Generalized $m$-Closed Sets in Biminimal Structure Spaces

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Abstract

In this paper, we introduce the concept of generalized $g_m$-closed sets in biminimal structure spaces. We obtain some properties of generalized $m$-closed sets. Applying generalized $m$-closed set, we investigate the notion of $m^{(i,j)}-T_{\frac{1}{2}}$ space. Also, we introduce $gM^{(i,j)}$-continuous functions and investigate some of their characterizations.

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1 Introduction

Generalized closed (briefly g-closed) sets in a topological space were introduced by Levin [5] in order to extend many of the important properties of closed sets to a larger family. For instance, it was shown that compactness, normality and completeness in a uniform space are inherited by generalized closed subsets. The study of bitopological spaces was first initiated by Kelly [4] and thereafter a large number of papers have been done to generalize the topological concepts to bitopological setting. Fukutake [3] introduced generalized closed sets and pairwise generalized closure operator in bitopological spaces. He defined a set $A$ of a bitopological space $X$ to be $\tau_1\tau_2$-generalized closed sets if $\tau_j$-$\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_i$-open in $X$. Also, he defined a new closure operator and strongly pairwise $T_{\frac{1}{2}}$-space. Recently, the present authors [7], [9]
have introduced the notions of $m$-structure, $m$-spaces and $M$-continuity. In [8], the first author introduced the notion of generalized $m$-closed (briefly $gm$-closed) sets and tried to unify certain types of modifications of g-closed sets. Boonpok [1] introduced the concept of biminimal structure spaces and studied $m^1_Xm^2_X$-closed sets and $m^1_Xm^2_X$-open sets in biminimal structure spaces.

The purpose of this paper is to introduce the concepts of $gm^{(i,j)}$-closed sets, $m^{(i,j)}-T_1$-spaces and $gM^{(i,j)}$-continuity for biminimal structure spaces and investigate some of their properties.

# 2 Preliminaries

**Definition 2.1.** [7] Let $X$ be a nonempty set and $\mathcal{P}(X)$ the power set of $X$. A subfamily $m_X$ of $\mathcal{P}(X)$ is called a minimal structure (briefly $m$-structure) on $X$ if $\emptyset \in m_X$ and $X \in m_X$.

By $(X, m_X)$, we denote a nonempty set $X$ with an $m$-structure $m_X$ on $X$ and it is called an $m$-space. Each member of $m_X$ is said to be $X$-open and the complement of an $m_X$-open set is said to be $m_X$-closed.

**Definition 2.2.** [7] Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$. For a subset $A$ of $X$, the $m_X$-closure of $A$ and the $m_X$-interior of $A$ are defined as follows:

1. $m_X$-Cl($A$) = $\cap\{F : A \subseteq F, X - F \in m_X\}$,
2. $m_X$-Int($A$) = $\cup\{U : U \subseteq A, U \in m_X\}$.

**Lemma 2.3.** [6] Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold:

1. $m_X$-Cl($X - A$) = $X - (m_X$-Int($A$)) and $m_X$-Int($X - A$) = $X - (m_X$-Cl($A$)),
2. If $(X - A) \in m_X$, then $m_X$-Cl($A$) = $A$ and if $A \in m_X$, then $m_X$-Int($A$) = $A$,
3. $m_X$-Cl($\emptyset$) = $\emptyset$, $m_X$-Cl($X$) = $X$, $m_X$-Int($\emptyset$) = $\emptyset$ and $m_X$-Int($X$) = $X$,
4. If $A \subseteq B$, then $m_X$-Cl($A$) $\subseteq m_X$-Cl($B$) and $m_X$-Int($A$) $\subseteq m_X$-Int($B$),
5. $A \subseteq m_X$-Cl($A$) and $m_X$-Int($A$) $\subseteq A$,
6. $m_X$-Cl($m_X$-Cl($A$)) = $m_X$-Cl($A$) and $m_X$-Int($m_X$-Int($A$)) = $m_X$-Int($A$).

**Lemma 2.4.** [6] Let $X$ be a nonempty set with a minimal structure $m_X$ and $A$ a subset of $X$. Then $x \in m_X$-Cl($A$) if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing $x$. 


Definition 2.5. [6] An $m$-structure $m_X$ on a nonempty set $X$ is said to have property $B$ if the union of any family of subsets belonging to $m_X$ belongs to $m_X$.

