Precise Asymptotics in Complete Moment Convergence of the Uniform Empirical Process

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Abstract

In this paper, we investigate the Precise asymptotics in complete moment convergence of the uniform empirical process.

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1 Introduction and main results

Let \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\} be a sequence of independent and identically distributed U[0, 1] random variables. Define the uniform empirical process

\[ F_n(t) = n^{-\frac{1}{2}} \sum_{i=1}^{n} (I_{\{\varepsilon_i \leq t\}} - t), \quad 0 \leq t \leq 1, \quad \|F_n\| = \sup_{0 \leq t \leq 1} |F_n(t)|. \]

Let U be the Brownian bridge of D[0, 1], also let log \(x = \ln(x \vee e)\) and C denote absolute positive constants, whose value can differ in different places.

Also let \(X, X_1, X_2, \ldots\) be i.i.d. random variables, and set \(S_n = X_1 + X_2 + \cdots + X_n, n \geq 1\). Hsu and Robbins (1947) introduced the concept of complete convergence and prove that the sequence of arithmetic means converges completely provided the mean and the variance exist. The converse was proved by
Erdős (1949, 1950), more generally, it was shown in Baum and Katz (1965) that for $0 < p < 2$ and $r \geq p$,

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P(|S_n| \geq \varepsilon n^{\frac{1}{p}}) < \infty, \quad \varepsilon > 0,$$

(1)

if and only if $E|X|^r < \infty$, and, when $r \geq 1$, $EX = 0$.

Another extension departs from the observation that the sum tends to infinity as $\varepsilon \searrow 0$. A first result in this direction was Heyde (1975), who proved that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n^{\frac{1}{p}-1}) = \frac{p}{r-p} E\|U\|^{\frac{2(r-p)}{r-p}}.$$

(2)

whenever $EX = 0$ and $EX^2 < \infty$. For analogous result in the more general case, see Chen (1978), Spătaru (1999) and Gut and Spătaru (2000). Recently, Zhang and Yang (2008) established the Precise asymptotics for the uniform empirical process, and the results are as follows:

**Theorem 1** For $1 \leq p < 2$, $r > p$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P\{\|F_n\| \geq \varepsilon n^{\frac{1}{p}-\frac{1}{2}}\} = \frac{p}{r-p} E\|U\|^{\frac{2(r-p)}{r-p}}.$$

**Theorem 2** For any $d > -1$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P\{\sup_{0 \leq t \leq 1} F_n(t) \geq \varepsilon n^{\frac{1}{p}-\frac{1}{2}}\} = \frac{p}{2-\frac{r-p}{2}} \frac{1}{\Gamma\left(\frac{r-p}{2-p}\right)}.$$

**Theorem 3** For $1 \leq p < 2$, $r > p$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P\{\sup_{0 \leq t \leq 1} F_n(t) \geq \varepsilon n^{\frac{1}{p}-\frac{1}{2}}\} = \frac{p}{2-\frac{r-p}{2}} \frac{1}{\Gamma\left(\frac{r-p}{2-p}\right)}.$$

**Theorem 4** For any $d > -1$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P\{\sup_{0 \leq t \leq 1} F_n(t) \geq \varepsilon n^{\frac{1}{p}-\frac{1}{2}}\} = \frac{p}{2d+1} \Gamma(d+1).$$
Also for i.i.d. random variables, Chow (1998) discussed the complete moment convergence, and obtained the following results.

**Theorem 5** Let \( \{X, X_k; k \geq 1\} \) be a sequence of i.i.d. random variables with \( E[X] = 0 \). Suppose that \( p \geq 1, \alpha > \frac{1}{2}, p\alpha > 1, E\{|X|^p + |X| \log(1 + |X|)\} < \infty \). Then for any \( \varepsilon > 0 \), we have

\[
\sum_{n=1}^{\infty} n^{p\alpha - 2 - \alpha} E \left\{ \max_{j \leq n} \left| \sum_{k=1}^{j} X_k \right| - \varepsilon n^\alpha \right\}_+ < \infty,
\]

where \( \{x\}_+ = \max(x, 0) \).

In this paper, we investigate the Precise asymptotics in complete moment convergence of the uniform empirical process. Our main results are as follows.

