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Wavelet Galerkin Solutions of Ordinary Differential Equations

Vinod Mishra[1] and Sabina[2]

Sant Longowal Institute of Engineering and Technology
Longowal 148 106 Punjab, India
[1]vinodmishra.2011@rediffmail.com, [2]sabinajindal8@gmail.com

Abstract. Advantage of wavelet Galerkin method over finite difference or element method has led to tremendous applications in science and engineering. In recent years there has been increasing attempt to find solutions of differential equations using wavelet techniques. In this paper, we elaborate the wavelet techniques and apply Galerkin procedure to analyse one dimensional harmonic wave equation as a test problem using fictitious boundary approach; overcoming Dianfeng et al. (1996) reservation at higher resolution. This could have been possible only after evaluating connection coefficients at various scales.

Keywords. Wavelet Galerkin Method; Condition Number; Connection Coefficients; Moments; Harmonic Wave Equation.

1. Introduction

Wavelet functions have generated significant interest from both theoretical and applied research over the last few years. The name wavelet comes from the requirement that they should integrate to zero, waving above and below *x*-axis. The concepts for understanding wavelets were provided recently by Meyer, Mallat, Daubechies, and many others. Since then, the number of applications where wavelets have been used has exploded.

Many different types of wavelet functions have been presented over the past few years. In this paper, the Daubechies family of wavelets will be considered due to their useful properties.

Since the contribution of orthogonal bases of compactly supported wavelet by Daubechies (1988) and multiresolution analysis based fast wavelet transform algorithm by Belkin (1991), wavelet based approximation of ordinary differential equations gained momentum in attractive way. Wavelets have the capability of representing the solutions at different levels of resolutions, which make them particularly useful for developing hierarchical solutions to engineering problems.

Among the approximations, wavelet Galerkin technique is the most frequently used scheme these days. Daubechies wavelets as bases in a Galerkin method to solve differential equations require a computational domain of simple shape. This has become possible due to the remarkable work by Latto et al. [1], Xu et al. [4], Williams et al. [5 & 6] and Amartunga et al. [10]. Yet there is difficulty in dealing with boundary conditions. So far problems with periodic boundary conditions or periodic distribution have been dealt successfully. Fictitious boundary approach with Dirichlet boundaries has been applied by Dianfeng et al. [11] in analysing SH wave, but found difficulty near resonance response. In this paper, it is overcome by finding connection coefficients at different scales. Remind that Latto et al. [1] provides connection coefficients for j=0 and N=6 only. A detailed exposition of problem with distinct boundaries is also dealt by Jordi Besora [8].

Wavelet $\psi(x)$**:** An oscillatory function $\psi(x) \in L^2(R)$ with zero mean is a wavelet if it has the desirable properties:

1. ***Smoothness*****:** $\psi(x)$ is n times differentiable and that their derivatives are continuous.

2. ***Localization*****:** $\psi(x)$ is well localized both in time and frequency domains, i.e., $\psi(x)$ and its derivatives must decay very rapidly. For frequency localization $\hat{\psi}(\omega)$ must decay sufficiently fast as $|\omega| \to \infty$ and that $\hat{\psi}(\omega)$ becomes flat in the neighborhood of $\omega = 0$. The flatness is associated with number of vanishing moments of $\psi(x)$, i.e.

$$\int_{-\infty}^{\infty} x^k \psi(x)dk = 0 \text{ or equivalently } \frac{d^k}{d\omega^k}\hat{\psi}(\omega) = 0 \text{ for } k = 0, 1, ..., n$$

in the sense that larger the number of vanishing moments more is the flatness when ω is small.

3. The ***admissibility condition***

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$$

suggests that $|\hat{\psi}(\omega)|^2$ decays at least as $|\omega|^{-1}$ or $|x|^{\in -1}$ for $\in > 0$.

Daubechies Wavelet. Daubechies wavelets are compactly supported functions. This means that they have non zero values within a finite interval and have a zero value everywhere else. That's why it is useful for representing the solution of differential equation. In 1988, Ingrid Daubechies defined scaling function as

$$\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x-k),$$

where N denotes the genus of the Daubechies wavelet. The functions generated with these coefficients will have $\mathrm{supp}\,\varphi = [0, N-1]$ and $(N/2-1)$ vanishing wavelet moments.

Sometimes the scaling functions are defined as

$$\varphi(x) = \sqrt{2}\sum_{k=0}^{N-1} c_k \varphi(2x-k),$$

where $a_k = \sqrt{2}c_k$, with the property that $\sum_{k=0}^{N-1} c_k = \sqrt{2}$.