Lemma 2.6. [7] Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$ satisfying property $B$. For a subset $A$ of $X$, the following properties hold:

1. $A \in m_X$ if and only if $m_X$-$\text{Int}(A) = A$,
2. $A$ is $m_X$-closed if and only if $m_X$-$\text{Cl}(A) = A$,
3. $m_X$-$\text{Int}(A) \in m_X$ and $m_X$-$\text{Cl}(A)$ is $m_X$-closed.

Definition 2.7. [2] Let $(X, m_X)$ be an $m$-space. A subset $A$ of $X$ is said to be generalized $m$-closed (briefly $gm$-closed) if $m_X$-$\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $m_X$-open.

Definition 2.8. [2] An $m$-space $(X, m_X)$ is called an $m$-$T_{\frac{1}{2}}$-space if every $gm$-closed set of $(X, m_X)$ is $m_X$-closed.

Definition 2.9. [10] A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M$-continuous if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(U) \subseteq V$.

Definition 2.10. [2] A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M$-open if $f(U)$ is an $m_Y$-open set of $(Y, m_Y)$, for every $m_X$-open set $U$ of $(X, m_X)$.

Definition 2.11. [1] Let $X$ be a nonempty set and let $m^1_X$, $m^2_X$ be minimal structures on $X$. A triple $(X, m^1_X, m^2_X)$ is called a biminimal structure space (briefly bim-space).

Let $(X, m^1_X, m^2_X)$ be a biminimal structure space and let $A$ be a subset of $X$. The $m_X$-closure and $m_X$-interior of $A$ with respect to $m^i_X$ are denoted by $m^i_X$-$\text{Cl}(A)$ and $m^i_X$-$\text{Int}(A)$, respectively for $i = 1, 2$.

The family of all $m^i_X$-open (resp. $m^i_X$-closed) sets of $(X, m^1_X, m^2_X)$ is denoted by $m^i_X\text{C}(X)$ (resp. $m^i_X\text{O}(X)$) for $i = 1, 2$.

3 Generalized $m$-closed sets

In this section, we introduce the concept of $gm^{(i,j)}$-closed sets in biminimal structure spaces and study some of their properties.

Definition 3.1. A subset $A$ of a biminimal structure space $(X, m^1_X, m^2_X)$ is said to be $m^{(i,j)}_X$-closed if $m^i_X$-$\text{Cl}(m^j_X$-$\text{Cl}(A)) = A$, where $i, j = 1, 2$ and $i \neq j$. The complement of a $m^{(i,j)}_X$-closed set is said to be $m^{(i,j)}_X$-open.
The family of all \(m^{(i,j)}\)-closed (resp. \(m^{(i,j)}\)-open) sets of \((X, m^1_X, m^2_X)\) is denoted by \(m^{(i,j)}\)C(X) (resp. \(m^{(i,j)}\)O(X)), where \(i, j = 1, 2\) and \(i \neq j\).

A subset \(A\) of a biminimal structure space \((X, m^1_X, m^2_X)\) is said to be \(m\)-closed if \(A\) is \(m^{(1,2)}\)-closed and \(m^{(2,1)}\)-closed. The complement of a \(m\)-closed set is said to be \(m\)-open.

**Example 3.2.** Let \(X = \{a, b, c\}\), \(m^1_X = \{\emptyset, \{b, c\}, X\}\) and \(m^2_X = \{\emptyset, \{b\}, \{c\}, X\}\). Then \(\{a\}\) is \(m^{(1,2)}\)-closed and \(m^{(2,1)}\)-closed. Hence, \(\{a\}\) is \(m\)-closed.

Let \((X, m^1_X, m^2_X)\) be a biminimal structure space and \(A\) be a subset of \(X\). Then \(A\) is \(m^{(i,j)}\)-closed if and only if \(m^i_X\)-Cl\((A) = A\) and \(m^j_X\)-Cl\((A) = A\), where \(i, j = 1, 2\) and \(i \neq j\).

The following statement is evident:

**Proposition 3.3.** Let \(A\) be a subset of a biminimal structure space \((X, m^1_X, X^2)\), where \(m^1_X, m^2_X\) have property \(B\). Then \(A\) is \(m^{(i,j)}\)-closed if and only if \(A\) is both \(m^i_X\)-closed and \(m^j_X\)-closed, where \(i, j = 1, 2\) and \(i \neq j\).

**Proposition 3.4.** For subsets \(A\) and \(B\) of a biminimal structure space \((X, m^1_X, m^2_X)\). If \(A\) and \(B\) are \(m^{(i,j)}\)-closed, then \(A \cap B\) is \(m^{(i,j)}\)-closed, where \(i, j = 1, 2\) and \(i \neq j\).

