenching in (1.1) is non-simultaneous with \(u\) being strictly positive; If \(m \geq q + 1, n < p + 1\), then any quenching in (1.1) is non-simultaneous with \(v\) being strictly positive.

**Theorem 1.3** If \(m < q + 1, n < p + 1\), both simultaneous and non-simultaneous quenching may occur.
By Theorems 1.1-1.3 of this paper, the complete classification for simultaneous and non-simultaneous quenching of (1.1) is obtained, which can be seen clearly by the help of the following figure.

Moreover, it is interesting that the simultaneous and non-simultaneous quenching phenomena depend sensitively on the choosing of the initial data in the region $m < q + 1$ and $n < p + 1$. That is to say, small $u_0$ ($v_0$) leads to the quenching of $u(v)$, and in some betweenes, there also exist initial data such that simultaneous quenching occurs. Now give the quenching rates for simultaneous and non-simultaneous quenching.

**Theorem 1.4** If quenching is non-simultaneous and, for instance $u(x, t)$ is the unique quenching component, then we have $u(0, t) \sim (T - t)^{\frac{1}{p+1}}$.

If quenching is simultaneous, then: (i) $u(0, t) \sim (T - t)^{\frac{1}{m-n-(p+1)(q+1)}}$, $v(0, t) \sim (T - t)^{\frac{1}{n-p-1}}$ if $m > q + 1, n > p + 1$ or $m < q + 1, n < p + 1$; (ii) $u(0, t) \sim (T - t)^{\frac{1}{p+m+1}}$, $v(0, t) \sim (T - t)^{\frac{1}{m-q+1}}$ if $m = q + 1, n = p + 1$; (iii) $u(0, t) \sim |\log(T - t)|^{\frac{1}{m-p-1}}$, $v(0, t) \sim (T - t)^{\frac{1}{m-q+1}|\log(T - t)|^{\frac{1}{n-p-1}}}$ if $m = q + 1, n > p + 1$.

**2 Proofs**

Firstly, we give a lemma that gives control of the time derivatives, which play an important role in our arguments.

**Lemma 2.1** If $(u_0, v_0)$ satisfies (H), then, for $\delta$ is the same as in (H),

$$u_t \leq -\delta u^{-p} v^{-m}, \quad v_t \leq -\delta u^{-n} v^{-q}.$$ (2.1)
Proof. Construct $I(x, t) = u_t + \delta u^{-p}v^{-m}$, $J(x, t) = v_t + \delta u^{-n}v^{-q}$. A straightforward computation shows $I_t - I_{xx} \leq pu^{-p-1}v^{-m}I + mu^{-p}v^{-m-1}J$, where the last inequality following from $u_xv_x \geq 0$ because of the monotonicity of $u, v$. Similarly, we get $J_t - J_{xx} \leq qu^{-q-1}v^{-q}I + nu^{-n-1}v^{-q}J$. Note that $I_x(0, t) = J_x(0, t) = 0$, $I_x(1, t) = J_x(1, t) = 0$ and $I(x, 0) \leq 0$, $J(x, 0) \leq 0$. Then (2.1) holds by the comparison principle.

Since the initial data are nondecreasing and satisfy (H), we follow that $u$ and $v$ are nondecreasing in space and nonincreasing in time. Then we have $\min_{0 \leq x \leq 1} u(x, t) = u(0, t)$, $\min_{0 \leq x \leq 1} v(x, t) = v(0, t)$. For $u_{xx}(0, t), v_{xx}(0, t) \geq 0$, we get $u'(0, t) \geq -u^{-p}(0, t)v^{-m}(0, t)$, $v'(0, t) \geq -u^{-n}(0, t)v^{-q}(0, t)$. Taking together with (2.1), we have the estimates

$$-u^{-p}(0, t)v^{-m}(0, t) \leq u'(0, t) \leq -\delta u^{-p}(0, t)v^{-m}(0, t), \quad (2.2)$$

$$-u^{-n}(0, t)v^{-q}(0, t) \leq v'(0, t) \leq -\delta u^{-n}(0, t)v^{-q}(0, t). \quad (2.3)$$

