Abstract

We investigate the dynamical behavior of the following fifth-order rational difference equation

\[ x_{n+1} = \frac{x_n x_{n-3} x_{n-4} + x_n + x_{n-3} + x_{n-4} + a}{x_n x_{n-3} + x_n x_{n-4} + x_{n-3} x_{n-4} + 1 + a}, \quad n = 0, 1, 2, \ldots, \]

where \( a \in [0, \infty) \) and the initial values \( x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty) \).

We find that the successive lengths of positive and negative semicycles of nontrivial solutions of the above equation occur periodically. We also show that the positive equilibrium of the equation is globally asymptotically stable.

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1 Introduction and Preliminaries

It is well known that many authors argued that the discrete time models governed by difference equations are more appropriate than the continuous

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ones when the populations have nonoverlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. The study of rational difference equations of order than one is quite challenging and rewarding because some prototypes for the development of the basic theory of global behavior of nonlinear difference equations of order than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equation of order greater than one is worth further consideration [1-6]. We believe that nonlinear rational difference equations are of paramount importance in their own right, and furthermore that results about these equations are also useful in analyzing the equations in the mathematical models of various biological systems and other applications.

In this paper, we consider the following fifth-order rational difference equation

\[
x_{n+1} = \frac{x_n x_{n-3} x_{n-4} + x_n + x_{n-3} + x_{n-4} + a}{x_n x_{n-3} + x_n x_{n-4} + x_{n-3} x_{n-4} + 1 + a}, \quad n = 0, 1, 2, \ldots,
\]

where \( a \in [0, \infty) \) and the initial values \( x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty) \).

It is easy to see that the positive equilibrium \( \mathcal{P} \) of Eq.(1.1) satisfies

\[
\mathcal{P} = \frac{\mathcal{P}^3 + 3\mathcal{P} + a}{3\mathcal{P}^2 + 1 + a}
\]

from which one can see that Eq.(1.1) has a unique positive equilibrium \( \mathcal{P} = 1 \).

2 Two lemmas

In this section, we establish two lemmas which will be useful in the proof of our main results.

**Lemma 2.1.** A positive solution of Eq.(1.1) is eventually equal to 1 if and only if

\[
(x_{-4} - 1)(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) = 0.
\]

Proof. Assume that (2.1) holds. Then, according to Eq.(1.1), we have

\[
x_{n+1} - 1 = \frac{(x_n - 1)(x_{n-3} - 1)(x_{n-4} - 1)}{x_n x_{n-3} + x_n x_{n-4} + x_{n-3} x_{n-4} + 1 + a}, \quad n = 0, 1, 2, \ldots,
\]
then by method of induction, we can easily obtain the following conclusions:

1) if $x_{-4} = 1$, then $x_n = 1$ for $n \geq 1$;
2) if $x_{-3} = 1$, then $x_n = 1$ for $n \geq 1$;
3) if $x_{-2} = 1$, then $x_n = 1$ for $n \geq 2$;
4) if $x_{-1} = 1$, then $x_n = 1$ for $n \geq 3$;
5) if $x_0 = 1$, then $x_n = 1$ for $n \geq 0$.

Conversely, assume that

$$(x_{-4} - 1)(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) \neq 0$$

Then, one can show that

$$x_n \neq 1 \text{ for any } n \geq 1.$$ 

In fact, assume the contrary that for some $N \geq 1$, such that

$$x_N = 1 \text{ and that } x_n \neq 1 \text{ for } -4 \leq n \leq N - 1. \quad (2.2)$$

It is to see that

$$1 = x_N = \frac{x_{N-1}x_{N-4}X_{N-5} + X_{N-1} + X_{N-4} + X_{N-5} + a}{x_{N-1}X_{N-4} + X_{N-1}X_{N-5} + X_{N-4}X_{N-5} + 1 + a},$$

which implies that $(x_{N-5} - 1)(x_{N-4} - 1)(x_{N-1} - 1) = 0$. Obviously, this contradicts (2.2). The proof is complete.

**Lemma 2.2.** Let $\{x_n\}_{n=-4}^{\infty}$ be nontrivial positive solution of Eq.(1.1). Then the following four conclusions are true for $n \geq 0$:

(a) $(x_{n+1} - 1)(x_n - 1)(x_{n-3} - 1)(x_{n-4} - 1) > 0$;
(b) $(x_{n+1} - x_n)(x_n - 1) < 0$;
(c) $(x_{n+1} - x_{n-3})(x_{n-3} - 1) < 0$;
(d) $(x_{n+1} - x_{n-4})(x_{n-4} - 1) < 0$.