**Theorem 1.1** For \( 1 \leq p < 2, d > -1 \), we have

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \frac{2(1+d)}{2-p} \sum_{n=3}^{\infty} \frac{(\log n)^d}{n} E \left\{ \|F_n\| - \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \right\}_+ = \frac{1}{1+d} \frac{1}{2^d p (1+d) + \frac{1}{2}} \Gamma \left( \frac{p}{2-p} (1+d) + \frac{1}{2} \right).
\]

**Theorem 1.2** For \( 1 \leq p < 2, r > p \), we have

\[
\lim_{\varepsilon \to 0} \varepsilon^{2(r-p)} \sum_{n=1}^{\infty} n^{r-2} E \left\{ \|F_n\| - \varepsilon n^{\frac{1}{p} - \frac{1}{2}} \right\}_+ = \frac{p}{r-p} \frac{1}{2^\frac{r-p}{2-p} + \frac{1}{2}} \Gamma \left( \frac{r-p}{2-p} + \frac{1}{2} \right).
\]

**Theorem 1.3** For any \( d > -1 \), we have

\[
\lim_{\varepsilon \to 0} \varepsilon^{2(1+d)} \sum_{n=1}^{\infty} \frac{(\log n)^d}{n} E \left\{ \|F_n\| - \varepsilon (\log n)^{\frac{1}{2}} \right\}_+ = \frac{1}{1+d} \frac{1}{2^{d+\frac{1}{2}}} \Gamma \left( d + \frac{3}{2} \right).
\]
Theorem 1.4  For \(d > 0\), \(\frac{1}{2} < b + \frac{1}{d} < 1\), we have

\[
\lim_{\varepsilon \to 0} \varepsilon^{2b+\frac{d}{2}-1} \sum_{n=1}^{\infty} \frac{(\log n)^{bd-d}}{n} E \left\{ \sup_{0 \leq t \leq 1} U(t) - \varepsilon (\log n)^\frac{d}{2} \right\}^+
\]
\[
= \frac{1}{2bd + 2 - 2d} \left( \frac{1}{2} - \frac{1}{d} + \frac{1}{2} \Gamma \left( b + \frac{1}{d} + \frac{1}{2} \right) \right).
\]

Theorem 1.5  For \(1 \leq p < 2\), \(r > p\), we have

\[
\lim_{\varepsilon \to 0} \varepsilon^{2(r-p)-2} \sum_{n=1}^{\infty} n^{r-2p} E \left\{ \sup_{0 \leq t \leq 1} U(t) - \varepsilon n^{\frac{r}{2} - \frac{1}{2}} \right\}^+
\]
\[
= \frac{p}{r-p} \frac{1}{2 - \frac{r}{2} + \frac{1}{2}} \Gamma \left( \frac{r}{2} - \frac{p}{2} + \frac{1}{2} \right).
\]

2  Some lemmas and theorems

Lemma 2.1 (Lemma 2.1 of Zhang and Yang (2008))  For any \(x > 0\), \(y > 0\), \(r > 0\), we have \(E\|U\|^r < \infty\), and

\[
|P\{\|U\| \geq x\} - P\{\|U\| \geq y\}| \leq 2P\{x \land y \leq \sup_{0 \leq t \leq 1} U(t) < x \lor y\}, \quad (3)
\]
\[
P\{\sup_{0 \leq t \leq 1} U(t) \geq x\} = e^{-2x^2}, \quad (4)
\]
\[
E\left( \sup_{0 \leq t \leq 1} U(t) \right)^r = \frac{r}{2^r + 1} \Gamma \left( \frac{r}{2} \right), \quad (5)
\]
\[
P\{\|U\| \geq x\} \leq 2e^{-2x^2}. \quad (6)
\]