Proof of Theorem 1.1. Suppose for contradiction that there exists $c > 0$ such that $u > c > 0$ in $[0, 1] \times [0, T)$. We know from (2.3) that $v'(0, t) \geq -c^{-n}v^{-q}(0, t)$. Integrating in $(t, T)$, we get $v(0, t) \leq c_1(T - t)^{-\frac{m}{p+1}}$. In view of (2.2), $u'(0, t) \leq -\delta c_1^{-m}u^{-p}(0, t)(T - t)^{-\frac{m}{p+1}}$. Integrate in $(0, T)$ to obtain

$$c_2 \int_0^T (T - t)^{-\frac{m}{p+1}} dt \leq u_0^{p+1}(0) - u^{p+1}(0, T). \quad (2.4)$$

The convergence of the integral in (2.4) implies that $m < q + 1$.

Proof of Theorem 1.2. It follows from (2.2) and (2.3) that

$$\frac{1}{T}u^{-(m-q)}(0, t)v'(0, t) \leq u^{-(n-p)}(0, t)u'(0, t) \leq \delta v^{-(m-q)}(0, t)v'(0, t). \quad (2.5)$$

For $m < q + 1$, $n > p + 1$, integrating the first inequality in (2.5) from $(0, t)\leq c_1 - c_2u^{p+1-n}(0, t)$ with two positive constants $c_1$ and $c_2$, which requires that $u$ remains positive up to the quenching time. The cases $m < q + 1, n = p + 1$ and $m > q + 1, n < p + 1$ can be treated in an analogous way.

We know from $u_t, v_t \leq 0$ that

$$u'(0, t) \leq -\delta u^{-p}(0, t)v_0^{-m}(0), \quad v'(0, t) \leq -\delta u_0^{-n}(0)v^{-q}(0, t). \quad (2.6)$$

Integrate on $(t, T)$ to get a preliminary estimate for $(u(0, t), v(0, t))$:

$$u(0, t) \geq c_1(T - t)^{\frac{1}{p+1}}, \quad v(0, t) \geq c_2(T - t)^{\frac{1}{q+1}}, \quad (2.7)$$

where $T \leq \min\{\frac{u_0^{p+1}(0)}{c_1}, \frac{v_0^{q+1}(0)}{c_2}\}$.

Define $U = \{(u_0, v_0) : u_0, v_0$ are nondecreasing and and satisfy (H)$\}$. 

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Lemma 2.2 If \( m < q + 1 \), then there exists \((u_0, v_0) \in U\) such that \(v\) quenches while \(u\) remains bounded away from zero.

**Proof.** If \((u_0, v_0) \in U\), one can check \((\lambda u_0, \mu v_0) \in U\) for any \(\lambda, \mu \in (0, 1)\). Fixed \(\lambda_0 \in (0, 1)\) and let \((u_{\lambda_0}, v_\mu)\) be the solution with initial data \((\lambda_0 u_0, \mu v_0)\) for any \(\mu \in (0, 1)\). We see from (2.7) that \(v_\mu(0, t) \geq c_2(T - t)^{-\frac{m}{q+1}}\) and \(T \leq \frac{\mu^{q+1} v^q(0)}{c_2^{q+1}}\), and we have \((u^{p+1}_{\lambda_0}(0, t)))' \geq -(p + 1)c_2^{-m}(T - t)^{-\frac{m}{q+1}}\) in terms of (2.2). Integrating this inequality in \((0, T)\), we obtain that

\[
u^{p+1}_{\lambda_0}(0, T) \geq u^{p+1}_{\lambda_0}(0, 0) - cT^{1-\frac{m}{q+1}} = \lambda_0^{p+1}u_0^{p+1} - cT^{1-\frac{m}{q+1}},
\]

where \(c\) is independent of \(\mu\). Clearly \(\mu v_0\) (and hence \(T\)) can be arbitrarily small if \(\mu\) small enough. Consequently, the right-hand side of (2.8) is strictly positive since \(m < q + 1\).