Proof. It follows by virtue of Eq.(1.1) that

$$x_{n+1} - 1 = \frac{(x_n - 1)(x_{n-3} - 1)(x_{n-4} - 1)}{x_nx_{n-3} + x_nx_{n-4} + x_{n-3}x_{n-4} + 1 + a}, \quad n = 0, 1, 2, ..., $$

and

$$x_{n+1} - x_n = \frac{(x_n - 1)[x_{n-3}(1 + x_n) + x_{n-4}(1 + x_n) + a]}{x_nx_{n-3} + x_nx_{n-4} + x_{n-3}x_{n-4} + 1 + a}, \quad n = 0, 1, 2, ...$$

From which we can see that inequalities (a) and (b) hold. The proofs for inequalities (c) and (d) are similar to the one for inequality (b). So we omit their here. The proof is complete.
3 Main results

First we analyze the trajectory structure of the semicycles of nontrivial solutions of Eq. (1.1). Here, we confine us to consider the situation of the strictly oscillatory solutions about $\mathcal{F} = 1$ of Eq. (1.1).

**Theorem 3.1.** Let $\{x_n\}_{n=-4}^\infty$ be a strictly oscillatory solutions of Eq. (1.1). Then the “rule of the trajectory structure” of nontrivial solutions of Eq. (1.1) is: ..., $4^-, 4^+, 4^-, 4^+, 4^-, 4^+, 4^-, ...$, or ..., $3^+, 1^-, 3^+, 1^-, 3^+, 1^-, ..., or ..., 2^+, 2^-, 2^-, 2^+, 2^-, 2^+, 2^-, ...$, or ..., $2^-, 1^-, 2^+, 1^-, 1^+, 2^-, 1^-, 2^+, 1^-, 1^+, 2^-, 1^-, 1^+, 2^+, 1^-, 2^+, 1^-, 1^+, 2^+, 1^-, 2^+,$ $1^-, 1^+, ..., or ..., 1^+, 3^-, 1^+, 3^-, 1^+, 3^-, 1^+, 3^-, ..., or ..., 1^-, 1^+, 2^-, 1^-, 1^+, 2^-, 1^-, 2^+, 1^+, 1^+, 2^+, 1^-, 2^+, 1^+, 2^+, 1^+, 2^+, 1^+, 2^+, 1^+, 2^+, 1^+, 2^+, 1^+, 2^+, 1^+, 2^+, 1^+, 2^+, 1^+, 2^+, 1^+,...$.

Proof. By Lemma 2.2 (a), one can see the lengths of a positive or a negative semicycle is at most 4. By virtue of the character of the strictly oscillatory, one can see that, for some integer $p \geq 0$, one of the following eight cases must occur:

Case 1. $x_{p-4} < 1, x_{p-3} > 1, x_{p-2} > 1, x_{p-1} > 1$ and $x_p > 1$.

Case 2. $x_{p-4} < 1, x_{p-3} > 1, x_{p-2} > 1, x_{p-1} > 1$ and $x_p < 1$.

Case 3. $x_{p-4} < 1, x_{p-3} > 1, x_{p-2} > 1, x_{p-1} < 1$ and $x_p < 1$.

Case 4. $x_{p-4} < 1, x_{p-3} > 1, x_{p-2} > 1, x_{p-1} < 1$ and $x_p > 1$.

Case 5. $x_{p-4} < 1, x_{p-3} > 1, x_{p-2} < 1, x_{p-1} < 1$ and $x_p < 1$.

Case 6. $x_{p-4} < 1, x_{p-3} > 1, x_{p-2} < 1, x_{p-1} < 1$ and $x_p > 1$.

Case 7. $x_{p-4} < 1, x_{p-3} > 1, x_{p-2} < 1, x_{p-1} > 1$ and $x_p < 1$.

Case 8. $x_{p-4} < 1, x_{p-3} > 1, x_{p-2} < 1, x_{p-1} > 1$ and $x_p > 1$.

