On Critical Point Theorems without Compactness

H. Boukhrisse
Department of Mathematics and Computer Sciences
University Mohamed Premier, Faculty of Sciences, Oujda, Morocco
boukhrisse@yahoo.fr

M. Moussaoui
Superior School of Technologie
University Mohamed Premier, Oujda, Morocco

Abstract

In this paper, we establish an abstract critical point theorem for a $C^1(E, \mathbb{R})$ functional where $E = V \oplus W$ is a reflexive Banach space, $V$ and $W$ are two closed subspaces of $E$. Our theorem doesn’t require any compactness condition of Palais Smale type and assume that the operator $\nabla \Phi(w + .) : V \mapsto V$ is strongly monotone for all $w \in W$. We improve a critical point theorem proved in [7] which generalizes many previous theorems. Our result is based specially on the important critical point theorem of Moussaoui [13].

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1 Introduction and origins of our results

Abstract critical point theorems are important tool in Nonlinear Analysis. Most of them are based on the compactness assumptions of Palais Smale type, see for example [1, 3, 14]. In this paper, we improve our critical point theorem proved in [7] which doesn’t require any compactness condition and suppose instead convexity assumptions. Some authors were interested in critical point theorems without compactness [10, 4, 13, 8] and most of them were interested to generalize the following abstract critical point theorem of type minimax established by A.C. Lazer, E.M.Landesman and D.R. Meyers [9]:

Theorem 1.1 Let $X$ and $Y$ be two closed subspaces of a real Hilbert space $H$ such that $X$ is finite dimensional and $H = X \oplus Y$ ($X$ and $Y$ are not necessarily orthogonal). Let $\Phi : H \to \mathbb{R}$ be a $C^2$ functional and let $\nabla \Phi$ and $D^2 \Phi$ denote the gradient and Hessian of $\Phi$, respectively. Suppose that there exist two positive constants $m_1$ and $m_2$ such that
\[
(D^2 \Phi(u)h, h) \leq -m_1 \|h\|^2,
\]
\[
(D^2 \Phi(u)k, k) \geq m_2 \|k\|^2
\]
for all $u \in H$, $h \in X$ and $k \in Y$. Then $\Phi$ has a unique critical point, i.e., there exists a unique $v_0 \in H$ such that $\nabla \Phi(v_0) = 0$. Moreover, this critical point is characterized by the equality
\[
\Phi(v_0) = \max_{x \in X} \min_{y \in Y} \Phi(x + y).
\]

Bates and Ekeland [4], V.L.Shapiro [15], Stepan and Tersian [16] and Manasevich [10] generalized Theorem 1.1 by dropping the condition that $V$ is finite dimensional. Via a reduction method, Manasevich supposed in [10] weaker conditions on Hessian of $\Phi$. On the other hand, Stepan and Tersian [16] studied the case where $X$ and $Y$ are not necessarily finite dimensional, $\nabla \Phi : H \to H$ is everywhere defined and hemicontinuous on $H$, which means that
\[
\lim_{t \to 0} \nabla \Phi(u + tv) = \nabla \Phi(u) \quad \forall u, v \in H.
\]
and instead of conditions on Hessian of $\Phi$, they supposed that
\[
(\nabla \Phi(h_1 + y) - \nabla \Phi(h_2 + y), h_1 - h_2) \leq -m_1 \|h_1 - h_2\|^2, \quad h_1, h_2 \in X, y \in Y \quad (1)
\]
\[
(\nabla \Phi(x + k_1) - \nabla \Phi(x + k_2), k_1 - k_2) \geq m_2 \|k_1 - k_2\|^2, \quad k_1, k_2 \in Y, x \in X, \quad (2)
\]
where $H = X \oplus Y$, $m_1$ and $m_2$ are strictly positives numbers. Their result rests heavily upon two theorems on $\alpha$-convex functionals and an existence theorem for a class of monotone operators due to Browder.

