On Quasi-Conformally Flat Quasi-Einstein Spaces with Recurrent Curvature

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Abstract

The present paper deals with a study of quasi-conformally flat quasi-Einstein spaces with recurrent curvature. We obtain various necessary and sufficient conditions for such a space to be recurrent (resp. concircularly recurrent, projectively recurrent and conformally recurrent). Also the existence of a quasi-Einstein space is ensured by an example which is neither quasi-conformally flat nor quasi-conformally symmetric.

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1 Introduction

The notion of quasi-Einstein spaces arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein spaces. It is well known that a connected Riemannian space \((M^n, g), n > 2\), is Einstein if its Ricci tensor \(R_{ij}\) of type \((0, 2)\) is of the form \(R_{ij} = \alpha g_{ij}\), where \(\alpha\) is a constant, which turns into \(R_{ij} = \frac{R}{n} g_{ij}\), \(R\) being the scalar curvature (constant) of the space. Let \((M^n, g), n > 2\), be a connected Riemannian space. Let \(U = \{x \in M: R_{ij} \neq \frac{R}{n} g_{ij} \text{ at } x\}\). Then \((M^n, g)\) is said to be quasi-Einstein space ([2],[10], [11]) if on \(U \subset M\), the relation

\[R_{ij} - \alpha g_{ij} = \beta A_i A_j \] (1)
holds, where $A_i$ is a unit covariant vector on $U$ and $\alpha, \beta$ are some scalars on $U$. It is obvious that the covariant vector $A_i$ as well as the function $\beta$ are non-zero at every point on $U$. Also every Einstein space is quasi-Einstein but not conversely (see Example 1 of section 4). Especially, every Ricci-flat space (e.g. Schwarzschild spacetime) is quasi-Einstein. The scalars $\alpha, \beta$ are known as the associated scalars of the space and the unit covariant vector $A_i$ is called the generator of the space. Such an $n$-dimensional quasi-Einstein space is denoted by $(QE)_n$.

In 1968 Yano and Sawaki [14] defined and studied a new curvature tensor on a Riemannian space of dimension $n$ which includes both the conformal and concircular curvature tensor as special cases. This curvature tensor is known as quasi-conformal curvature tensor. The quasi-conformal curvature tensor $W_{ijk}^h$ of type (1,3) of a Riemannian space of dimension $n(>3)$ [This condition is assumed throughout the paper as for $n = 3$, the conformal curvature tensor vanishes ] is defined by

$$W_{ijk}^h = -(n - 2)bC_{ijk}^h + [a + (n - 2)b]\tilde{C}_{ijk}^h, \quad (2)$$

where $a, b$ are arbitrary constants not simultaneously zero, $C_{ijk}^h$ and $\tilde{C}_{ijk}^h$ are conformal and concircular curvature tensor of type $(1,3)$ respectively.

In particular, if $a = 1$ and $b = -\frac{1}{n-2}$, then the quasi-conformal curvature tensor reduces to conformal curvature tensor. Again if $a = 1$ and $b = 0$, then the quasi-conformal curvature tensor reduces to concircular curvature tensor.

The conformal curvature tensor and concircular curvature tensor $C_{ijk}^h, \tilde{C}_{ijk}^h$ are respectively given by

$$C_{ijk}^h = R_{ijk}^h - \frac{1}{(n-2)}[\delta_k^h R_{ij}^h - \delta_j^h R_{ik}^h + R_{k[ij}^h g_{ik]} - R_{k[ij}^h g_{ik]}^h] \quad (3)$$

and

$$\tilde{C}_{ijk}^h = R_{ijk}^h - \frac{R}{n(n-1)}[\delta_k^h g_{ij} - \delta_j^h g_{ik}], \quad (4)$$

where $R$ denotes the scalar curvature of the space. Using (3) and (4) in (2) we get

$$W_{ijk}^h = aR_{ijk}^h + b[\delta_k^h R_{ij}^h - \delta_j^h R_{ik}^h + R_{k[ij}^h g_{ik]} - R_{k[ij}^h g_{ik]}^h]$$

$$- \frac{R}{n} \left( \frac{a}{n - 1} + 2b \right) [\delta_k^h g_{ij} - \delta_j^h g_{ik}]. \quad (5)$$