Theorem 2.1 (lemma 3.2.3 of stout (1995, p. 120))  Let \(\{a_{ni}\}\) be a matrix of real numbers and \(\{x_i\}\) a sequence of real numbers. Let \(x_i \to x\) as \(i \to \infty\), then

\[
\sum_{i=1}^{\infty} |a_{ni}| \leq M < \infty \quad \text{for all} \quad n \geq 1,
\]

\[
\sum_{i=1}^{\infty} a_{ni} \to 1 \quad \text{as} \quad n \to \infty,
\]

and \(a_{ni} \to 0\) as \(n \to \infty\) for each \(i \geq 1\), imply that

\[
\sum_{i=1}^{\infty} a_{ni}x_i \to x \quad \text{as} \quad n \to \infty.
\]
Theorem 2.2 (Lemma 2.4 of Huang and Zhang (2005)) For \( n \geq 1 \), let \( a_n(\varepsilon) > 0 \), \( \beta_n(\varepsilon) > 0 \), and \( f(\varepsilon) > 0 \) satisfy
\[
\alpha_n(\varepsilon) \sim \beta_n(\varepsilon), \text{ as } n \to \infty \text{ and } \varepsilon \to \varepsilon_0,
\]
and
\[
f(\varepsilon)\beta_n(\varepsilon) \to 0, \text{ as } \varepsilon \to \varepsilon_0, \forall \ n \geq 1.
\]
Then
\[
\limsup_{\varepsilon \to \varepsilon_0} f(\varepsilon) \sum_{n=1}^{\infty} \alpha_n(\varepsilon) = \limsup_{\varepsilon \to \varepsilon_0} f(\varepsilon) \sum_{n=1}^{\infty} \beta_n(\varepsilon).
\]

3 Proofs

In this section, we will present the proof of theorem 1.1. Let \( C \) denote positive constants whose values may vary from place to place. In the sequel, set \( a(\varepsilon) = \exp \left\{ \frac{M}{\varepsilon^{2-p}} \right\} \), \( M > 1 \) and \( \varepsilon \) is small enough.

Our proof is based on the following propositions.

Proposition 3.1 For \( 1 \leq p < 2, d > -1 \), we have
\[
\lim_{\varepsilon \to 0} \varepsilon^{2d(1+d)\frac{2p}{2-p}} \sum_{n=3}^{\infty} \frac{(\log n)^d}{n} E \left\{ \|U\| - \varepsilon(\log n)^{\frac{1}{p} - \frac{1}{2}} \right\} = \frac{1}{1 + d} \frac{1}{2^{2p(1+d)+\frac{1}{2}}} \Gamma \left( \frac{p}{2} \right) \Gamma \left( \frac{p}{2} - 1 \right).
\]

Proof. Note that from (3) and (4) and change of variable, we have
\[
\lim_{\varepsilon \to 0} \varepsilon^{2d(1+d)\frac{2p}{2-p}} \sum_{n=3}^{\infty} \frac{(\log n)^d}{n} E \left\{ \|U\| - \varepsilon(\log n)^{\frac{1}{p} - \frac{1}{2}} \right\} = \lim_{\varepsilon \to 0} \int_{e}^{\infty} \frac{(\log t)^d}{t} \int_{0}^{\infty} 2 \exp \left\{ -2 \left( x + \varepsilon(\log t)^{\frac{1}{p} - \frac{1}{2}} \right)^2 \right\} dx dt
\]
\[
= \lim_{\varepsilon \to 0} \int_{e}^{\infty} \frac{(\log t)^d}{t} \int_{\varepsilon(\log t)^{\frac{1}{2}}}^{\infty} 2 \exp \left\{ -2z^2 \right\} dz dt
\]
\[
= 2 \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \exp \left\{ -2z^2 \right\} \int_{\varepsilon}^{z} \frac{2p}{2-p} g(y^{2p(1+d)} - 1) dy dz
\]
\[
\begin{align*}
= \frac{4p}{2-p} \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \exp \left\{ -2z^2 \right\} \int_\varepsilon^z y^{2p/(1+d)-1} dy dz \\
= \frac{2}{1+d} \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \exp \left\{ -2z^2 \right\} z^{2p/(1+d)} dz \\
= \frac{1}{1+d} \frac{1}{2^{2p/(1+d)+\frac{1}{2}}} \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \exp \left\{ -w \right\} w^{p/(1+d)-\frac{1}{2}} dw \\
= \frac{1}{1+d} \frac{1}{2^{2p/(1+d)+\frac{1}{2}}} \Gamma \left( \frac{p}{2} (1+d) + \frac{1}{2} \right).
\end{align*}
\]