Lemma 2.3 If \(m < q + 1\), then the set of initial data in \(U\) such that \(v\) quenches and \(u\) remains strictly positive is open in the \(L^\infty\)-topology.

**Proof.** Let \((u, v)\) be a solution of (1.1) with initial data \((u_0, v_0)\) such that \(v\) quenches in finite time \(T\) while \(u > c > 0\). For any small \(\epsilon > 0\), there exists \(\sigma\) such that the solution \((\overline{u}, \overline{v})\) of (1.1) has the quenching time \(\overline{T} \in (T - \epsilon, T + \epsilon)\) and \(\overline{u}(0, T - \epsilon) \geq \frac{1}{2} u(0, T - \epsilon) \geq \frac{\sigma}{2}\) whenever the initial data \((\overline{u}_0, \overline{v}_0) \in \{(\overline{u}_0, \overline{v}_0) : ||\overline{u}_0 - u_0||_\infty + ||\overline{v}_0 - v_0||_\infty \leq \sigma\} \in U\). Similarly to the argument used in the proof of Lemma 2.2, we get \(\overline{u}^{p+1}(0, \overline{T}) \geq \overline{u}^{p+1}(0, T - \epsilon) - c_1(T - (T - \epsilon))^{1-\frac{m}{q+1}} \geq (\frac{\sigma}{2})^{p+1} - c_1(2\epsilon)^{1-\frac{m}{q+1}}\), where \(c_1\) is independent of \(\epsilon\). Therefore, \(\overline{u}\) cannot quench if we take \(\epsilon\) sufficiently small.

Lemma 2.4 If \(m < q + 1, n < p + 1\), then there exists \((u_0, v_0) \in U\) such that simultaneous quenching occurs.

**Proof.** For any \(\lambda \in (0, 1)\), we denote by \((u_\lambda, v_{1-\lambda})\) the solution of (1.1) with initial data \((\lambda u_0, (1-\lambda) v_0)\). Define \(M_\lambda = \{\lambda \in (0, 1) : u_\lambda\) quenches, \(v_{1-\lambda}\) remains positive\}\), \(N_\lambda = \{\lambda \in (0, 1) : u_\lambda\) remains positive, \(v_{1-\lambda}\) quenches\}\). We know from the proof of Lemma 2.2 that if \(\lambda\) is small enough, then \(v_{1-\lambda}\) remains strictly positive when \(u_\lambda\) quenches. Analogously \(\lambda \in N_\lambda\) if \(\lambda\) is close to 1 sufficiently. So \(M_\lambda, N_\lambda \neq \phi\). On the other hand, Lemma 2.3 says that the two sets \(M_\lambda\) and \(N_\lambda\) are open. This concludes that there exists initial data \((u_0, v_0)\) such that \(u\) and \(v\) quench simultaneously.

We get the proof of Theorem 1.3 by combining Lemmas 2.2 and 2.4.

**Proof of Theorem 1.4.** If \(u\) quenches while \(u > c > 0\), we get from (2.2) that 

\[-c^{-m} u^{-p}(0, t) \leq -u^{-p}(0, t)v^{-m}(0, t) \leq u'(0, t) \leq -\delta u^{-p}(0, t)v_0^{-m}(0)\]

Integrating in \((t, T)\), we have \(u(0, t) \sim (T - t)^{\frac{1}{p+1}}\).

Now assume that \(u, v\) simultaneously quench. For \(m > q + 1, n > p + 1\), by (2.5) we have \(c_1 v^{q+1-m}(0, t) + c_2 \leq v^{p+1-n}(0, t) \leq C_1 v^{q+1-m}(0, t) + C_2\). For
t close to T, the additive constants \(c_2, C_2\) become negligible. Hence, introducing this in (2.2) and (2.3), we get \(u'(0, t) \sim -u(0, t) \frac{\alpha \frac{m-n-q-pq}{mn-n-q-pq}}{m-(q+1)}, v'(0, t) \sim -v(0, t) \frac{\alpha \frac{m-n-q-pq}{mn-n-q-pq}}{n-(p+1)}.\) The rates are straightforward.