Our purpose in this paper is to generalize the previous results. We will be interested in the case that $X$ and $Y$ may be of infinite dimensional and our approach is based essentially on a new version of Theorem 1.1 established by Moussaoui [13], where he supposed that $\Phi$ is of class $C^1$ and assume weakly convexity conditions. His theorem is the following:

Theorem 1.2 Let $H$ be a Hilbert space such that $H = V \oplus W$ where $V$ is finite dimensional subspace of $H$ and $W$ its orthogonal. Let $\Phi : H \to \mathbb{R}$ a functional such that:
(i) $\Phi$ is of class $C^1$.
(ii) $\Phi$ is coercive on $W$.  

(iii) For fixed $w \in W$, $v \mapsto \Phi(v + w)$ is concave on $V$.
(iv) For fixed $w \in W$, $\Phi(v + w) \to -\infty$ as $\|v\| \to +\infty$, $v \in V$; and the convergence is uniform on bounded subsets of $W$.
(v) for all $v \in V$, $\Phi$ is weakly lower semi-continuous on $W + v$.
Then $\Phi$ admits a critical point in $H$.

In [6] and [7], we proved some theorems that generalized Theorem 1.2 when $V$ isn’t necessary finite dimensional. In this paper, we improve one of these results, we obtain a main theorem and a corollary that we will present in section 2. We note that our theorems are of type min-max.

2 Main theorem

**Theorem 2.1** Let $E$ be a reflexive Banach space such that $E = V \oplus W$ where $V$ and $W$ are two closed subspaces of $E$. Suppose that $\Phi \in C^1(E, \mathbb{R})$, $\Phi$ is weakly upper semi-continuous on $W + v$ for all $v \in V$ and $\Phi$ is anti-coercive on $W$, that is,
\[
\Phi(w) \to -\infty \quad \text{as} \quad \|w\| \to +\infty.
\]
Moreover, Assume that there exists $a > 0$ such that:
\[
(\nabla \Phi(w + v_1) - \nabla \Phi(w + v_2), v_1 - v_2) \geq a \|v_1 - v_2\|^2 \tag{3}
\]
for all $v_1, v_2$ in $V$ and $w$ in $W$. Then $\Phi$ admits at least one critical point $u$ such that
\[
\Phi(u) = \max_{w \in W} \min_{v \in V} \Phi(v + w).
\]

**Remark 2.2** Condition (3) implies that $\nabla \Phi(w + .) : V \mapsto V$ is strongly monotone or a monotone. We note that monotone operators were used at first as necessary condition to resolve differential equations by Amann in [2].

**Remark 2.3** Theorem 2.1 generalizes Theorem 1.1 in three aspects. Firstly our theorem requires the spaces being reflexive Banach spaces instead of Hilbert spaces, secondly it requires the functionals being $C^1$ instead of $C^2$, thirdly Theorem 2.1 requires the monotonicity of $\nabla \Phi$ only on a part of the underlying space.

**Remark 2.4** We note that our theorem doesn’t require the "uniform convergence on bounded subsets of $W"$ which supposed in condition (iv) of Theorem 1.2. That will be of big interest to improve the resolution of some nonlinear differential problems.
In the proof of our theorem, we inspired by the proof of Theorem 1.2 [13], our proof is based on some properties of convex analysis and we use in particular the least action principle. Note that this principle has been used for example by J.Mawhin, M.Willem and Ward [11, 12] to obtain necessary and sufficient conditions for the existence of solutions for some differential equations. Using the same principle and by a variant method, Chun-Lei Tang and Xing-Ping Wu proved in [17] a similar result of our main theorem. Recall a general formulation of the least action principle which can be found for example in [5]

The least action principle : Suppose that $V$ is a reflexive Banach space and $\varphi : V \mapsto \mathbb{R}$ is weakly lower semi-continuous. Assume that $\varphi$ is coercive, that is,

$$\varphi(v) \to +\infty \quad \text{as} \quad \|v\| \to +\infty.$$ 

Then $\varphi$ has at least one minimum.

The proof of Theorem 1.2 is based on lemmas 2.1 and 2.2.