If Case 1 occurs, it follows from Lemmas 2.2 (a) that $x_{p+1} < 1, x_{p+2} < 1$, $x_{p+3} < 1, x_{p+4} < 1, x_{p+5} > 1, x_{p+6} > 1, x_{p+7} > 1, x_{p+8} > 1, x_{p+9} < 1$, $x_{p+10} < 1, x_{p+11} < 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} > 1, x_{p+15} > 1, x_{p+16} > 1$, $x_{p+17} < 1, x_{p+18} < 1, x_{p+19} < 1, x_{p+20} < 1, x_{p+21} > 1, x_{p+22} > 1, x_{p+23} > 1$, $x_{p+24} > 1, x_{p+25} < 1, x_{p+26} < 1, x_{p+27} < 1, x_{p+28} < 1, x_{p+29} < 1, x_{p+30} > 1, x_{p+31} > 1, x_{p+32} > 1, ...

If Case 2 occurs, then Lemma 2.2 (a) implies that $x_{p+1} > 1, x_{p+2} > 1, x_{p+3} < 1, x_{p+4} < 1, x_{p+5} > 1, x_{p+6} > 1, x_{p+7} > 1, x_{p+8} < 1, x_{p+9} > 1, x_{p+10} > 1, x_{p+11} > 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} > 1, x_{p+15} > 1, x_{p+16} < 1, x_{p+17} > 1, x_{p+18} > 1, x_{p+19} > 1, x_{p+20} < 1, x_{p+21} < 1, x_{p+22} > 1, x_{p+23} > 1, x_{p+24} < 1, x_{p+25} > 1, x_{p+26} > 1, x_{p+27} > 1, x_{p+28} < 1, x_{p+29} > 1, x_{p+30} > 1, x_{p+31} > 1, x_{p+32} < 1, ...

If Case 3 occurs, it follows from Lemma 2.2 (a) that $x_{p+1} > 1, x_{p+2} > 1, x_{p+3} < 1, x_{p+4} < 1, x_{p+5} > 1, x_{p+6} > 1, x_{p+7} < 1, x_{p+8} < 1, x_{p+9} > 1, x_{p+10} > 1, x_{p+11} < 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} > 1, x_{p+15} < 1, x_{p+16} < 1, x_{p+17} > 1, x_{p+18} > 1, x_{p+19} < 1, x_{p+20} < 1, x_{p+21} > 1, x_{p+22} > 1, x_{p+23} < 1, x_{p+24} < 1, x_{p+25} < 1, x_{p+26} > 1, x_{p+27} > 1, x_{p+28} < 1, x_{p+29} < 1, x_{p+30} > 1, x_{p+31} > 1, x_{p+32} < 1, ...
x_{p+24} < 1, x_{p+25} > 1, x_{p+26} > 1, x_{p+27} < 1, x_{p+28} < 1, x_{p+29} > 1, x_{p+30} > 1, x_{p+31} < 1, x_{p+32} < 1, \ldots.

If Case 4 occurs, then Lemma 2.2 (a) implies that: $x_{p+1} < 1, x_{p+2} < 1, x_{p+3} > 1, x_{p+4} < 1, x_{p+5} > 1, x_{p+6} > 1, x_{p+7} < 1, x_{p+8} > 1, x_{p+9} < 1, x_{p+11} > 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} > 1, x_{p+15} < 1, x_{p+16} > 1, x_{p+17} < 1, x_{p+18} < 1, x_{p+19} < 1, x_{p+20} < 1, x_{p+21} > 1, x_{p+22} < 1, x_{p+23} < 1, x_{p+24} < 1, x_{p+25} > 1, x_{p+26} < 1, x_{p+27} > 1, x_{p+28} < 1, x_{p+29} > 1, x_{p+30} < 1, x_{p+31} < 1, x_{p+32} > 1, \ldots.$

If Case 5 occurs, then Lemma 2.2 (a) implies that: $x_{p+1} > 1, x_{p+2} < 1, x_{p+3} < 1, x_{p+4} < 1, x_{p+5} > 1, x_{p+6} < 1, x_{p+7} < 1, x_{p+8} < 1, x_{p+9} > 1, x_{p+10} < 1, x_{p+11} < 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} < 1, x_{p+15} < 1, x_{p+16} < 1, x_{p+17} > 1, x_{p+18} < 1, x_{p+19} < 1, x_{p+20} < 1, x_{p+21} > 1, x_{p+22} < 1, x_{p+23} > 1, x_{p+24} > 1, x_{p+25} < 1, x_{p+26} > 1, x_{p+27} < 1, x_{p+28} < 1, x_{p+29} > 1, x_{p+30} < 1, x_{p+31} > 1, x_{p+32} > 1, \ldots.