A Riemannian space of dimension $n(> 3)$ is said to be quasi-conformally flat if its quasi-conformal curvature tensor vanishes identically.
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It is well known that a Riemannian space $M$ is said to be locally symmetric due to Cartan if the covariant derivative of its curvature tensor vanishes, i.e., $R^h_{ijkl} = 0$, where ',' denotes the covariant differentiation with respect to the coordinates. The notion of locally symmetric spaces has been weakened by many authors in several ways such as recurrent manifolds by A. G. Walker [13], semi-symmetric manifolds by Z. I. Szabó [15], weakly symmetric manifolds by L. Tamássy and T. Q. Binh [12]. A non-flat Riemannian space is said to be recurrent [13] if there exists a non-zero covariant vector $\mu_l$ such that

$$R^h_{ijkl} = \mu_l R^h_{ijkl}. \quad (6)$$

The object of the present paper is to study a quasi-conformally flat (QE)$_n$ with recurrent curvature. (QE)$_n$ is studied in [3], [5], [6], [8], [9], [10] and also references therein.

In section 2 we investigate a necessary and sufficient condition for a quasi-conformally flat (QE)$_n$ to be recurrent. Also in a quasi-conformally flat (QE)$_n$, the form of the metric is determined (see Theorem 2.5). Section 3 deals with necessary and sufficient conditions for a quasi-conformally flat (QE)$_n$ to be concircularly (resp., projectively, conformally) recurrent.

The last section deals with a proper example of (QE)$_n$.

2 Condition for a quasi-conformally flat (QE)$_n$ to be recurrent

Let us consider a (QE)$_n$ which is quasi-conformally flat. Then from (5) it follows that

$$R^h_{ijkl} = X[\delta^h_k R_{ij} - \delta^h_j R_{ik} + R^h_k g_{ij} - R^h_j g_{ik}] + Y R[\delta^h_k g_{ij} - \delta^h_j g_{ik}], \quad (7)$$

where

$$X = -\frac{b}{a} \quad \text{and} \quad Y = \frac{1}{an(n-1)}[a + (n-1)2b] \quad \text{provided that} \quad a \neq 0. \quad (8)$$

Since the space under consideration is (QE)$_n$, its Ricci tensor $R_{ij}$ of type (0,2) can be written as (1), which yields,

$$R = n\alpha + \beta. \quad (9)$$

Using (8), (1) and (9) in (7) we get

$$R^h_{ijk} = P[\delta^h_k g_{ij} - \delta^h_j g_{ik}] + Q[\delta^h_k A_i A_j - \delta^h_j A_i A_k + g_{ij} A^h A_k - g_{ik} A^h A_j], \quad (10)$$

where $P = \frac{na^2 + b}{n(n-1)} + \frac{2b\beta}{an}$, $Q = -\frac{b}{a}\beta$ are scalars and $g^{ij} A_i = A^j$.

In [1], Amur and Maralabhavi proved that a quasi-conformally flat space
is either conformally flat or Einstein. So a non-Einstein quasi-conformally flat space is conformally flat. Again in [3], Debnath and Konar proved the following:

**Theorem 2.1** ([3]) A conformally flat quasi-Einstein space of dimension \( n \) is locally isometrically immersed in an Euclidean space of dimension \( n + 1 \) provided \( \alpha \neq 0 \) and \( \beta > (n - 2)\alpha \).

Hence in view of Theorem 2.1, we can state the following:

**Theorem 2.2** A non-Einstein quasi-conformally flat \((QE)\) with \( \alpha \neq 0 \) of dimension \( n \) is locally isometrically immersed in an Euclidean space of dimension \( n + 1 \) provided \( \alpha \neq 0 \) and \( \beta > (n - 2)\alpha \).

Again in [4] it is proved that the dimension of a \((QE)\) hypersurface of \( \mathbb{R}^{n+1} \) is odd if \( \alpha + \beta = 0 \). Hence we can state the following:

**Corollary 2.3** A non-Einstein quasi-conformally flat \((QE)\) with \( \alpha \neq 0 \) satisfying \( \alpha + \beta = 0 \) and \( \beta > (n - 2)\alpha \) is odd dimensional.