We have computed the Daubechies coefficients for N=4, 6, 8, 10, 12 in the Table given below:

Table 1.1: Daubechies Wavelet Filter Coefficients c_k

k	N=4	N=6	N=8	N=10	N=12	N=14	N=16
0	0.4830	0.3327	0.2304	0.1601	0.1115	0.0779	0.0544
1	0.8365	0.8069	0.7148	0.6038	0.4946	0.3965	0.3129
2	0.2241	0.4599	0.6309	0.7243	0.7511	0.7291	0.6756
3	-0.1294	-0.1350	-0.0280	0.1384	0.3153	0.4698	0.5854
4		-0.0854	-0.1870	-0.2423	-0.2263	-0.1439	-0.0158
5		0.0352	0.0308	-0.0322	-0.1298	-0.2240	-0.2840
6			0.0329	0.0776	0.0975	0.0713	0.0005
7			-0.0106	-0.0062	0.0275	0.0806	0.1287
8				-0.0126	-0.0316	-0.0380	-0.0174
9				0.0033	0.0006	-0.0166	-0.0441
10					0.0048	0.0126	0.0140
11					-0.0011	0.0004	0.0087
12						-0.0018	-0.0049
13						0.0004	-0.0004
14							0.0007
15							-0.0001

The associated wavelet function is given by

$$\psi(x) = \sum_{k=2-N}^{1} (-1)^k a_{1-k} \varphi(2x-k).$$

2. Wavelet Galerkin Technique

Galerkin Method. It was Russian engineer V.I. Galerkin who proposed a projection method based on weak form. In it a set of test functions are selected such that residual of differential equation becomes orthogonal to test functions [8].

Consider one-dimensional differential equation [12, pp. 451-479]

$$Lu(x) = f(x), \quad 0 \le x \le 1 \tag{2.1}$$

with Dirichlet boundary conditions

$$u(0) = a, u(1) = b.$$

f is real valued and continuous functions of x on [0,1]. L is a uniformly elliptic differential operator.

$L^2([0, 1])$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dt.$$

Suppose that $\{v_j\}$ is a complete orthonormal system for L^2 ([0, 1]) and that every v_j is C^2 on [0, 1] such that

$$v_j(0) = a, v_j(1) = b.$$

Select a finite set Λ of indices j and consider the subspace

$$S = span\{v_j : j \in \Lambda\}$$

Let the approximate solution u_s of the given equation be

$$u_s = \sum_{k \in \Lambda} x_k v_k \in S. \tag{2.2}$$

We would like to determine x_k in a way that u_s behaves as if is a true solution on S, i.e.

$$\langle Lu_s, v_j \rangle = \langle f, v_j \rangle \ \forall \ j \in \Lambda \tag{2.3}$$

such that the boundary conditions $u_s(0) = u_s(1) = 0$ are satisfied. Substituting u_s in (2.3),

$$\sum_{k \in \Lambda} \langle Lv_k, v_j \rangle x_k = \langle f, v_j \rangle \ \forall \ j \in \Lambda. \tag{2.4}$$

Let X and Y denote the vectors $(x_k)_{k \in \Lambda}$ and $(y_k)_{k \in \Lambda} = \langle f, v_k \rangle,$ and A the matrix

$$A = \lfloor a_{j,k} \rfloor_{j,k \in \Lambda}, \text{ where } a_{j,k} = \langle Lv_k, v_j \rangle.$$

(2.4) reduces to the system of linear equations

$$\sum_{k\in\Lambda} a_{j,k}\, x_k = y_j \text{ or equivalently } AX = Y. \tag{2.5}$$

Thus in Galerkin method, for each subset Λ, we find an approximation u_s in S to u by solving (2.5) for X and then substituting its components in (2.2). It is expected that as we increase Λ in some systematic way, u_s converges to u, the actual solution.

Condition Number of a Matrix. We know that a linear system $AX = Y$ has a unique solution X for every Y if a square matrix A is invertible. It is often observed that for two close values of Y and Y', X and X' are far apart. Such a linear system is called *badly conditioned*. Thus data Y is expected to be fairly accurate. *Condition number* of A is given by

$$C_\#(A) = \|A\|\ \|A^{-1}\|, \qquad (C_\#(A) \ge 1).$$

Thus $C_\#(A)$, is the measure of stability of the linear system under perturbation of the data Y. Small condition number near 1 is desirable. In case it is high, replace the system by equivalent system $BAX = BY$, B is a preconditioning matrix such that

$$C_\#(BA) < C_\#(A).$$

To facilitate easy calculation, A is considered to be *sparse*, i.e., A should have high proportion of entries 0. The best one is when A is in diagonal form.