**Proposition 3.2** For any \(M > 1, 1 \leq p < 2, d > -1\), we have

\[
\lim_{\varepsilon \to 0} \varepsilon^{2p(1+d) - \frac{1}{2}} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} \left| E \left\{ \|F_n\| - \varepsilon (\log n)^{\frac{1}{p}-\frac{1}{2}} \right\} \right| + E \left\{ \|U\| - \varepsilon (\log n)^{\frac{1}{p}-\frac{1}{2}} \right\} = 0.
\]

**Proof.** By theorem 2 on page 96 from Pollard (1984), we know that \(\{F_n(t)\}\) converge in distribution to the Brownian bridge \(U\), then by corollary 1 on page 31 from Billingsley (1968), we know that \(\{\|F_n\|\}\) converge in distribution to \(\|U\|\). As we know, the distribution function of \(\|U\|\) is continuous, by theorem 4 on page 172 from Vladimir (1997), we obtain

\[
\lim_{n \to \infty} \Delta_n = \lim_{n \to \infty} \sup_x \left| P\{\|F_n\| \geq x\} - P\{\|U\| \geq x\} \right| = 0.
\]  
(7)

By theorem 1 on page 173 from Kiefer and Wolfowitz (1958), we know that

\[
P\{\|F_n\| \geq x\} \leq ce^{-cx^2},
\]  
(8)

where \(c\) denote absolute positive constants, whose values can differ in different places. Then

\[
\begin{align*}
exponentially & \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} \left| E \left\{ \|F_n\| - \varepsilon (\log n)^{\frac{1}{p}-\frac{1}{2}} \right\} \right| \\
- E \left\{ \|U\| - \varepsilon (\log n)^{\frac{1}{p}-\frac{1}{2}} \right\} \} \\
= \varepsilon^{2p(1+d) - \frac{1}{2}} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} \left| \int_0^\infty P\{\|F_n\| \geq x + \varepsilon (\log n)^{\frac{1}{p}-\frac{1}{2}} \} dx \right| \\
- \int_0^\infty P\{\|U\| \geq x + \varepsilon (\log n)^{\frac{1}{p}-\frac{1}{2}} \} dx.
\end{align*}
\]
\[
\leq \varepsilon \frac{2^{p(1+d)}}{2-p} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} (P_{n1} + P_{n2}),
\]

where

\[
P_{n1} = \left| \int_{0}^{\Delta_n^{-\frac{1}{2}}} P \{ \|F_n\| \geq x + \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \} \, dx - \int_{\Delta_n^{-\frac{1}{2}}}^{\Delta_n^{\frac{1}{2}}} P \{ \|U\| \geq x + \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \} \, dx \right|
\]

\[
P_{n2} = \left| \int_{\Delta_n^{\frac{1}{2}}}^{\infty} P \{ \|F_n\| \geq x + \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \} \, dx - \int_{\Delta_n^{\frac{1}{2}}}^{\infty} P \{ \|U\| \geq x + \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \} \, dx \right|
\]

Thus, for \( P_{n1} \), applying theorem 2.1, we have

\[
\varepsilon \frac{2^{p(1+d)}}{2-p} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} P_{n1}
\leq \varepsilon \frac{2^{p(1+d)}}{2-p} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} \Delta_n^{-\frac{1}{2}} \Delta_n
\]

\[
= \varepsilon \frac{2^{p(1+d)}}{2-p} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} \Delta_n^{\frac{1}{2}}
\]

\[
\leq M^{1+d} \frac{1}{(\log a(\varepsilon))^{1+d}} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} \Delta_n^{\frac{1}{2}} \rightarrow 0, \text{ as } \varepsilon \to 0 \text{ and } n \to \infty. \quad (9)
\]

For \( P_{n2} \), by markov inequality and theorem 2.1 and (7) and (8), we have for large enough \( n \),

\[
\varepsilon \frac{2^{p(1+d)}}{2-p} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} P_{n2}
\leq c \varepsilon \frac{2^{p(1+d)}}{2-p} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} \int_{\Delta_n^{\frac{1}{2}}}^{\infty} e^{-cx^2} \, dx \rightarrow 0, \text{ as } n \to \infty. \quad (10)
\]