**Proof.** Let \(A\) and \(B\) be \(m^{(i,j)}\)-closed. Then \(m^i_X\)-Cl\(((m^i_X\)-Cl\((A)) = A\) and \(m^i_X\)-Cl\(((m^i_X\)-Cl\((B)) = B\). Since \(A \cap B \subseteq A\) and \(A \cap B \subseteq B\), \(m^i_X\)-Cl\(((m^i_X\)-Cl\((A \cap B)) \subseteq m^i_X\)-Cl\(((m^i_X\)-Cl\((A)) \subseteq m^i_X\)-Cl\(((m^i_X\)-Cl\((B))\). Therefore, \(m^i_X\)-Cl\(((m^i_X\)-Cl\((A \cap B)) \subseteq m^i_X\)-Cl\(((m^i_X\)-Cl\((A)) \cap m^i_X\)-Cl\(((m^i_X\)-Cl\((B)) = A \cap B\). But \(A \cap B \subseteq m^i_X\)-Cl\(((m^i_X\)-Cl\((A \cap B))\). Consequently, \(m^i_X\)-Cl\(((m^i_X\)-Cl\((A \cap B)) = A \cap B\). Hence, \(A \cap B\) is \(m^{(i,j)}\)-closed.

**Remark 1.** The union of two \(m^{(i,j)}\)-closed sets is not a \(m^{(i,j)}\)-closed set in general as can be seen from the following example.

**Example 3.5.** Let \(X = \{1, 2, 3\}\), \(m^1_X = \{\emptyset, \{1, 3\}, \{2, 3\}, X\}\) and \(m^2_X = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, X\}\). Then \(\{1\}\) and \(\{2\}\) are \(m^{(1,2)}\)-closed but \(\{1\} \cup \{2\} = \{1, 2\}\) is not \(m^{(1,2)}\)-closed.

**Proposition 3.6.** Let \((X, m^1_X, m^2_X)\) be a biminimal structure space and let \(A\) be a subset of \(X\). Then \(A\) is \(m^{(i,j)}\)-open if and only if \(A = m^i_X\)-Int\(((m^i_X\)-Int\((A))\), where \(i, j = 1, 2\) and \(i \neq j\).

**Proof.** Let \(A\) be a \(m^{(i,j)}\)-open set. Then \(X - A\) is \(m^{(i,j)}\)-closed. Therefore, \(m^i_X\)-Cl\(((m^i_X\)-Cl\((X - A)) = X - A\). By Lemma 2.3(1), \(X - (m^i_X\)-Int\(((m^i_X\)-Int\((A))\)) = X - A\). Consequently, \(A = m^i_X\)-Int\(((m^i_X\)-Int\((A))\).

Conversely, let \(A = m^i_X\)-Int\(((m^i_X\)-Int\((A))\). Therefore, \(X - A = X - (m^i_X\)-Int\(((m^i_X\)-Int\((A))\))\). By Lemma 2.3(1), \(X - A = m^i_X\)-Cl\(((m^i_X\)-Cl\((X - A))\). Hence, \(X - A\) is \(m^{(i,j)}\)-closed. Consequently, \(A\) is \(m^{(i,j)}\)-open. 

\(\Box\)
Proposition 3.7. For subsets $A$ and $B$ of a biminimal structure space $(X, m^1_X, m^2_X)$. If $A$ and $B$ are $m^{(i,j)}_X$-open, then $A \cup B$ is $m^{(i,j)}_X$-open, where $i, j = 1, 2$ and $i \neq j$.

Proof. Let $A$ and $B$ be $m^{(i,j)}_X$-open. Then $m^i_X \setminus \text{Int}(m^i_X \setminus \text{Int}(A)) = A$ and $m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(B)) = B$. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(A)) \subseteq m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(A \cup B))$ and $m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(B)) \subseteq m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(A \cup B))$. Therefore, $A \cup B = m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(A)) \cup m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(B)) \subseteq m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(A \cup B))$. But $m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(A \cup B)) \subseteq A \cup B$. Consequently, $m^i_X \setminus \text{Int}(m^j_X \setminus \text{Int}(A \cup B)) = A \cup B$. Hence, $A \cup B$ is $m^{(i,j)}_X$-open. \qed

Remark 2. The intersection of two $m^{(i,j)}_X$-open sets is not a $m^{(i,j)}_X$-open set in general as can be seen from the following example.