Wavelet Galerkin method

Let $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ be a wavelet basis for $L^2([0,1])$ with boundary conditions

$$\psi_{j,k}(0) = \psi_{j,k}(1) = 0.$$

For each $(j,k) \in \Lambda$, $\psi_{j,k}$ is C^2.

The scale of ψ approximates 2^{-j} and is centralized near point $2^{-j}k$ and equates to zero outside the interval centred at $2^{-j}k$ of length proportional to 2^{-j}.

In Wavelet Galerkin method equations (2.2) and (2.3) may thus be replaced by

$$u_s = \sum_{j,k\in\Lambda} x_{j,k}\psi_{j,k}$$

and

$$\sum_{j,k\in\Lambda} \langle L\psi_{j,k}, \psi_{l,m}\rangle x_{j,k} = \langle f, \psi_{l,m}\rangle \quad \forall\ (l,m) \in \Lambda.$$

So that $AX = Y$. (2.6)

Where A= $\lfloor a_{l,m;j,k} \rfloor_{(l,m),(j,k)\in\Lambda}$, $X = (x_{j,k})_{(j,k)\in\Lambda}$, $Y = (y_{l,m})_{(l,m)\in\Lambda}$.

In it $a_{l,m;j,k} = < L\psi_{j,k}, \psi_{l,m} >$, $y_{l,m} = < f, \psi_{l,m} >$.

The pairs *(l,m)* and *(j,k)* represent respectively row and column of *A*.

Consider *A* to be sparse. Represent $AX = Y$ by equivalent

$$MZ = V \tag{2.7}$$

in which case M has relatively low condition number, if A has not. This system is now well conditioned. Again M to be sparse is desirable.

The matrix M in the preconditioned system (4.6) has condition number bounded independently of Λ. So as we increase Λ to approximate solution with more accuracy, the condition number maintains it boundedness, which is much better than the finite difference method in which case condition number grows as N^2. Thus the data errors, may be due to rounding off, has no effect in wavelet Galerkin solution over [0, 1] as we approach for better and better accuracy.

3. Connection Coefficients

In order to find the solution of differential equation by using wavelet Galerkin method there is need to find the connection coefficients as explored in Latto et al. [1] as

$$\Omega_{l_1 l_2}^{d_1 d_2} = \int_{-\infty}^{\infty} \varphi_{l_1}^{d_1}(x)\varphi_{l_2}^{d_2}(x)dx. \tag{3.1}$$

Taking derivatives of the scaling function d times. we get

$$\varphi^{d}(x) = 2^{d}\sum_{k=0}^{N-1} a_k \varphi_k^{d}(2x-k). \tag{3.2}$$

After simplification and considering it for all $\Omega_{l_1 l_2}^{d_1 d_2}$, gives a system of linear equations with $\Omega^{d_1 d_2}$ as unknown vector:

$$T\Omega^{d_1 d_2} = \frac{1}{2^{d-1}}\Omega^{d_1 d_2},$$

where $d = d_1 + d_2$ and $T = \sum_i a_i a_{q-2l+i}$.

The moments M_i^k of φ_i are defined as

$$M_i^k = \int_{-\infty}^{\infty} x^k \varphi_i(x)dx$$

with $M_0^0 = 1$.

Latto et al. [1] derives a formula to compute the moments by induction on *k*.

$$M_i^j = \frac{1}{2(2^j - 1)}\sum_{t=0}^{j}\binom{j}{t} i^{j-t}\sum_{l=0}^{t-1}\binom{t}{l}\left(\sum_{i=0}^{N-1} a_i i^{t-l}\right),$$

where the a_i s are the Daubechies wavelet coefficients.

Finally the system will be

$$\begin{pmatrix} T - \frac{1}{2^{d-1}} I \\ M^d \end{pmatrix} \Omega^{d_1 d_2} = \begin{pmatrix} 0 \\ d! \end{pmatrix}.$$

Matlab software is used to compute the connection coefficients and moments at different scales. We have computed the connection coefficients by substituting the values of Daubechies coefficients in the matrix T and by evaluating moments by using the programme that is given in [8].
Latto et al. [1] computed the connection coefficients at j=0 and N=6 only. We have computed the connection coefficients at all values of j and N. The values of connection coefficients, for example, at j=0, 4, 7 and N=6 and at j=5 and N=12 are shown in Table 3.1.