From (1) we have

\[
R_{i,j} - \mu_{l} R_{ij} = g_{ij}(\alpha_{,l} - \mu_{l}\alpha) + (\beta_{,l} - \mu_{l}\beta)A_{i}A_{j} + \beta(A_{i, l}A_{j} + A_{i}A_{j, l}),
\]

which yields

\[
R_{i,l} - \mu_{l} R = n(\alpha_{,l} - \mu_{l}\alpha) + (\beta_{,l} - \mu_{l}\beta) + \beta(A_{i, l}A_{i} + A_{j}A_{j, l}).
\]

By virtue of (9) we obtain

\[
R_{i,l} - \mu_{l} R = n(\alpha_{,l} - \mu_{l}\alpha) + (\beta_{,l} - \mu_{l}\beta).
\]

Combining (12) and (13) we get

\[
A_{i}A_{i, l} = 0,
\]

since \( \beta \neq 0 \). From the last relation, it follows that

\[
A_{i, l} = 0,
\]

i.e., \( A_{i} \) is parallel. Thus the space admits a non-null local unit parallel vector field. In ([7], p. 160) Ruse, et. al. proved the following:

**Theorem 2.4** If a Riemannian space \((M^{n}, g)\) admits a non-null local parallel vector field \( \lambda \), a coordinate system \((z^{i})\) can be chosen locally so that the metric of \((M^{n}, g)\) takes the form

\[
ds^{2} = \sum_{i, j=1}^{n-1} g_{ij}dz^{i}dz^{j} + (dz^{n})^{2}
\]

where \( g_{ij} \) is a non-singular symmetric \((n - 1) \times (n - 1)\) matrix independent of \( z^{n} \).
Since in a quasi-conformally flat \((QE)_n\), the generator \(A_i\) is a unit non-null parallel vector field, by virtue of Theorem 2.4, we can state the following:

**Theorem 2.5** In a quasi-conformally flat \((QE)_n\) with \(a \neq 0\), a coordinate system \((z^i)\) can be chosen locally so that the metric of \((M^n, g)\) takes the form (15).

Now we seek a necessary and sufficient condition for a quasi-conformally flat \((QE)_n\) to be recurrent. If the space under consideration is recurrent, then from (6) it follows that

\[ R_{ij,l} - \mu_l R_{ij} = 0, \quad (16) \]

and

\[ R_{,l} - \mu_l R = 0. \quad (17) \]

In view of (12), (13) and (17) we get (14). From (10) we have

\[
R_{ijk,l}^h - \mu_l R_{ijk}^h = [P_{,l} - \mu_l P][\delta_k^h g_{ij} - \delta_j^h g_{ik}]
+ [Q_{,l} - \mu_l Q][\delta_k^h A_i A_j - \delta_k^h A_j A_i + g_{ij} A^h A_k - g_{ik} A^h A_j]
+ Q[\delta_k^h (A_i, l A_j + A_j, l A_i) - \delta_j^h (A_i, l A_k + A_k, l A_i)]
+ Q[g_{ij} (A^h A_k, l + A^h A_k, l) - g_{ik} (A^h A_j, l + A^h A_j, l)].
\]

Using (17), in (13) we obtain

\[ n(\alpha_{,l} - \mu_l \alpha) + (\beta_{,l} - \mu_l \beta) = 0. \quad (19) \]

We suppose that \(Q_{,l} - \mu_l Q = 0\). Then we get

\[ \beta_{,l} = \mu_l \beta \] provided that \( a \neq 0 \).

Using (20) in (19) we get

\[ \alpha_{,l} = \mu_l \alpha. \quad (21) \]

From (20) and (21) it follows that

\[ \alpha \beta_{,l} - \alpha_{,l} \beta = 0, \] provided that \( a \neq 0 \).

Next, we assume that in a quasi-conformally flat \((QE)_n\) the relation (22) holds. Then by virtue of (14), (18) and (22) we get

\[ R_{ijk,l}^h = \frac{\beta_{,l}}{\beta} R_{ijk}^h \quad (23) \]

i.e.,

\[ R_{ijk,l}^h = \lambda_l R_{ijk}^h, \]

where \( \lambda_l = \frac{\beta_{,l}}{\beta} = \frac{\alpha_{,l}}{\alpha} \) is a covariant vector, i.e., the space under consideration is recurrent. Thus we can state the following:
Theorem 2.6 A quasi-conformally flat \((QE)_n\) with the relation (22) is recurrent if and only if the generator \(A_i\) is parallel.