If Case 6 occurs, it follows from Lemma 2.2 (a) that: $x_{p+1} < 1, x_{p+2} > 1, x_{p+3} > 1, x_{p+4} < 1, x_{p+5} > 1, x_{p+6} < 1, x_{p+7} > 1, x_{p+8} < 1, x_{p+9} > 1, x_{p+10} > 1, x_{p+11} > 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} < 1, x_{p+15} > 1, x_{p+16} < 1, x_{p+17} > 1, x_{p+18} < 1, x_{p+19} > 1, x_{p+20} < 1, x_{p+21} > 1, x_{p+22} < 1, x_{p+23} > 1, x_{p+24} < 1, x_{p+25} > 1, x_{p+26} < 1, x_{p+27} > 1, x_{p+28} < 1, x_{p+29} > 1, x_{p+30} < 1, x_{p+31} > 1, x_{p+32} < 1, \ldots.

If Case 7 occurs, then Lemma 2.2 (a) implies that: $x_{p+1} > 1, x_{p+2} < 1, x_{p+3} > 1, x_{p+4} < 1, x_{p+5} > 1, x_{p+6} < 1, x_{p+7} > 1, x_{p+8} < 1, x_{p+9} > 1, x_{p+10} > 1, x_{p+11} > 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} < 1, x_{p+15} < 1, x_{p+16} > 1, x_{p+17} < 1, x_{p+18} > 1, x_{p+19} > 1, x_{p+20} < 1, x_{p+21} > 1, x_{p+22} < 1, x_{p+23} > 1, x_{p+24} > 1, x_{p+25} < 1, x_{p+26} > 1, x_{p+27} > 1, x_{p+28} < 1, x_{p+29} > 1, x_{p+30} < 1, x_{p+31} < 1, x_{p+32} > 1, \ldots.

If Case 8 occurs, it follows from Lemma 2.2 (a) that: $x_{p+1} < 1, x_{p+2} > 1, x_{p+3} > 1, x_{p+4} < 1, x_{p+5} > 1, x_{p+6} < 1, x_{p+7} < 1, x_{p+8} > 1, x_{p+9} < 1, x_{p+10} > 1, x_{p+11} > 1, x_{p+12} < 1, x_{p+13} > 1, x_{p+14} < 1, x_{p+15} < 1, x_{p+16} > 1, x_{p+17} < 1, x_{p+18} > 1, x_{p+19} > 1, x_{p+20} < 1, x_{p+21} > 1, x_{p+22} < 1, x_{p+23} < 1, x_{p+24} > 1, x_{p+25} < 1, x_{p+26} > 1, x_{p+27} > 1, x_{p+28} < 1, x_{p+29} > 1, x_{p+30} < 1, x_{p+31} < 1, x_{p+32} > 1, \ldots.$

We may now see that the lengths of positive and negative semicycles of nontrivial solutions of Eq.(1.1) occur periodically and can be expressed in the "form" $4^{-}, 4^{+}, 4^{-}, 4^{+}, 4^{-}, 4^{+}, 4^{-}, \ldots$, or $3^{+}, 1^{-}, 3^{+}, 1^{-}, 3^{+}, 1^{-}, 3^{+}, 1^{-}, \ldots$, or $2^{+}, 2^{-}, 2^{+}, 2^{-}, 2^{+}, 2^{-}, 2^{+}, \ldots$, or $2^{-}, 1^{-}, 2^{+}, 1^{-}, 1^{+}, 2^{-}, 1^{-}, \ldots$, or $1^{+}, 1^{-}, 1^{+}, 1^{-}, 1^{+}, 1^{-}, 1^{+}, 1^{-}, \ldots$, or $1^{+}, 1^{-}, 1^{+}, 1^{-}, 1^{+}, 1^{-}, 1^{+}, 1^{-}, \ldots$. Therefore, the proof is complete.
Theorem 3.2. Assume that \(a \in [0, \infty)\). Then the equilibrium of Eq.(1.1) is globally asymptotically stable.