Lemma 2.1 Let $V$ and $W$ be reflexive Banach spaces and $\Phi \in C^1(E, \mathbb{R})$ such that $E = V \oplus W$ and satisfies:

1) $\forall w \in W$, $\Phi$ is weakly lower semi-continuous on $V + w$ and

$$\Phi(v + w) \to +\infty \quad \text{as} \quad \|v\| \to +\infty.$$ 

2) $\forall v \in V$, $\Phi$ is weakly upper semi-continuous on $W + v$ and

$$\Phi(w) \to -\infty \quad \text{as} \quad \|w\| \to +\infty.$$ 

Then the set

$$V(w) = \{v \in V : \varphi(w) = \Phi(v + w) = \min_{g \in V} \Phi(g + w)\}$$

is nonempty and $\varphi : W \mapsto \mathbb{R}$ is bounded above and achieves its maximum at some $w_0 \in W$.

Proof. By the least action principle, $\Phi$ admits a minimum on $V + w$ for all $w \in W$. Then the set $V(w)$ is nonempty. There exists a sequence $u_n = v_n + w_n$ such that $\Phi(u_n) \to \sup_W \varphi = b$ with $w_n \in W$ and $v_n \in V(w_n)$. Claim:

$$\|w_n\| \leq c.$$ 

Otherwise, since

$$\Phi(u_n) = \Phi(v_n + w_n) \leq \Phi(w_n),$$
then by the anti-coercivity of $\Phi$ on $W$, we conclude that $\Phi(w_n) \to -\infty$. Hence $\Phi(u_n) \to -\infty$. A contradiction. So, there exists a subsequence, still denoted $(w_n)$ such that $w_n \to w$. Take $v$ in $V$, we have

$$\Phi(v + w) \geq \limsup_n \Phi(v + w_n) \geq \limsup_n \Phi(v_n + w_n) = b.$$ 

This is true for all $v \in V$, in particular for $v \in V(w)$. Then $\varphi$ is bounded above and achieves its maximum at some point $w_0 \in W$.

**Lemma 2.2** Consider the assumptions of Lemma 2.1, and assume moreover that $\Phi$ is continuously differentiable on $E$ and there exists $a > 0$ such that

$$(\nabla \Phi(w + v_1) - \nabla \Phi(w + v_2), v_1 - v_2) \geq a \|v_1 - v_2\|^2 \tag{4}$$

for all $w \in W, v_1, v_2 \in V$. Then the mapping $f : W \mapsto V$ such that $f(w) \in V(w)$ is single-valued and continuous.

**Proof.** By lemma 2.1, $V(w)$ is nonempty and we affirm that $V(w)$ is a singleton. Otherwise we suppose that there exist $v_1$ and $v_2$ such that

$$\Phi(v_1 + w) = \Phi(v_2 + w) = \min_{g \in V} \Phi(g + w).$$

Let $v_\lambda = \lambda v_1 + (1 - \lambda)v_2$ and $0 < \lambda < 1$. By condition (4), $\Phi$ is strictly convex. Then

$$\Phi(v_\lambda + w) < \lambda \Phi(v_1 + w) + (1 - \lambda) \Phi(v_2 + w) = \Phi(v_1 + w) = \Phi(v_2 + w).$$

Absurd. We suppose that $f$ is not continuous, thus there exists $\delta > 0$ and a sequence $(w_n)$ converging to $w \in W$ and an integer $N$ large enough such that

$$\|f(w_n) - f(w)\| \geq \delta, \quad \forall n \geq N.$$ 

We have

$$\Phi(w + f(w) + tg) - \Phi(w + f(w)) \geq 0 \quad \forall g \in V.$$ 

We divide by $t$ and let $t$ converge to $0$, we obtain:

$$(\nabla \Phi(w + f(w)), g) \geq 0 \quad \forall g \in V.$$ 

Then

$$(\nabla \Phi(w + f(w)), g) = 0 \quad \forall g \in V. \tag{5}$$

Let $P$ be the projection of $H$ onto $V$ defined by $P(v + w) = v$, and let $P^*$ be the operator adjoint of $P$. Then for each $n$ we obtain

$$(\nabla \Phi(w_n + f(w)), f(w_n) - f(w)) = (\nabla \Phi(w_n + f(w)), P(f(w_n) - f(w))) = (P^*(\nabla \Phi(w_n + f(w))), f(w_n) - f(w)).$$
Consequently,
\[ \| P^*(\nabla \Phi(w_n + f(w))) \| \| f(w_n) - f(w) \| \geq - (\nabla \Phi(w_n + f(w)), f(w_n) - f(w)) . \]