Such a space is locally symmetric if and only if \(\alpha_{,l} = 0, \beta_{,l} = 0\). Hence we can state the following:

Corollary 2.7 A quasi-conformally flat \((QE)_n\) with constant associated scalars is locally symmetric if and only if the generator of the space \(A_i\) is parallel.

3 Conditions for a quasi-conformally flat \((QE)_n\) to be concircularly, projectively and conformally recurrent

We suppose that a Riemannian space \((M^n, g)\) satisfies the condition

\[
F^h_{ijk, l} - \mu_l F^h_{ijk} = 0,
\]

where \(F^h_{ijk}\) is a non-vanishing tensor of type \((1, 3)\) and \(\mu_l\) is a non-zero covariant vector. Then the Riemannian space is said to be concircularly (resp., projectively, conformally) recurrent if \(F^h_{ijk} = \bar{C}^h_{ijk}\) (resp., \(P^h_{ijk}, C^h_{ijk}\)), where \(P^h_{ijk}\) is the projective curvature tensor of type \((1, 3)\).

Let us consider a quasi-conformally flat \((QE)_n\) which is concircularly recurrent. Then from (24), for \(F^h_{ijk} = \bar{C}^h_{ijk}\), we have

\[
\bar{C}^h_{ijk, l} = \mu_l \bar{C}^h_{ijk},
\]

which yields

\[
\bar{C}_{ij, l} = \mu_l \bar{C}_{ij},
\]

where \(\bar{C}_{ij}\) is given by

\[
\bar{C}_{ij} = R_{ij} - \frac{R}{n} g_{ij}.
\]

Using (27) in (26) we obtain

\[
R_{ij, l} - \mu_l R_{ij} = \frac{g_{ij}}{n} (R_{,l} - \mu_l R).
\]

Using (11) and (13) in (28) and then transvecting with \(g^{ij}\), we get

\[
A_{i, l} A_i^i + A^i A_{j, l} = 0,
\]

since \(\beta \neq 0\). From (29) it follows that the relation (14) holds.

Again if \(A_i\) is parallel, then by virtue of (13), (14), (18) and (22), we have

\[
\bar{C}^h_{ijk, l} = \frac{\beta_{,l}}{\beta} \bar{C}^h_{ijk}.
\]
i.e., the space under consideration is concircularly recurrent. Hence we can state the following:

**Theorem 3.1**  A quasi-conformally flat \((QE)_n\) with the relation (22) is concircularly recurrent if and only if the generator \(A_i\) is parallel.

The projective curvature tensor \(P^h_{ijk}\) of type \((1, 3)\) of a Riemannian space \((M^n, g)\) is given by

\[
P^h_{ijk} = R^h_{ijk} - \frac{1}{n-1} [R_{ij} \delta^h_k - R_{ik} \delta^h_j].
\]  

(31)

We suppose that a quasi-conformally flat \((QE)_n\) is projectively recurrent. Then

\[
P^h_{ijk, l} = \mu_i P^h_{ijk},
\]  

(32)

where \(\mu_i\) is a non-zero covariant vector. Using (31) in (32) we obtain

\[
R^h_{ijk, l} - \mu_i R^h_{ijk} = \frac{1}{n-1} [\delta^h_k (R_{ij, l} - \mu_l R_{ij}) - \delta^h_j (R_{ik, l} - \mu_l R_{ik})].
\]  

(33)

Using (11) and (13) in (33), we obtain (14). Then in view of (11), (14), (18) and (22), we get

\[
P^h_{ijk, l} = \frac{\beta_i}{\beta} P^h_{ijk},
\]  

(34)

i.e., the space under consideration is projectively recurrent. Hence we can state the following:

**Theorem 3.2**  A quasi-conformally flat \((QE)_n\) with the relation (22) is projectively recurrent if and only if the generator \(A_i\) is parallel.

Finally, we suppose that a quasi-conformally flat \((QE)_n\) is conformally recurrent. Then we have

\[
C^h_{ijk, l} = \mu_i C^h_{ijk},
\]  

(35)

where \(\mu_i\) is a non-zero covariant vector.