Proof. We must prove that the positive equilibrium of Eq.(1.1) \(\bar{\Phi}\) is both locally asymptotically stable and globally attractive. The linearized equation of Eq.(1.1) about the positive equilibrium \(\bar{\Phi} = 1\) is

\[
y_{n+1} = 0 \cdot y_n + 0 \cdot y_{n-1} + 0 \cdot y_{n-2} + 0 \cdot y_{n-3} + 0 \cdot y_{n-4}, \quad n = 0, 1, 2, \ldots.
\]

By virtue of [2, Remark 1.3.1], \(\bar{\Phi}\) is locally asymptotically stable. It remains to verify that every positive \(\{x_n\}_{n=-4}^{\infty}\) of Eq.(1.1) converges to 1 as \(n \to \infty\). Namely, we want to prove

\[
\lim_{n \to \infty} x_n = 1 \quad (3.1)
\]

If the initial values of the solutions satisfy (2.1), then Lemma 2.1 says the solution is eventually equal to 1 and, of course, (3.1) holds. Therefore, we assume in the following that the initial values of the solution do not satisfy (2.1). Then by Remark 2.1 we know, for any solution \(\{x_n\}_{n=-4}^{\infty}\) of Eq.(1.1), \(x_n \neq 1\) for \(n \geq -4\).

If the solution is nonoscillatory about the positive equilibrium \(\bar{\Phi}\) of Eq.(1.1), then we know from Lemma 2.2 (b) that the solution is monotonic and bounded. So, the limit \(\lim_{n \to \infty} x_n = L\) exists and is finite. Taking the limit on both sides of Eq.(1.1), we obtain

\[
L = \frac{L^3 + 3L + a}{3L^2 + 1 + a}
\]

from which one can obtain that \(L = 1\), hence, (3.1) is true.

Thus, it suffices to prove that (3.1) holds for the solution to be strictly oscillatory.

Consider now \(\{x_n\}_{n=-4}^{\infty}\) to be strictly oscillatory about the positive equilibrium \(\bar{\Phi}\) of Eq.(1.1). By virtue of Theorem 3.1, one understands that the lengths of positive and negative semicycles which occur successively is: \(\ldots, 4^-, 4^+, 4^-, 4^+, 4^-, \ldots\), or, \(3^+, 1^-, 3^+, 1^-, 3^+, 1^-, \ldots\), or, \(2^+, 2^-, 2^+, 2^-, 2^+, 2^-, \ldots\), or, \(2^-, 1^+, 1^-, 2^+, 1^-, 1^+, 2^-, 1^+, 1^-, 2^+, 1^-, 1^+, 2^-, 1^+, 1^-, 2^+, 1^-, 1^+, 2^-, 1^+, 1^-, 2^+, 1^-, 1^+, 2^-, 1^+, 1^-, 2^+, 1^-, 1^+, 2^-, 1^+, 1^-, 2^+, 1^-, 1^+, 2^-, 1^+, 1^-, 2^+, 1^-, 1^+, 2^-, 1^+, 1^-, 2^+, \ldots\).
First, we investigate the case where the rule for the lengths of positive and negative semicycles which occur successively is: ..., $4^-, 4^+,...$.

For the convenience of state, for some nonnegative integer $p$, we denote by $\{x_p, x_{p+1}, x_{p+2}, x_{p+3}\}^-$, and $\{x_{p+4}, x_{p+5}, x_{p+6}, x_{p+7}\}^+$ the terms of a negative semicycle and a positive semicycle of length four respectively. So, negative and positive semicycles to occur successively can be periodically expressed as follows:

$$\{x_{p+8n}, x_{p+8n+1}, x_{p+8n+2}, x_{p+8n+3}\}^-, \{x_{p+8n+4}, x_{p+8n+5}, x_{p+8n+6}, x_{p+8n+7}\}^+, n = 0, 1,\ldots$$

The following results (1) and (2) can be easily obtained from Lemma 2.2 (b) and (d).

1. $x_{p+8n} < x_{p+8n+1} < x_{p+8n+2} < x_{p+8n+3} < x_{p+8n+8} < 1$;
2. $1 < x_{p+8n+12} < x_{p+3n+7} < x_{p+8n+6} < x_{p+8n+5} < x_{p+8n+4}$.

From which we can see that $\{x_{p+8n}\}_{n=0}^\infty$ is a monotonically increasing sequence with upper bound 1 and $\{x_{p+8n+4}\}_{n=0}^\infty$ is a monotonically decreasing sequence with lower 1. So the limits

$$\lim_{n\to\infty} x_{p+8n} = \lim_{n\to\infty} x_{p+8n+1} = \lim_{n\to\infty} x_{p+8n+2} = \lim_{n\to\infty} x_{p+8n+3} = L$$

and

$$\lim_{n\to\infty} x_{p+8n+4} = \lim_{n\to\infty} x_{p+8n+5} = \lim_{n\to\infty} x_{p+8n+6} = \lim_{n\to\infty} x_{p+8n+7} = M$$

exist and are finite.