Example 3.8. Let $X = \{1, 2, 3\}$, $m^1_X = \emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, X$ and $m^2_X = \emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, X$. Then $\{1, 3\}$ and $\{2, 3\}$ are $m^{(1,2)}_X$-open but $\{1, 3\} \cap \{2, 3\} = \{3\}$ is not $m^{(1,2)}_X$-open.

Definition 3.9. A subset $A$ of a biminimal structure space $(X, m^1_X, m^2_X)$ is said to be $(i, j)$-generalized $m$-closed (briefly $gm^{(i,j)}_X$-closed) if $m^i_X \setminus \text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in m^j_X$, where $i, j = 1, 2$ and $i \neq j$. The complement of a $gm^{(i,j)}_X$-closed set is said to be $gm^{(i,j)}_X$-open.

The family of all $gm^{(i,j)}_X$-closed (resp. $gm^{(i,j)}_X$-open) sets of $(X, m^1_X, m^2_X)$ is denoted by $gm^{(i,j)}_X(X)$ (resp. $gm^{(i,j)}_X(O)$), where $i, j = 1, 2$ and $i \neq j$.

A subset $A$ of a biminimal structure space $X$ is said to be pairwise $gm$-closed if $A$ is $gm^{(1,2)}_X$-closed and $gm^{(2,1)}_X$-closed. The complement of a pairwise $gm$-closed set is said to be pairwise $gm$-open.

Remark 3. Every $m^{(i,j)}_X$-closed set is $gm^{(i,j)}_X$-closed. The converse is not true as can be seen from the following example.

Example 3.10. Let $X = \{1, 2, 3\}$, $m^1_X = \emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, X$ and $m^2_X = \emptyset, \{1\}, \{2\}, X$. Then $\{1, 2\}$ is $gm^{(1,2)}_X$-closed but it is not $m^{(1,2)}_X$-closed.

Proposition 3.11. For a subset $A$ of a biminimal structure space $(X, m^1_X, m^2_X)$, if $A$ is both $m^{(i,j)}_X$-open and $gm^{(i,j)}_X$-closed, then $A$ is $m^{(i,j)}_X$-closed, where $i, j = 1, 2$ and $i \neq j$.

The intersection of two $gm^{(i,j)}_X$-closed sets need not be $gm^{(i,j)}_X$-closed as seen from the following example.

Example 3.12. Let $X = \{a, b, c\}$, $m^1_X = \emptyset, \{a\}, \{c\}, \{a, b\}, X$ and $m^2_X = \emptyset, \{a, c\}, X$. Then $\{a, c\}$ and $\{b, c\}$ are $gm^{(1,2)}_X$-closed but $\{a, c\} \cap \{b, c\} = \{c\}$ is not $gm^{(1,2)}_X$-closed.
Proposition 3.13. Let $m^1_X$ and $m^2_X$ be minimal structures on $X$ have property $\mathcal{B}$. If $A$ is $gm^{(i,j)}_X$-closed and $F$ is $m^{(i,j)}_X$-closed, then $A \cap F$ is $gm^{(i,j)}_X$-closed.

Proof. Let $A \cap F \subseteq U$ and let $U$ be $m^i_X$-open. Then $A \subseteq U \cup (X - F)$ and so $m^i_X$-$Cl(A) \subseteq U \cup (X - F)$. Therefore, $m^i_X$-$Cl(A) \cap F \subseteq U$. Since $F$ is $m^i_X$-closed, we obtain $m^i_X$-$Cl(A) \cap F \subseteq U$. Hence, $A \cap F$ is $gm^{(i,j)}_X$-closed. $\square$

Remark 4. $gm^{(1,2)}(X)$ is generally not equal to $gm^{(2,1)}(X)$. For example $gm^{(1,2)}(X) \neq gm^{(2,1)}(X)$ in example 3.12.

Proposition 3.14. Let $m^1_X$ and $m^2_X$ be m-structures on $X$. If $m^1_X \subseteq m^2_X$, then $gm^{(1,2)}(X) \subseteq gm^{(2,1)}(X)$.

The converse of the above proposition is not true as seen from the following example.

Example 3.15. Let $X = \{a, b, c\}$, $m^1_X = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $m^2_X = \{\emptyset, \{a\}, X\}$. Then $gm^{(1,2)}(X) \subseteq gm^{(2,1)}(X)$ but $m^1_X$ contained in $m^2_X$.