Then continuing in a similar way as in the previous case, we can state the following:

**Theorem 3.3**  A quasi-conformally flat \((QE)_n\) with the relation (22) is conformally recurrent if and only if the generator \(A_i\) is parallel.
4 An example of \((QE)_4\)

Example 4.1 Let \(M^4 = \mathbb{R}^4\) be endowed with the metric

\[
ds^2 = g_{ij}dx^i dx^j = f \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] + (dx^4)^2,
\]

\((i, j = 1, 2, ..., 4),\) where \(f\) is a non-constant and nowhere vanishing continuously differentiable function of \(x^4\) such that \(f_{..} \neq 0, f_{..} - f_{.2} \neq 0, f^2 f_{..} - 3 f_{..} f_{..} + 2 f_{..} \neq 0, 0 < x^4 < \infty; x^1, ..., x^4\) are the standard coordinates of \(\mathbb{R}^4,\) where \(f_{..}\) means the third order partial differentiation with respect to \(x^4.\)

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor, the scalar curvature and the quasi-conformal curvature tensor are as follows

\[
\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = - \frac{1}{2} f_{..}, \quad \Gamma^4_{14} = \Gamma^4_{24} = \Gamma^4_{34} = \frac{1}{2f} f.
\]

\[
R_{1411} = R_{2442} = R_{4334} = \frac{1}{2} f_{..} - \frac{1}{4f} (f_{.})^2, \quad R_{2112} = R_{3113} = R_{2332} = \frac{1}{4} (f_{.})^2
\]

\[
R_{11} = R_{22} = R_{33} = \frac{1}{2} f_{..} + \frac{1}{4f} (f_{.})^2, \quad R_{44} = \frac{3}{f} \left[ \frac{1}{2} f_{..} - \frac{1}{4f} (f_{.})^2 \right],
\]

\[
R = \frac{3}{f} f_{..}, \quad W_{i44i} = \frac{1}{4} (a + 2b) \left[ f_{..} - \frac{(f_{.})^2}{f} \right], \quad i = 1, 2, 3,
\]

\[
W_{2112} = W_{3113} = W_{2332} = \frac{1}{4} (a + 2b) \left[ (f_{.})^2 - ff_{..} \right].
\]

Therefore \(\mathbb{R}^4\) with the considered metric is a Riemannian space \((M^4, g)\) of non-vanishing scalar curvature. We shall now show that this \(M^4\) is a \((QE)_4.\)

Let us now consider the associated scalars, and the components of the covariant vector \(A_i\) as follows:

\[
\alpha = \frac{1}{f} \left[ \frac{1}{2} f_{..} + \frac{1}{4f} (f_{.})^2 \right], \quad \beta = \frac{1}{f} \left[ f_{..} - \frac{1}{f} (f_{.})^2 \right]
\]

\[
A_i(x) = \begin{cases} 1 & \text{for } i = 4, \\ 0 & \text{otherwise}, \end{cases}
\]

at any point \(x \in M.\) In our \(M^4,\) \((1)\) reduces with these associated scalars and the components of the covariant vector \(A_i\) to the following equations:

\[
R_{ii} = \alpha g_{ii} + \beta A_i A_i \text{ for } i = 1, 2, 3, 4
\]

since for the cases other than \((39)\) the components of each term of \((1)\) vanishes identically and the relation holds trivially. For \(i = 1,\) in view of \((37)\) and \((38),\)
R.H.S. of (39) = \alpha g_{11} + \beta A_1 A_1 = \frac{1}{2} f \ldots + \frac{1}{17} (f_\ldots)^2 = R_{11} = \text{L.H.S. of (39)}.

Similarly it can easily be shown that (39) holds for \( i = 2, 3, 4 \). Therefore, \((M^4, g)\) is \((QE)_4\) which is neither quasi-conformally flat nor quasi-conformally symmetric. Hence we can state the following:

**Theorem 4.2** Let \((M^4, g)\) be a Riemannian space endowed with the metric given by (36). Then \((M^4, g)\) is a \((QE)_4\) with non-vanishing scalar curvature which is neither quasi-conformally flat nor quasi-conformally symmetric.

**References**


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