Taking the limits on both sides of the following equations

$$x_{p+8n+5} = \frac{x_{p+8n+4}x_{8n+1}x_{8n} + x_{p+8n+4} + x_{p+8n+1} + x_{p+8n} + a}{x_{p+8n+4}x_{p+8n+1} + x_{p+8n+4}x_{p+8n} + x_{p+8n+8}x_{p+8n+1} + 1 + a}$$

and

$$x_{p+8n+9} = \frac{x_{p+8n+8}x_{8n+5}x_{8n+4} + x_{p+8n+8} + x_{p+8n} + x_{p+8n+5} + x_{p+8n+4} + a}{x_{p+8n+8}x_{8n+5}x_{8n+4} + x_{p+8n+8}x_{p+8n+4} + x_{p+8n+8}x_{p+8n+4} + 1 + a}$$

give rise to

$$M = \frac{ML^2 + M + 2L + a}{2ML + L^2 + 1 + a} \quad \text{and} \quad L = \frac{LM^2 + L + 2M + a}{2LM + M^2 + 1 + a}$$

From which, we get $L = M = 1$.

So,
\[
\lim_{n \to \infty} x_{p+8n+k} = 1, \quad k = 0, 1, 2, \ldots, 7.
\]

Which implies that (3.1) is true.

Second, we investigate the case where the rule for the lengths of positive and negative semicycles which occur successively is: \(\ldots, 3^+, 1^-, \ldots\). Similar to the first case, positive and negative semicycles to occur successively can be periodically expressed as follows:

\[
\{x_{p+4n}, x_{p+4n+1}, x_{p+4n+2}, \}^+, \{x_{p+4n+3}\}^-, \quad n = 0, 1, 2, \ldots.
\]

The following results (1) and (2) can be easily obtained from Lemma 2.2 \((b)\) and \((c)\).

1. \(1 < x_{p+4n+4} < x_{p+4n}\);
2. \(x_{p+4n+3} < x_{p+4n+7} < 1\).

From which one can see \(\{x_{p+4n}\}_{n=0}^{\infty}\) is a monotonically decreasing sequence with lower 1 and \(\{x_{p+4n+3}\}_{n=0}^{\infty}\) is a monotonically increasing sequence with upper 1, therefore,

\[
\lim_{n \to \infty} x_{p+4n} = L, \quad \text{and} \quad \lim_{n \to \infty} x_{p+4n+3} = M
\]

exist and are finite.

Taking the limit on both sides of the following equation

\[
x_{p+4n+4} = \frac{x_{p+4n+3}x_{p+4n}x_{p+4n-1} + x_{p+4n+3} + x_{p+4n} + x_{p+4n-1} + a}{x_{p+4n+3}x_{p+4n} + x_{p+4n+3}x_{p+4n-1} + x_{p+4n}x_{p+4n-1} + 1 + a}.
\]

We have

\[
L = \frac{M^2L + 2M + L + a}{M^2 + 2ML + 1 + a}
\]

From which we have \(L = 1\). Noting that \(x_{p+4n} > x_{p+4n+1} > x_{p+4n+2} > 1\), from which we have

\[
\lim_{n \to \infty} x_{p+4n} = \lim_{n \to \infty} x_{p+4n+1} = \lim_{n \to \infty} x_{p+4n+2} = 1.
\] (3.2)

Again, taking the limit on both sides of the following equation

\[
x_{p+4n+7} = \frac{x_{p+4n+6}x_{p+4n+3}x_{p+4n+2} + x_{p+4n+6} + x_{p+4n+3} + x_{p+4n+2} + a}{x_{p+4n+6}x_{p+4n+3} + x_{p+4n+6}x_{p+4n+2} + x_{p+4n+3}x_{p+4n+2} + 1 + a}.
\]

yield
\[ M = \frac{2M + 2 + a}{2M + 2 + a} = 1. \]  

(3.3)

(3.2) and (3.3) implies that (3.1) is true.

Similarly, we can prove other six cases. So we obtain that all the eight cases verify (3.1). The proof is complete.

**References**


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