Proposition 3.16. For each element $x$ of a space $(X, m^1_X, m^2_X)$, \{\{x\}\} is $m^i_X$-closed or $X - \{x\}$ is $gm^{(i,j)}_X$-closed, where $i, j = 1, 2$ and $i \neq j$.

Proof. Let $x \in X$ and the singleton \{\{x\}\} be not $m^i_X$-closed. Then $X - \{x\}$ is not $m^i_X$-open and $X$ is the only $m^i_X$-open set which contains $X - \{x\}$ and $X - \{x\}$ is $gm^{(i,j)}_X$-closed. $\square$

Proposition 3.17. Let $A$ be a subset of a biminimal structure space $(X, m^1_X, m^2_X)$. If $A$ is $gm^{(i,j)}_X$-closed, then $m^i_X$-$Cl(A) - A$ contains no nonempty $m^i_X$-closed set, where $i, j = 1, 2$ and $i \neq j$.

Proof. Let $A$ be an $gm^{(i,j)}_X$-closed set and $F$ is a $m^i_X$-closed set such that $F \subseteq m^i_X$-$Cl(A) - A$. Since $A \in gm^{(i,j)}(X)$, we have $m^i_X$-$Cl(A) \subseteq X - F$. Thus $F \subseteq m^i_X$-$Cl(A) \cap (X - (m^i_X$-$Cl(A))) = \emptyset$. $\square$

The converse of the above proposition is not true as seen from the following example.

Example 3.18. Let $X = \{a, b, c\}$, $m^1_X = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $m^2_X = \{\emptyset, \{a\}, X\}$. If $A = \{b\}$, then $m^2_X$-$Cl(A) - A = \{c\}$ does not contain any nonempty $m^i_X$-closed set. But $A$ is not $gm^{(1,2)}_X$-closed.

Corollary 3.19. Let $m^1_X$ and $m^2_X$ be minimal structures on $X$ satisfying property $\mathcal{B}$. If $A$ is $gm^{(i,j)}_X$-closed set in $(X, m^1_X, m^2_X)$, then $A$ is $m^i_X$-closed if and only if $m^i_X$-$Cl(A) - A$ is $m^i_X$-closed, where $i, j = 1, 2$ and $i \neq j$. 
Proof. If $A$ is $m_X^j$-closed, then $m_X^j$-$Cl(A) = A$. i.e., $m_X^j$-$Cl(A) - A = \emptyset$ and hence $m_X^j$-$Cl(A) - A$ is $m_X^j$-closed.

Conversely, if $m_X^j$-$Cl(A) - A$ is $m_X^j$-closed, then by Proposition 3.17, $m_X^j$-$Cl(A) - A = \emptyset$, since $A$ is $gm_X^{(i,j)}$-closed. Therefore, $A$ is $m_X^j$-closed. \[\square\]

**Proposition 3.20.** Let $m_X^1$ and $m_X^2$ be minimal structures on $X$ satisfying property $B$. If $A$ is a $gm_X^{(i,j)}$-closed set of $(X, m_X^1, m_X^2)$, then $m_X^j$-$Cl(\{x\}) \cap A \neq \emptyset$ holds for each $x \in m_X^j$-$Cl(A)$, where $i,j = 1, 2$ and $i \neq j$.

Proof. Let $x \in m_X^j$-$Cl(A)$. Suppose that $m_X^j$-$Cl(\{x\}) \cap A = \emptyset$. Then $A \subseteq X - (m_X^j$-$Cl(\{x\}))$. Since $A$ is $gm_X^{(i,j)}$-closed and $X - (m_X^j$-$Cl(\{x\}))$ is $m_X^i$-open, $m_X^j$-$Cl(A) \subseteq X - (m_X^j$-$Cl(\{x\}))$. Consequently, $m_X^j$-$Cl(A) \cap m_X^j$-$Cl(\{x\})$. This is a contradiction. \[\square\]

**Proposition 3.21.** If $A$ is a $gm_X^{(i,j)}$-closed set of $(X, m_X^1, m_X^2)$ such that $A \subseteq B \subseteq m_X^j$-$Cl(A)$, then $B$ is also a $gm_X^{(i,j)}$-closed set of $(X, m_X^1, m_X^2)$, where $i,j = 1, 2$ and $i \neq j$.