Table 3.1: 2- term connection coefficients

Connection Coefficients at N=6, j=0, $d1$=2, $d2$=0

$\Omega[-4] =$ 5.357142857144194e-003
$\Omega[-3] =$ 1.142857142857108e-001
$\Omega[-2] =$ -8.761904761904359e-001
$\Omega[-1] =$ 3.390476190476079e+000
$\Omega[0] =$ -5.267857142857051e+000
$\Omega[1] =$ 3.390476190476190e+000
$\Omega[2] =$ -8.761904761904867e-001
$\Omega[3] =$ 1.142857142857139e-001
$\Omega[4] =$ 5.357142857141956e-003

Connection Coefficients at N=6, j=7, $d1$=2, $d2$=0

$\Omega[-4] =$ 8.777142857143141e+001
$\Omega[-3] =$ 1.872457142857136e+003
$\Omega[-2] =$ -1.435550476190484e+004
$\Omega[-1] =$ 5.554956190476204e+004
$\Omega[0] =$ -8.630857142857112e+004
$\Omega[1] =$ 5.554956190476144e+004
$\Omega[2] =$ -1.435550476190458e+004
$\Omega[3] =$ 1.872457142857130e+003
$\Omega[4] =$ 8.777142857143379e+001

Connection Coefficients at N=12, j=5, d=2

$\Omega[10] =$ -1.294452960894988e-008
$\Omega[9] =$ 2.693100643685327e-005
$\Omega[8] =$ -3.549272110888916e-003
$\Omega[7] =$ -5.566810017417083e-002
$\Omega[6] =$ -6.727300176318085e-002
$\Omega[5] =$ 6.633534506945016e+000
$\Omega[4] =$ -5.054628929484702e+001
$\Omega[3] =$ 2.098232006991124e+002
$\Omega[2] =$ -6.458708407940719e+002
$\Omega[1] =$ 2.367351361198240e+003
$\Omega[0] =$ -3.774529005718791e+003
$\Omega[-1] =$ 2.367351361198260e+003
$\Omega[-2] =$ -6.458708407940832e+002
$\Omega[-3] =$ 2.098232006991146e+002
$\Omega[-4] =$ -5.054628929484773e+001
$\Omega[-5] =$ 6.633534506945424e+000
$\Omega[-6] =$ -6.727300176361031e-002
$\Omega[-7] =$ -5.566810017416311e-002
$\Omega[-8] =$ -3.549272110869743e-003
$\Omega[-9] =$ 2.693100643242748e-005
$\Omega[-10] =$ -1.294445771987477e-008

4. Wavelet Methods for ODE

Consider the equation [Amaratunga et al.]

$$\frac{\partial^2 u}{\partial x^2} + \alpha u = f, \tag{4.1}$$

where u, f are periodic in x of period $d \in Z$.

The Wavelet Galerikin solution of periodic problem is slightly more complicated than the finite difference approach as the former involves solving a set of

simultaneous equations in wavelet space and not in physical space. The solution in wavelet space is then transformed back to physical space by FTT.
Let the wavelet expansion $u(x)$ at scale j be

$$u(x) = \sum_k c_k 2^{j/2} \varphi(2^j x - k), \quad k \in Z, \tag{4.2}$$

where c_k s are periodic wavelet coefficients of u, periodic in x.
Put $y = 2^j x$ so that

$$U(y) = u(x) = \sum_k C_k \varphi(y-k), \; C_k = 2^{j/2} c_k .$$

If d is the period of u, then $U(y)$ and so also C_k is periodic in y with period $2^j d$.
Let us discretize $U(y)$ at all dyadic points $x = 2^{-j} \; y, \; y \in Z$

$$U_i = \sum_k C_k \varphi_{i-k} = \sum_k C_{i-k} \varphi_k, \; i = 0, 1, 2, \ldots, n-1.$$

The matrix representation is $U = k_\varphi * C$, where k_φ is the convolution kernal, i.e. the first column of the scaling function matrix.
Similarly the wavelet expansion for $f(x)$,

$$f(x) = \sum_k d_k 2^{j/2} \varphi(2^j x - k), \quad k \in Z. \tag{4.3}$$

$$F(y) = f(x) = \sum_k D_k \varphi(y-k), \; D_k = 2^{j/2} d_k .$$

And the matrix representation is

$$F = k_\varphi * D.$$

Substitute the expansions of $u(x)$ and $f(x)$ in (4.1) and then take inner product on both sides with $\varphi(y-j), \; j \in Z$.
Use $\Omega_{j-k} = \int \varphi^{(n)}(y-k)\varphi(y-j)dy$ and $\delta_{jk} = \int \varphi(y-k)\varphi(y-j)dx$, we obtain $k_\Omega . C = g.$ Now take Fourier Transforms

$$\hat{U} = \hat{k}_\varphi . \hat{C}.$$

$$\hat{F} = \hat{k}_\varphi . \hat{D}.$$

$$\hat{k}_\Omega . \hat{C} = \hat{g}.$$

Subsequently, $\hat{U} = \hat{F} / \hat{k}_\Omega$. Inverse FT will give U.
Wherein . and / denote component by component multiplication and division respectively.