For \(m < q + 1, n < p + 1\), we deduce the quenching rate by a bootstrap argument. First of all, from (2.6), we get \((u^{p+1}(0, t))' \leq -\delta(p+1)v_0^m(0)\), which means \(u(0, t) \geq c_1(T-t)^{\frac{m-n}{m-p}}\). Combining with (2.3) we get \((v^{q+1}(0, t))' \geq -(q+1)c_1^{-k}(T-t)^{\frac{q}{p+1}}\), i.e., \(v(0, t) \leq C_1(T-t)^{\frac{q}{p+1}}\). Iterating this procedure, we get \(u(0, t) \geq c_k(T-t)^{\alpha_k}, v(0, t) \leq C_k(T-t)^{\beta_k}\), where \(\alpha_k, \beta_k\) satisfy

\[
\alpha_0 = \frac{1}{p+1}, \quad \beta_0 = \frac{p+1-k}{q(p+1)}, \quad \alpha_{k+1} = \frac{1-m\beta_k}{p+1}, \quad \beta_{k+1} = \frac{1-k\alpha_k}{q+1}, \quad k = 1, 2, \ldots.
\]

One can check that \(\alpha_k \to -\frac{m-q-1}{mn-(p+1)(q+1)}, \beta_k \to -\frac{n-p-1}{mn-(p+1)(q+1)}\), and that \(C_k \leq C < \infty\) and \(c_k \geq c > 0\). Therefore, passing to the limit, we get \(u(0, t) \geq c(T-t)^{\frac{m-n}{m-n-pq}}\) and \(v(0, t) \leq C(T-t)^{\frac{m-n}{m-n-pq}}\). The reverse inequalities can be obtained in the same way.

For \(m = q + 1, n = p + 1\), let \(w = cu - v\). Taking \(c\) big enough, we get \(w(x, 0) > 0\). On the other hand, \(w\) satisfies the equation \(w_t = w_{xx} - \frac{w}{w^{q+1}}, 0 < x < 1, t > 0\) with \(w_x(0, t) = w_x(1, t) = 0, t > 0\). Thus, by the comparison principle we have \(w \geq 0\), i.e., \(cu \geq v\). A similar argument shows that \(Cv \geq u\), hence, \(u \sim v\). Using the (2.2) and (2.3), we get the quenching rate immediately.

For \(m = q + 1, n > p + 1\), we know from (2.5) that \(v^{-1}(0, t)u'(0, t) \sim u^{p-n}(0, t)u'(0, t)\). After integrating, we get \(v(0, t) \sim e^{\frac{-c}{u^{p+1-n}(0, t)}}\). From (2.2), we have

\[
u^p(0, t)u_t(0, t) \sim -e^{mcu^{p+1-n}(0, t)}.
\]

Integrating (2.9) yields

\[
-\int_t^T e^{-mcu^{p+1-n}(0, s)}u^p(0, s)u_s(0, s)ds \sim (T-t).
\]

Letting \(mcu^{p+1-n}(0, s) = y\) in (2.10), we have \(\int_0^\infty y^n e^{-y}dy \sim (T-t)\). It is known that the incomplete Gamma function \(\Gamma(a, z) = \int_0^\infty y^{a-1}e^{-y}dy\) satisfies \(\Gamma(a, z) \sim z^{a-1}e^{-z}\) for \(z \to \infty\). Then we obtain \((u^{p+1-n}(0, t))^{m-1}e^{-mcu^{p+1-n}(0, t)} \sim (T-t)\). Hence \(u(0, t) \sim |\log(T-t)|^{-\frac{1}{n-p-1}}\). The behavior for \(v\) is now immediate by using (2.3).

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## References


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