Proof. Suppose that $A$ is $gm_X^{(i,j)}$-closed and $A \subseteq B \subseteq m_X^j$-$Cl(A)$. Let $B \subseteq U$ and $U$ is $m_X^j$-open. Then $A \subseteq U$. Since $A$ is $gm_X^{(i,j)}$-closed, we have $m_X^j$-$Cl(A) \subseteq U$. Since $B \subseteq m_X^j$-$Cl(A)$, $m_X^j$-$Cl(B) \subseteq m_X^j$-$Cl(A) \subseteq U$. Hence, $B$ is $gm_X^{(i,j)}$-closed. \[\square\]

**Theorem 3.22.** For a biminimal structure space $(X, m_X^1, m_X^2)$, where $(m_X^1, m_X^2)$ have property $B$. Then $m_X^j$-$O(X) \subseteq m_X^j$-$C(X)$ if and only if every subset of $X$ is a $gm_X^{(i,j)}$-closed set, where $i,j = 1, 2$ and $i \neq j$.

Proof. Suppose that $m_X^j$-$O(X) \subseteq m_X^j$-$C(X)$. Let $A$ be a subset of $X$ such that $A \subseteq U$, where $U \in m_X^j$-$O(X)$. Then $m_X^j$-$Cl(A) \subseteq m_X^j$-$Cl(U) = U$ and hence $A$ is $gm_X^{(i,j)}$-closed.

Conversely, suppose that every subset of $X$ is $gm_X^{(i,j)}$-closed. Let $U \in m_X^j$-$O(X)$. Since $U$ is $gm_X^{(i,j)}$-closed, we have $m_X^j$-$Cl(U) \subseteq U$. Therefore, $U \in m_X^j$-$C(X)$ and hence $m_X^j$-$O(X) \subseteq m_X^j$-$C(X)$.

**Theorem 3.23.** A subset $A$ of a biminimal structure space $(X, m_X^1, m_X^2)$ is $gm_X^{(i,j)}$-open if and only if $F \subseteq m_X^j$-$Int(A)$ whenever $F$ is $m_X^i$-closed and $F \subseteq A$, where $i,j = 1, 2$ and $i \neq j$.

Proof. Suppose that $A$ is $gm_X^{(i,j)}$-open. We shall show that $F \subseteq m_X^j$-$Int(A)$ whenever $F$ is $m_X^i$-closed and $F \subseteq A$. Let $A \subseteq F$ and $F$ is $m_X^i$-closed. Then $X - A \subseteq X - F$ and $X - F$ is $m_X^i$-open, we have $X - A$ is $gm_X^{(i,j)}$-closed. Hence, $m_X^j$-$Cl(X - A) \subseteq X - F$. Consequently, $X - (m_X^j$-$Int(A)) \subseteq X - F$. Therefore, $F \subseteq m_X^i$-$Int(A)$. \[\square\]
Conversely, suppose that $F \subseteq m_X^j - \text{Int}(A)$ whenever $F \subseteq A$ and $F$ is $m_X^i$-closed. Let $X - A \subseteq U$ and $U$ is $m_X^i$-open. Then $X - U \subseteq A$ and $X - U$ is $m_X^i$-closed. By our assumption, we have $X - U \subseteq m_X^i - \text{Int}(A)$. Hence, $X - (m_X^i - \text{Int}(A)) \subseteq U$. Therefore, $m_X^i - \text{Cl}(X - A) \subseteq U$. Consequently, $X - A$ is $gm_X^{(i,j)}$-closed. Hence, $A$ is $gm_X^{(i,j)}$-open.

**Proposition 3.24.** Let $A$ and $B$ be subsets of a biminimal structure space $(X, m_X^1, m_X^2)$ such that $m_X^i - \text{Int}(A) \subseteq B \subseteq A$. If $A$ is $gm_X^{(i,j)}$-open, then $B$ is $gm_X^{(i,j)}$-open, where $i, j = 1, 2$ and $i \neq j$.

**Proof.** Suppose that $A$ and $B$ subsets such that $m_X^i - \text{Int}(A) \subseteq B \subseteq A$. Let $A$ be $gm_X^{(i,j)}$-open. Let $F \subseteq B$ and $F$ is $m_X^i$-closed. Since $F \subseteq B$ and $B \subseteq A$, we have $F \subseteq A$. Therefore, $F \subseteq m_X^i - \text{Int}(A)$. Since $m_X^i - \text{Int}(A) \subseteq B$, we have $m_X^i - \text{Int}(m_X^i - \text{Int}(A)) \subseteq m_X^i - \text{Int}(B)$. Therefore, $m_X^i - \text{Int}(A) \subseteq m_X^i - \text{Int}(B)$. Consequently, $F \subseteq m_X^i - \text{Int}(B)$. Hence, $B$ is $gm_X^{(i,j)}$-open.