Fictitious Boundary Approach [Dianfeng et al.]
Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \alpha u = 0,\ x \in [0,1] \tag{4.4}$$

with Dirichlet's boundaries $u(0), u(1)$.
u in (4.2) is periodic in x of period $d \in Z$ (k varies from $-N+1$ to 2^j).
The boundaries of the support of (4.2) are $\frac{-N+1}{2^j}$ and $\frac{N-1+2^j}{2^j}$. Subsequently, the original boundaries 0 and 1 now changes to the Fictitious Boundaries, i.e. boundary on both sides of 0 and 1 are extended by an amount $\frac{N-1}{2^j}$,

$$\varphi(0) = u\left(\tfrac{-N+1}{2^j}\right)$$

$$\varphi(N-1) = u\left(\tfrac{N-1+2^j}{2^j}\right)$$

without affecting the solution within [0,1], the affected solution is within $[\frac{-N+1}{2^j},0]$ and $[1,\frac{N-1+2^j}{2^j}]$. The equation (4.4) now reduces to

$$2^{2j}\sum_{k=-N+1}^{2^j} C_k \varphi''(y-k) + \alpha \sum_{k=-N+1}^{2^j} C_k \varphi(y-k) = 0. \tag{4.5}$$

Inner product is taken on both sides of (4.5) with $\varphi(y-n)$ taking the integration limits to $\frac{-N+1}{2^j}, \frac{N-1+2^j}{2^j}$. We obtain

$$2^{2j}\sum_k C_k \Omega_{j-k} + \alpha C_j = 0. \tag{4.6}$$

The Dirichlet boundary conditions give equations

$$\sum_{k=-N+1}^{2^i} C_k \varphi(-k) = u(0). \tag{4.7a}$$

$$\sum_{k=-N+1}^{2^i} C_k \varphi(1-k) = u(1). \tag{4.7b}$$

Take inner product with $\varphi(-l)$ and $\varphi(1-l)$ respectively reducing the left boundary condition to $\sum_{k=-N+1}^{2^i} C_k \delta_{lk}(0) = u(0)$ and $\sum_{k=-N+1}^{2^i} C_k \delta_{lk}(1) = u(1).$ First and last equations are replaced by the following equations. The place of corresponding connection coefficients in first row and last row are determined by the delta function. Appropriate connection coefficients are used to solve the ill-conditioned system for C_k .

Capacitance Matrix and the Boundary Conditions [Amaratunga et al.]
Consider the equation (4.1) defined over [a,b] with Dirichlet boundary conditions

$$u(a) = u_a \, \& \, u(b) = u_b.$$

Let u and f are periodic with period d, $0 < a < b < d$.

If f is not periodic, it can be make periodic making it zero or extending smoothly outside $[a,b]$. Let $v(x)$ be the solution in $[a,b]$ with periodic boundary conditions. The solution $u(x)$ with Dirichlet boundary conditions is by adding another functions $w(x)$ such that

$$u = v + w. \tag{4.8}$$

From (18) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} + \alpha v + \alpha w = f$, i.e.

$$\frac{\partial^2 w}{\partial x^2} + \alpha w = 0 \ \text{ in } [a,b].$$

However on or outside $[a,b]$, w_{xx} may take such values as to make u satisfy the given boundary conditions. The desired effect may be achieved by placing sources (or delta functions) along the boundary $[0,d]$ which encompasses $[a,b]$. In other words, we need the solution w to

$$\frac{\partial^2 w}{\partial x^2} + \alpha w = X \ \text{ in } [0,d],$$

where $X = X(x) = X_a \delta(x-a) + X_b \delta(x-b)$.

X_a, X_b are constants, and δ stands for delta function.

Now Green's function $G(x)$ (or impulse response) of the differential equation is given by

$$\frac{\partial^2 G}{\partial x^2} + \alpha G = \delta(x).$$

So the solution $w = G(x) * X(x) = X_a G(x-a) + X_b G(x-b)$. (4.9)

From (4.8) and (4.9)

$$w(a) = X_a G(o) + X_b G(a-b) = u_a - v(a).$$

$$w(b) = X_a G(b-a) + X_b G(o) = u_b - v(b).$$

Equivalently,

$$\begin{bmatrix} G(o) & G(a-b) \\ G(b-a) & G(o) \end{bmatrix} \begin{bmatrix} X_a \\ X_b \end{bmatrix} = \begin{bmatrix} u_a - v(a) \\ u_b - v(b) \end{bmatrix}.$$

Solving for X_a, X_b, the solution w is obtained.