Since \((\nabla \Phi(w_n + f(w)), f(w_n) - f(w)) = 0\), we obtain
\[ -(\nabla \Phi(w_n + f(w)), f(w_n) - f(w)) = (\nabla \Phi(w_n + f(w)) - \nabla \Phi(w_n + f(w)), f(w) - f(w)). \]

Then, from (4), we obtain
\[ \| P^*(\nabla \Phi(w_n + f(w))) \| \| f(w_n) - f(w) \| \geq a \| f(w_n) - f(w) \|^2 . \]

For \(n\) large enough, we conclude that
\[ \| P^*(\nabla \Phi(w_n + f(w))) \| \geq a \| f(w_n) - f(w) \| \geq a\delta . \quad (6) \]

On the other hand, we have
\[ (\nabla \Phi(w_n + f(w)) \to \nabla \Phi(w + f(w)), \]
then by equation (5), for any \(v \in V\) we have
\[ (\nabla \Phi(w_n + f(w)), v) \to 0. \]

So
\[ (\nabla \Phi(w_n + f(w)), P(v + w)) \to 0 \quad \forall v \in V, w \in W, \]
then
\[ (P^*(\nabla \Phi(w_n + f(w))), h) \to 0 \quad \forall h \in E \]
We conclude that
\[ \| P^*(\nabla \Phi(w_n + f(w))) \| \to 0, \]
which contradicts the inequality (6), absurd.

**Proof of Theorem 2.1.** Let us prove that the function \(\Phi\) is coercive on \(V + w\)
for each \(w \in W\).

By relation (3), we have:
\[ (\nabla \Phi(w + sv) - \nabla \Phi(w), sv) \geq a\|sv\|^2. \]

Let \(0 < s \leq 1\), we divide the last inequality by \(s\), we obtain
\[ (\nabla \Phi(w + sv), v) \geq (\nabla \Phi(w), v) + as\|v\|^2. \]

So
\[ (\nabla \Phi(w + sv), v) \geq -\|\nabla \Phi(w)\|\|v\| + as\|v\|^2. \]
On the other hand, we have
\[ \Phi(w + v) = \Phi(w) + \int_0^1 \langle \nabla \Phi(w + sv), v \rangle ds. \]

Then
\[ \Phi(w + v) \geq \Phi(w) - \| \nabla \Phi(w) \| v \| + \int_0^1 as v^2 ds \]
\[ \geq \Phi(w) + \left( \frac{1}{2} a v \| - \| \nabla \Phi(w) \| \right) v \| . \]

Hence \( \Phi(w + v) \to +\infty \) when \( \| v \| \to +\infty \) for all \( w \) in \( W \). Let \( w \in W \) and \( u = v + w \) such that \( v \in V(w) \). We will prove that if \( \varphi \) attains its maximum on \( W \) at \( w \), then \( u \) is a critical point of \( \Phi \). We have
\[ (\nabla \Phi(u), v) = 0 \quad \forall v \in V. \]

So it suffices to prove that
\[ (\nabla \Phi(u), h) = 0 \quad \forall h \in W. \]

Take \( h \in W \) and \( w_t = w + th \) for \( |t| \leq 1 \). For each \( t \) such that \( 0 < |t| \leq 1 \), there exists a unique \( v_{tn} \in V(w_{tn}) \). Since \( w_{tn} \to w \) when \( n \to +\infty \), we deduce by lemma 2.2 that \( v_{tn} \) converges to a certain \( v_0 \) with \( v_0 \in V(w) \) and \( v_0 = v \).

