Generalization of a Fixed Point Theorem in Cone Metric Spaces

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Abstract
Let $P$ be a subset of a Banach space $E$ and $P$ is normal and regular cone on $E$, we prove the existence of the fixed point for multivalued maps in cone metric spaces and these theorems generalize the Bose and Mukerjee results and the results of varies authors.

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1 Introduction

In recent years, several authors(see[1-6]) have studied the strong convergence to a fixed point with contractive constant in cone metric spaces. Seong Hoon Cho and Mi Sun Kim [6]have proved certain fixed point theorems by using Multivalued mapping in the setting of contractive constant in metric spaces. We first recall definitions and Bose and Mukerjee results that are needed in the sequel.

2 Preliminary Notes

Let $E$ be a Banach space and a subset $P$ of $E$ is said to be a cone if it satisfies the following conditions,
(i) $P \neq \emptyset$ and $P$ is closed;

(ii) $ax + by \in P$ for all $x, y \in P$ and $a, b$ are non-negative real numbers;

(iii) $P \cap (-P) = \emptyset$.

The partial ordering $\leq$ with respect to the cone $P$ by $x \leq y$ if and only if $y - x \in P$. If $y - x \in$ interior of $P$, then it is denoted by $x \ll y$. The cone $P$ is said to be a Normal if a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The cone $P$ is called regular if every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent.

**Definition 2.1.** Let $X$ be a non-empty set, and suppose the mapping $d : X \times X \rightarrow E$ is said to be a Cone metric space if it satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

**Example 2.2.** Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}, X = R$ and $d : X \times X \rightarrow E$ defined by $d(x, y) = (|x - y|, \alpha |x - y|)$ where $\alpha \geq 0$ is a constant. Then $(X, d)$ is a cone metric space[1].

**Definition 2.3.** Let $(X, d)$ be a metric space, $x \in X$ and $\{x_n\}$ a sequence in $X$. Then

(i) $\{x_n\}$ converges to $x$ whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x) \ll c$ for all $n \geq N$.

(ii) $\{x_n\}$ is a cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

**Definition 2.4.** Let $(X, d)$ is said to be a complete cone metric space, if every cauchy sequence is convergent in $X$.

Let $(X, d)$ be a metric space. We denote by $CB(X)$ the family of nonempty closed bounded subset of $X$ and let $C(X)$ denote the set of all nonempty compact subsets of $X$. Let $H(., .)$ be the Hausdorff distance on $CB(X)$. That is, for $A, B \in CB(X)$,

$$H(A, B) = \max\{\sup_{a \in A}d(a, B), \sup_{b \in B}d(A, b)\}$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from the point $a$ to the subset $B$. A multivalued mapping $T : X \rightarrow CB(X)$ is called a contraction mapping if there exists $q \in (0, 1)$ such that $H(Tx, Ty) \leq qd(x, y)$ $\forall x, y \in X$ and $x \in X$ is said to be a fixed point of $T$ if $x \in T(X)$. 

Theorem 2.5. Bose and Mukherjee(1977)]. Let $X$ be a complete metric space and let $T_1, T_2 : X \to CB(X)$ be two multivalued mappings satisfying the following conditions: for any $x, y \in X$, $H(T_1(x), T_2(y)) \leq \alpha_1 d(x, T_1(x)) + \alpha_2 d(y, T_2(y)) + \alpha_3 d(x, T_1(x)) + \alpha_4 d(x, T_2(y)) + \alpha_5 d(x, y)$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are non negative real numbers and $\Sigma_{i=1}^{5} \alpha_i < 1$ and $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$. Then there exists an element $x \in X$ such that $x \in T_1(x), x \in T_2(x)$.

3 Main Results

Theorem 3.1. Let $(X, d)$ be a complete cone metric space and let mapping $T_1, T_2 : X \to C(X)$ satisfying the following conditions

(i) for each $x \in X$, $T_1(x), T_2(x) \in CB(X)$.

(ii) $H(T_1(x), T_2(y)) \leq \alpha_1 d(x, T_1(x)) + \alpha_2 d(y, T_2(y)) + \alpha_3 d(x, T_1(x)) + \alpha_4 d(x, T_2(y)) + \alpha_5 d(x, y)$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are non negative real numbers and $\Sigma_{i=1}^{5} \alpha_i < 1$ and $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$. Then there exists $p \in X$ such that $p \in T_1(x) \cap T_2(x)$.

Proof. Let $x_0 \in X$, $T_1(x_0)$ is a nonempty closed bounded subset of $X$. choose $x_1 \in T_1(x_0)$, for this $x_1$ by the same reason mentioned above $T_2(x_1)$ is nonempty closed bounded subset of $X$.