Offsetting the boundary sources. Placing the sources at the boundary in the wavelet method amounts to large error due to finite support of δ ; i.e, number of non zero wavelet coefficients. Actually, support of δ in wavelet space is equal to support of φ that is used to define it.

$$\delta(x) = \sum_k g_k \varphi(y-k),$$

where wavelet coefficients

$$g_k = 2^m \varphi(-k).$$

Number of wavelet coefficients $= 2^m d$.

The magnitude s of the offset is so chosen that number of discritization point involved is at least equal to the support of scaling function

Suppose

$$a_1 = a - s, \quad b_1 = b + s.$$

Then $X = X_a \delta(x - a_1) + X_b \delta(x - b_1)$ so that X_a, X_b are solutions of

$$\begin{bmatrix} G(a-a_1) & G(a-b_1) \\ G(b-a_1) & G(b-b_1) \end{bmatrix} \begin{bmatrix} X_a \\ X_b \end{bmatrix} = \begin{bmatrix} u_a - v(a) \\ u_b - v(b) \end{bmatrix}.$$

5. Test Problem

We apply fictitious boundary approach to analyse harmonic wave (differential) equation:

$$\frac{\partial^2 u}{\partial x^2} + \alpha u = 0, \; x \in [0,1] \tag{5.1}$$

with Dirichlet boundary conditions $u(0) = 1$ and $u(1) = 0$. $\alpha = (9.5\pi)^2 = 891$.

In [8], this problem has been treated with boundary $u(0) = 1$ & $u(1) = 0$.

Exact solution is: $u = cos\sqrt{\alpha}\, x - cot\sqrt{\alpha}\, sin\sqrt{\alpha}\, x.$

Let the solution be $u(x) = \sum_{k=-N+1}^{2^j} c_k 2^{j/2} \varphi(2^j x - k)$.

Fictitious boundary approach is applied in which original boundary [0,1] is extended on both ends by margin $\frac{N-1}{2^j}$. Proceeding as in Latto et al. [1], (5.1) reduces to

$$\sum_k C_k \Omega_{n,k} + \alpha \sum_k C_k \delta_{n,k} = 0 \tag{5.2}$$

where

$$\Omega_{n,k} = \int \varphi''(2^j x - n) \varphi(2^j x - k) dx$$

and

$$\delta_{n,k} = \int \varphi(2^j x - n)\varphi(2^j x - k)dx.$$

We compute the solution of equation (5.2) by replacing the first and last equations by the equations obtained by using the right and left boundaries which also represent the relations of the coefficients c_k as explained in Section 4. After replacing the rows, we get the following matrix form with N=6:

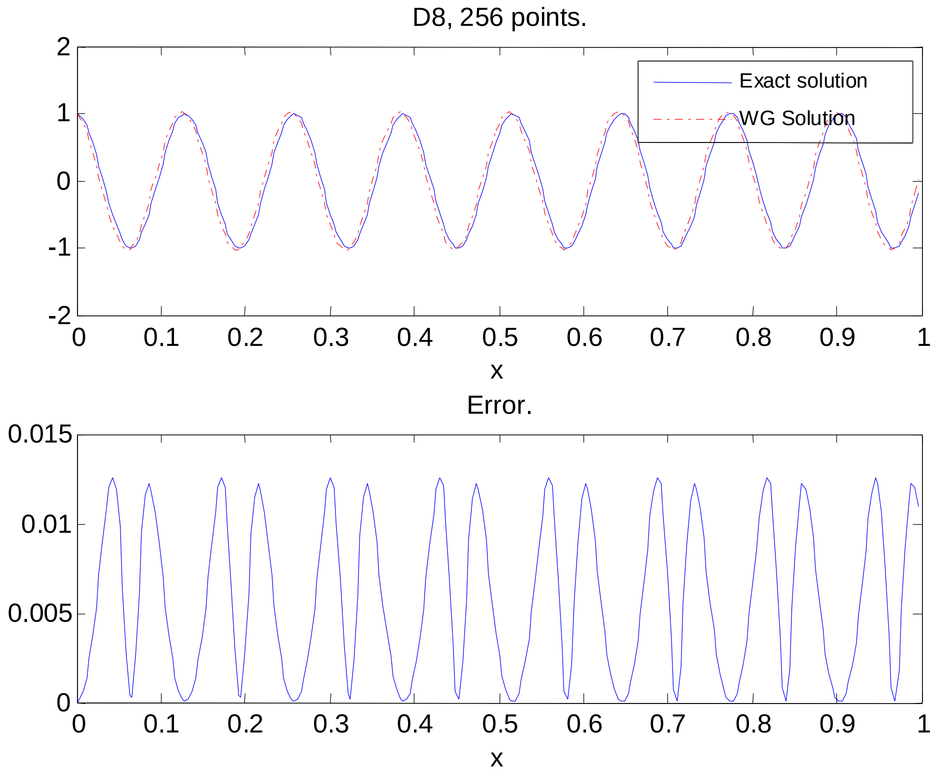
$$TC = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots 0 & 0 \\ \Omega_1 & \Omega_0+\alpha & \Omega_{-1} & \dots & 0 & \Omega_{-4} & 0 & \dots & 0 & 0 & 0 & \dots 0 & 0 \\ : & : & : & : & : & : & : & : & : & : & : & : \; : & : \\ \Omega_{-4} & \Omega_{-3} & \dots & \dots & \dots & \Omega_0+\alpha & \dots & \dots & \dots & 0 & 0 & \dots \; \dots & 0 \\ : & : & : & : & : & : & : & : & : & : & : & : \; : & : \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & \dots & 1 & 0 & 0 & \dots 0 & 0 \end{bmatrix}_{(2^j+4)*(2^j+6)}$$