Denote \( P_n = P_{n1} + P_{n2} \). Then, we have

\[
\frac{M^{1+d}}{(\log a(\varepsilon))^{1+d}} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} P_n \rightarrow 0, \text{ as } \varepsilon \to 0.
\]
Then
\[
\varepsilon^{2p(1+d)} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} \left( E \left\{ \|F_n\| - \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \right\} + 
- E \left\{ \|U\| - \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \right\} \right)
= \frac{M^{1+d}}{(\log a(\varepsilon))^{1+d}} \sum_{n \leq a(\varepsilon)} \frac{(\log n)^d}{n} P_n \to 0, \text{ as } \varepsilon \to 0.
\]

**Proposition 3.3** For \(1 \leq p < 2\), \(d > -1\), we have
\[
\lim_{M \to \infty} \varepsilon^{2p(1+d)} \sum_{n > a(\varepsilon)} \frac{(\log n)^d}{n} E \left\{ \|U\| - \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \right\} = 0.
\]

**Proof.** Note that, \(a(\varepsilon) - 1 \geq \sqrt{a(\varepsilon)}\), for \(M > 1\) and \(\varepsilon\) is small enough. Thus it follows from (4) and (6) that
\[
\varepsilon^{2p(1+d)} \sum_{n > a(\varepsilon)} \frac{(\log n)^d}{n} E \left\{ \|U\| - \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \right\} +
\leq c \varepsilon^{2p(1+d)} \int_{a(\varepsilon) - 1}^{\infty} \frac{(\log t)^d}{t} \int_{0}^{\infty} \exp \left\{ -2(x + \varepsilon (\log t)^{\frac{1}{p} - \frac{1}{2}})^2 \right\} dx dt
\leq c \varepsilon^{2p(1+d)} \int_{\sqrt{a(\varepsilon)}}^{\infty} \frac{(\log t)^d}{t} \int_{\varepsilon (\log t)^{\frac{1}{p} - \frac{1}{2}}}^{\infty} 2 \exp \{-2z^2\} dz dt
\leq c \int_{M^{-2p}}^{\infty} y^{2p(1+d)} - 1 \int_{y}^{\infty} \exp \{-2z^2\} dz dy
\leq c \int_{M^{-2p}}^{\infty} y^{2p(1+d)} - 1 \exp \{-2z^2\} dz \to 0, \text{ as } M \to \infty.
\]

**Proposition 3.4** For \(d > -1\), \(1 \leq p < 2\), we have
\[
\lim_{M \to \infty} \varepsilon^{2p(1+d)} \sum_{n > a(\varepsilon)} \frac{(\log n)^d}{n} E \left\{ \|F_n\| - \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \right\} = 0.
\]

**Proof.** For \(\varepsilon\) sufficiently small, from (8), we obtain
\[
\varepsilon^{2p(1+d)} \sum_{n > a(\varepsilon)} \frac{(\log n)^d}{n} E \left\{ \|F_n\| - \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \right\} +
= \varepsilon^{2p(1+d)} \sum_{n > a(\varepsilon)} \frac{(\log n)^d}{n} \int_{0}^{\infty} P \left\{ \|F_n\| \geq x + \varepsilon (\log n)^{\frac{1}{p} - \frac{1}{2}} \right\} dx
\]
\[ c = \varepsilon \int_{a(\varepsilon)^{-1}}^{\infty} \frac{(\log t)^d}{t} \int_{0}^{\infty} \exp \left\{ -c(x + \varepsilon(\log t)^{\frac{1}{2} - 1})^2 \right\} dx dt \]
\[ \leq c \varepsilon \int_{\sqrt{a(\varepsilon)}}^{\infty} \frac{(\log t)^d}{t} \int_{\varepsilon \sqrt{\log t}}^{\infty} \exp \left\{ -cz^2 \right\} dz dt \]
\[ \leq c \int_{M}^{\infty} y^{2(1+d)} \frac{2p}{2-p} - 1 \int_{y}^{\infty} \exp \left\{ -cz^2 \right\} dz dy \]
\[ \leq c \int_{M}^{\infty} z^{2(1+d)} \frac{2p}{2-p} \exp \left\{ -cz^2 \right\} dz \to 0, \text{ as } M \to \infty. \]

Thus, the proof of theorem 1.1 is completed.

The proof of other theorems are similar to proof of theorem 1.1, so we omit them.

References


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