**Proposition 3.25.** If a subset $A$ of a biminimal structure space $(X, m_X^1, m_X^2)$ is $gm_X^{(i,j)}$-closed, then $m_X^i - \text{Cl}(A) - A$ is $gm_X^{(i,j)}$-open, where $i, j = 1, 2$ and $i \neq j$.

**Proof.** Suppose that $A$ is $gm_X^{(i,j)}$-closed. We shall show that $m_X^i - \text{Cl}(A) - A$ is $gm_X^{(i,j)}$-open. Let $F \subseteq m_X^i - \text{Cl}(A) - A$ and $F$ is $m_X^i$-closed. Since $A$ is $gm_X^{(i,j)}$-closed, we have $m_X^i - \text{Cl}(A) - A$ does not contain nonempty $m_X^i$-closed by Proposition 3.17. Consequently, $F = \emptyset$. Therefore, $\emptyset \subseteq m_X^i - \text{Cl}(A) - A$, $\emptyset \subseteq m_X^i - \text{Int}(m_X^i - \text{Cl}(A) - A)$, we obtain $F \subseteq m_X^i - \text{Int}(m_X^i - \text{Cl}(A) - A)$. Hence, $m_X^i - \text{Cl}(A) - A$ is $gm_X^{(i,j)}$-open.

4 $m^{(i,j)} - T_{\frac{1}{2}}^\xi$ spaces and $m^{(i,j)} - \xi T_{\frac{1}{2}}$ spaces

In this section, we introduce $m^{(i,j)} - T_{\frac{1}{2}}^\xi$ and $m^{(i,j)} - \xi T_{\frac{1}{2}}$ biminimal structure spaces and in Theorem 4.20 we prove that the $m^{(i,j)} - \xi T_{\frac{1}{2}}$ spaces is the dual of the class of $m^{(i,j)} - T_{\frac{1}{2}}^\xi$ spaces to the class of $m^{(i,j)} - T_{\frac{1}{2}}$ spaces.

**Definition 4.1.** A subset $A$ of a biminimal structure space $(X, m_X^1, m_X^2)$ is said to be $\xi m_X^{(i,j)}$-closed if $m_X^i - \text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in gm_X^i O(X)$, where $i, j = 1, 2$ and $i \neq j$. The complement of a $\xi m_X^{(i,j)}$-closed set is said to be $\xi m_X^{(i,j)}$-open.

The family of all $\xi m_X^{(i,j)}$-closed (resp. $\xi m_X^{(i,j)}$-open) sets of $(X, m_X^1, m_X^2)$ is denoted by $\xi m_X^{(i,j)} C(X)$ (resp. $\xi m_X^{(i,j)} O(X)$), where $i, j = 1, 2$ and $i \neq j$. 
A subset $A$ of a biminimal structure space $(X, m_X^1, m_X^2)$ is said to be pairwise $\xi m_X$-closed if $A$ is $\xi m_X^{(1,2)}$-closed and $\xi m_X^{(2,1)}$-closed. The complement of a pairwise $\xi m_X$-closed set is said to be pairwise $\xi m_X$-open.

**Remark 5.** Every $\xi m_X^{(i,j)}$-closed set is $gm_X^{(i,j)}$-closed. The converse is not true as can be seen from the following example.

**Example 4.2.** Let $X = \{1, 2, 3\}$. Define minimal structures $m_X^1$ and $m_X^2$ on $X$ as follows: $m_X^1 = \{\emptyset, \{1\}, X\}$ and $m_X^2 = \{\emptyset, \{1, 2\}, X\}$. Then $\{2\}$ is $gm_X^{(1,2)}$-closed but it is not $\xi m_X^{(1,2)}$-closed.

**Proposition 4.3.** For each element $x$ of a space $(X, m_X^1, m_X^2)$, $\{x\}$ is $m_X^1$-gm-closed or $X - \{x\}$ is $\xi m_X^{(i,j)}$-closed, where $i, j = 1, 2$ and $i \neq j$.

**Proposition 4.4.** Let $A$ be a subset of a biminimal structure space $(X, m_X^1, m_X^2)$. If $A$ is $\xi m_X^{(i,j)}$-closed, then $m_X^j Cl(A) - A$ contains no nonempty $gm_X^i$-closed set, where $i, j = 1, 2$ and $i \neq j$.