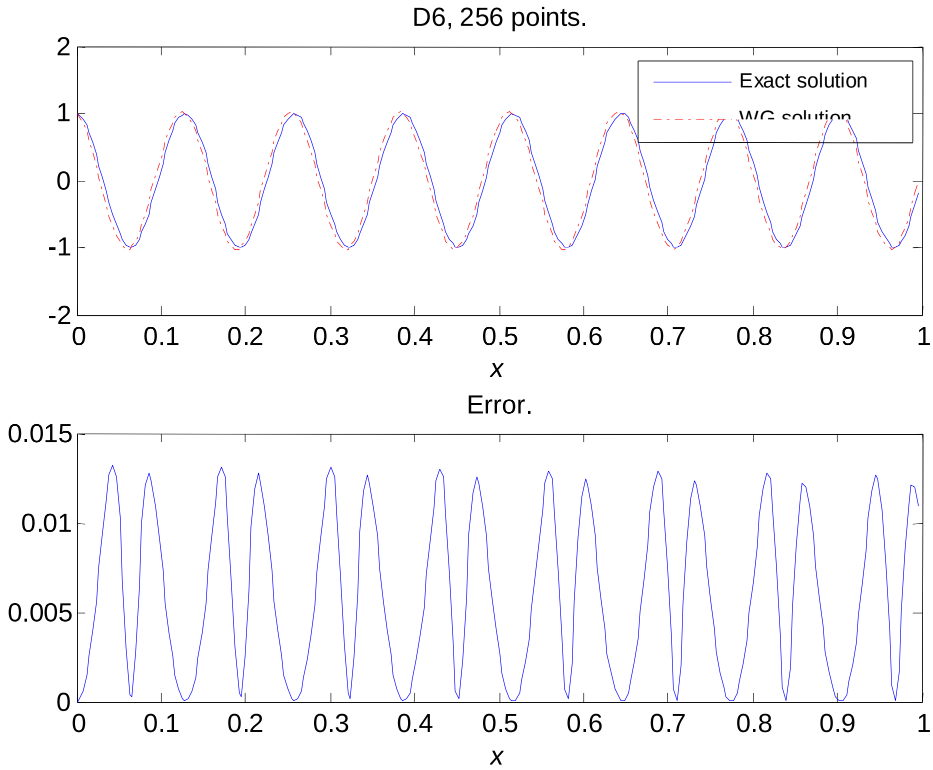
$$\text{where } C = \begin{bmatrix} C_{-5} \\ C_{-4} \\ C_{-3} \\ C_{-2} \\ C_{-1} \\ : \\ : \\ C_{2^j} \end{bmatrix}_{1*(2^j+6)}$$

Now by applying Gauss elimination method and using the programming of Matlab this matrix can be easily solved. Thus solution is obtained directly by substituting the values of c_k.