Since, $x_1 \in T_1(x_0)$ and $T_1(x_0)$ and $T_2(x_1)$ are closed bounded subset of $X$, $\exists x_2 \in T_2(x_1)$ such that $d(x_1, x_2) \leq H(T_1(x_0), T_2(x_1)) + q$ since $q = \max \{\frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3}, \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_4}\}$, hence $q \in (0, 1)$. Then we have

\[
d(x_1, x_2) \leq H(T_1(x_0), T_2(x_1)) + q \\
\leq \alpha_1 d(x_0, T_1(x_0)) + \alpha_2 d(x_1, T_2(x_1)) + \alpha_3 d(x_0, T_2(x_1)) + \alpha_4 d(x_1, T_1(x_0)) + \alpha_5 d(x_0, x_1) + q. \\
\leq \alpha_1 d(x_0, x_1) + \alpha_2 d(x_1, x_2) + \alpha_3 d(x_0, x_2) + \alpha_4 d(x_1, x_1) + \alpha_5 d(x_0, x_1) + q. \\
\leq \alpha_1 d(x_0, x_1) + \alpha_2 d(x_1, x_2) + \alpha_3 d(x_0, x_2) + \alpha_4 d(x_1, x_1) + \alpha_5 d(x_0, x_1) + q.
\]

\[
(1 - \alpha_2 - \alpha_3)d(x_1, x_2) \leq (\alpha_1 + \alpha_3 + \alpha_5)d(x_0, x_1) + q.
\]

\[
d(x_1, x_2) \leq \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3} d(x_0, x_1) + q.
\]

\[
d(x_1, x_2) \leq q d(x_0, x_1) + q.
\]

For this $x_2$, $T_1(x_2)$ is a nonempty closed bounded subset of $X$. Since $x_2 \in T_2(x_1)$ and $T_2(x_1)$ and $T_1(x_2)$ are closed bounded subset of $X$, $\exists x_3 \in T_1(x_2)$ such that
Continuing this process, we get a sequence \( \{x_n\} \) such that \( x_{n+1} \in T_2(x_n) \) or \( x_{n+1} \in T_1(x_n) \) and \( d(x_{n+1}, x_n) \leq q^n d(x_0, x_1) + nq^n \).

Let \( 0 \ll c \) be given, choose a natural number \( N_1 \) such that \( q^n d(x_0, x_1) + nq^n \ll c \) for all \( n \geq N_1 \) this implies \( d(x_{n+1}, x_n) \ll c \).

\( \cdot \) \( \{x_n\} \) is a cauchy sequence in \( (X, d) \) is a complete cone metric space, there exists \( p \in X \) such that \( x_n \to p \).

Choose a natural number \( N_2 \) such that \( d(x_n, p) \ll \frac{c(1-\alpha_1+\alpha_4)}{2m(1+\alpha_3)} \) and \( d(x_{n-1}, p) \ll \frac{c(1-\alpha_1+\alpha_4)}{2m(\alpha_4+\alpha_5)} \) for all \( n \geq N_2 \).

\[
\begin{align*}
d(x_2, x_3) & \leq H(T_1(x_2), T_2(x_1)) + q^2 \\
& \leq \alpha_1 d(x_2, T_1(x_2)) + \alpha_2 d(x_1, T_2(x_1)) + \alpha_3 d(x_2, T_2(x_1)) \\
& + \alpha_4 d(x_1, T_1(x_2)) + \alpha_5 d(x_1, x_2) + q^2. \\
& \leq \alpha_1 d(x_2, x_3) + \alpha_2 d(x_1, x_2) + \alpha_3 d(x_2, x_2) \\
& + \alpha_4 d(x_1, x_3) + \alpha_5 d(x_1, x_2) + q^2. \\
& \leq \alpha_1 d(x_2, x_3) + \alpha_2 d(x_1, x_2) + \alpha_3 d(x_1, x_2) \\
& + \alpha_4 d(x_2, x_3) + \alpha_5 d(x_1, x_2) + q^2. \\
(1 - \alpha_1 - \alpha_4) d(x_2, x_3) & \leq (\alpha_2 + \alpha_4 + \alpha_5) d(x_1, x_2) + q^2. \\
d(x_2, x_3) & \leq \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_4} d(x_1, x_2) + q^2. \\
& \leq q \{ q(d(x_0, x_1) + q) + q^2. \\
& \leq q^2 d(x_0, x_1) + 2q^2.
\end{align*}
\]
for all $n \geq N_2$, $d(T_1(p), p) \ll \frac{c}{m}$ for all $m \geq 1$, we get $\frac{c}{m} - d(T_1(p), p) \in P$, and as $m \to \infty$ we get $\frac{c}{m} \to 0$ and $P$ is closed $-d(T_1(p), p) \in P$, but $d(T_1(p), p) \in P$. therefore $d(T_1(p), p) = 0$. and so $p \in T_1(p)$.

Similarly we can prove that $p \in T_2(p)$. Hence $p \in T_1(p) \cap T_2(p)$.

**Corollary 3.2.** Let $(X, d)$ be a complete cone metric space and let mapping $T : X \to C(X)$ satisfying the following conditions

(i) for each $x \in X$, $T(x), T(y) \in CB(X)$.

(ii) $H(T_1(x), T_2(y)) \leq q d(x, y)$ for some $q \in [0, 1)$.

Then there exists an element $p \in X$ such that $p \in T(p)$.

**Proof.** The proof of the corollary immediately follows by putting $T_1 = T_2 = T$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, $\alpha_5 = q$ in the previous theorem.

**Corollary 3.3.** Let $X$ be a complete cone metric space and let mapping $T_1, T_2 : X \to C(X)$ be two multi-valued mappings satisfying the following conditions

For any $x, y \in X$, $H(T_1(x), T_2(y)) \leq \alpha_1 d(x, T_1(x)) + \alpha_2 d(y, T_2(y)) + \alpha_3 d(y, T_1(x)) + \alpha_4 d(x, T_2(y)) + \alpha_5 d(x, y)$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are non negative real numbers and $\Sigma_{i=1}^{5} \alpha_i < 1$ and $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$. Then there exists $x \in X$ such that $x \in T_1(x)$ and $x \in T_2(x)$.

**References**


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