For \( t > 0 \), we have:
\[ \frac{\Phi(w_t + v_t) - \Phi(v_t + w)}{t} \geq \frac{\Phi(w_t + v_t) - \Phi(v_0 + w)}{t} \geq 0. \]

Then,
\[ (\nabla \Phi(v_t + w + \lambda_t h), h) \geq 0, \quad 0 < \lambda_t < 1. \]

At the limit, we obtain
\[ (\nabla \Phi(u), h) = 0 \quad \forall h \in W. \]

Hence \( u \) is a critical point of \( \Phi \).

**Corollary 2.5** Suppose that \( E \) and \( \Phi \) are as in Theorem 2.1 and \( \Phi \) satisfies instead of (3) the following condition:
there exists a strictly increasing function \( \gamma : [0, +\infty[ \to [0, +\infty[ \) such that:
\[ \gamma(t) \to +\infty \quad \text{as} \quad t \to +\infty \]
and
\[ (\nabla \Phi(w + v_1) - \nabla \Phi(w + v_2), v_1 - v_2) \geq \gamma(\| v_1 - v_2 \|) \| v_1 - v_2 \| \]
(7)
for all \( w \in W, v_1, v_2 \in V \). Then, the conclusion of Theorem 2.1 holds true.
**Proof.** Condition (3) occurs in the proof of Theorem 2.1 to show that $f : W \mapsto V$ defined in lemma 2.2 is continuous and to prove the coercivity of $\Phi$ on $V + w$ for each $w$ in $W$.

We begin by proving the continuity of $f : W \mapsto V$ when conditions of corollary are verified. By the proof of lemma 2.2, for $n$ large enough, we have

$$\|P^*(\nabla \Phi(w_n + f(w)))\| \geq \gamma(\|f(w_n) - f(w)\|) \geq \gamma(\delta).$$

On the other hand, we established that $\|P^*(\nabla \Phi(w_n + f(w)))\| \to 0.$

then for $n$ sufficiently large

$$\|P^*(\nabla \Phi(w_n + f(w)))\| < \gamma(\frac{\delta}{2})$$

A contradiction since the function $\gamma$ is strictly increasing. Hence $f$ is continuous.

To prove that $\Phi$ is coercive on $V + w$ for each $w \in W$, we will follow the same method as in the proof of Theorem 2.1. Indeed, by condition (7), we have

$$(\nabla \Phi(w + sv) - \nabla \Phi(w), sv) \geq \gamma(\|sv\|)\|sv\|.$$ 

Let $0 < s \leq 1$, we divide the last inequality by $s$, we obtain then

$$(\nabla \Phi(w + sv), v) \geq (\nabla \Phi(w), v) + \gamma(\|sv\|)\|v\|.$$ 

On the other hand, we have:

$$\Phi(w + v) = \Phi(w) + \int_{0}^{1} < \nabla \Phi(w + sv), v > ds,$$

and

$$(\nabla \Phi(w), v) \geq -\|\nabla \Phi(w)\|\|v\|.$$ 

Then

$$(\nabla \Phi(w + sv), v) \geq -\|\nabla \Phi(w)\|\|v\| + \gamma(\|sv\|)\|v\|.$$ 

Hence

$$\Phi(w + v) \geq \Phi(w) - \|\nabla \Phi(w)\|\|v\| + \int_{0}^{1} \|v\|\|\gamma(\|sv\|)\|ds.$$

$$\geq \Phi(w) - \|\nabla \Phi(w)\|\|v\| + \int_{0}^{1} \|v\|\|\gamma(\|sv\|)\|ds.$$

Since $\gamma(t) \to +\infty$ as $t \to +\infty$, there exists $R > 0$ such that

$$\|t\| \geq R \quad \text{et} \quad \gamma(t) \geq 2\|\nabla \Phi(w)\|.$$ 

So for $\|v\| \geq 4R$, we have

$$\Phi(w + v) \geq \Phi(w) - \|\nabla \Phi(w)\|\|v\| + \int_{0}^{1} \|v\|\|\nabla \Phi(w)\|ds.$$

$$\geq \Phi(w) + \frac{1}{3}\|\nabla \Phi(w)\|\|v\|.$$ 

Then $\Phi(w + v) \to +\infty$ when $\|v\| \to +\infty$ for all $w$ in $W$. This completes the proof of Corollary 2.5.
References


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