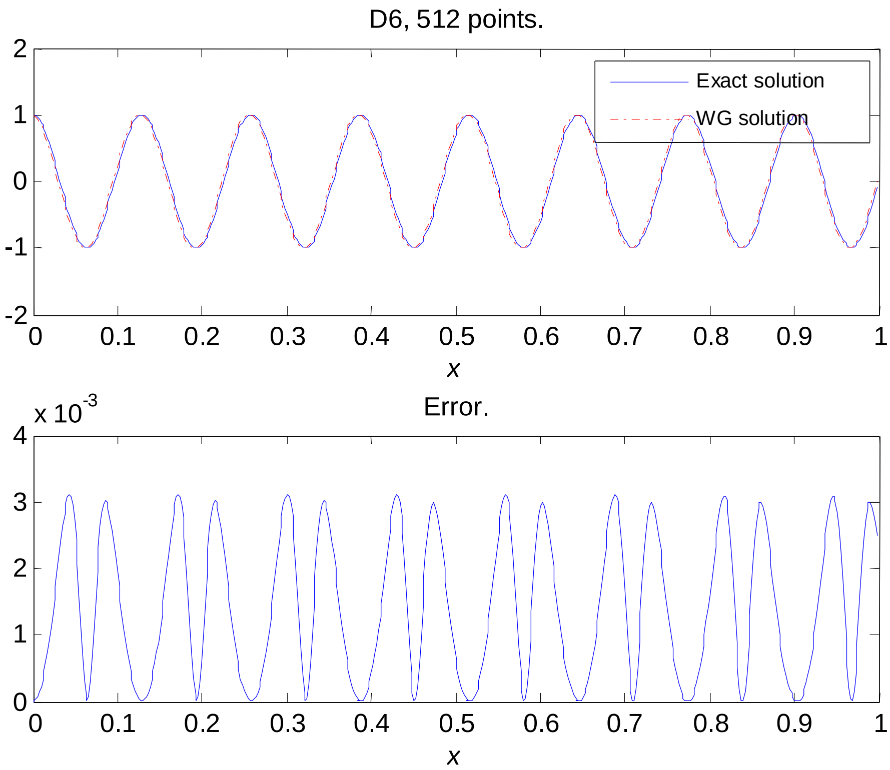
For N=8, j=8, the graph of solution of exact solution and wavelet solution



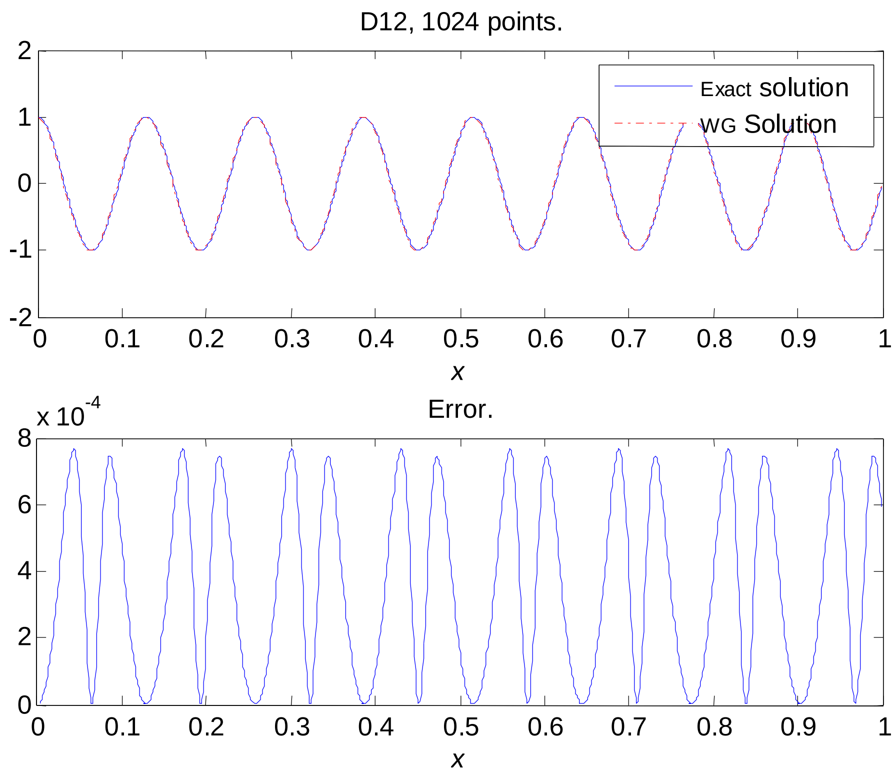
For N=6, j=8, the graph of solution of exact solution and wavelet solution



For N=6, j=9, graph shows that error decreases between exact and wavelet solution.



For N=12, j=10, graph shows that error decreases between exact and wavelet solution.



6. Conclusion

The wavelet method has been shown to be a powerful numerical tool for the fast and accurate solution of differential equations. Solutions obtained using the Daubechies 6, 8 and 12 coefficients wavelets have been compared with the exact solutions. In solving harmonic equation α is chosen to be 891. Matching solutions are obtained for N=6, j=9 and N=12, j=10. Condition numbers for D12 are constantly lower than for D6, less errors are shown for former. Dianfeng et al. [11] is silent for higher values of resolution near resonance point but here good solution is shown to exist for D12.

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