**Definition 4.5.** A biminimal structure space $(X, m_X^1, m_X^2)$ is said to be $m^{(i,j)}$-$T_{1/2}$ if for every $gm_X^{(i,j)}$-closed set is $m_X^j$-closed, where $i, j = 1, 2$ and $i \neq j$.

**Definition 4.6.** A biminimal structure space $(X, m_X^1, m_X^2)$ is said to be $m^{(i,j)}$-$T_{1/2}^\xi$ if for every $\xi m_X^{(i,j)}$-closed set is $m_X^j$-closed, where $i, j = 1, 2$ and $i \neq j$.

**Proposition 4.7.** If $(X, m_X^1, m_X^2)$ is $m^{(i,j)}$-$T_{1/2}$ space, then it is a $m^{(i,j)}$-$T_{1/2}^\xi$ space but not conversely.

**Example 4.8.** Let $X = \{a, b, c\}$, $m_X^1 = \{\emptyset, \{a\}, X\}$ and $m_X^2 = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $(X, m_X^1, m_X^2)$ is a $m^{(1,2)}$-$T_{1/2}^\xi$ space but not a $m^{(1,2)}$-$T_{1/2}$ space.

**Theorem 4.9.** A biminimal structure space $(X, m_X^1, m_X^2)$ is a $m^{(i,j)}$-$T_{1/2}$ space if and only if $\{x\}$ is $m_X^j$-open or $m_X^i$-closed for each $x \in X$, where $i, j = 1, 2$ and $i \neq j$.

*Proof.* Suppose that $\{x\}$ is not $m_X^j$-closed. Then $X - \{x\}$ is $gm_X^{(i,j)}$-closed by Proposition 3.16. Since $(X, m_X^1, m_X^2)$ is an $m^{(i,j)}$-$T_{1/2}$ space, $X - \{x\}$ is $m_X^j$-closed. Therefore, $\{x\}$ is $m_X^j$-open.

Conversely, let $F$ be a $gm_X^{(i,j)}$-closed set. By assumption, $\{x\}$ is $m_X^j$-open or $gm_X^i$-closed for any $x \in m_X^j Cl(F)$. Case (i) Suppose $\{x\}$ is $m_X^j$-open. Since $\{x\} \cap F \neq \emptyset$, we have $x \in F$. Case (ii) Suppose $\{x\}$ is $m_X^i$-closed. If $x \notin F$, then $\{x\} \subseteq m_X^i Cl(F) - F$, which is a contradiction to Proposition 4.4. Hence, $x \in F$. Thus in both cases, we conclude that $F$ is $m_X^j$-closed. Therefore, $(X, m_X^1, m_X^2)$ is an $m^{(i,j)}$-$T_{1/2}^\xi$ space. \(\square\)
Theorem 4.10. A biminimal structure space \((X, m_X^1, m_X^2)\) is a \(m^{(i,j)}-T_\frac{\xi}{2}\) space if and only if \(\{x\}\) is \(m_X^i\)-open or \(gm_X^i\)-closed for each \(x \in X\), where \(i, j = 1, 2\) and \(i \neq j\).

Definition 4.11. A biminimal structure space \((X, m_X^1, m_X^2)\) is said to be pairwise \(m^{(i,j)}-T_{\frac{1}{2}}\) space if it is both \(m^{(1,2)}-T_{\frac{1}{2}}\) space and \(m^{(2,1)}-T_{\frac{1}{2}}\) space.

Definition 4.12. A biminimal structure space \((X, m_X^1, m_X^2)\) is said to be pairwise \(m-T_{\frac{\xi}{2}}\) space if it is both \(m^{(1,2)}-T_{\frac{\xi}{2}}\) space and \(m^{(2,1)}-T_{\frac{\xi}{2}}\) space.

Proposition 4.13. If \((X, m_X^1, m_X^2)\) is a pairwise \(m-T_{\frac{1}{2}}\) space, then it is a pairwise \(m-T_{\frac{\xi}{2}}\) space but not conversely.

Example 4.14. Let \(X, m_X^1\) and \(m_X^2\) be as in Example 4.4. Then \((X, m_X^1, m_X^2)\) is also a \(m^{(2,1)}-T_{\frac{\xi}{2}}\) space and therefore it is a pairwise \(m-T_{\frac{1}{2}}\) space. But \((X, m_X^1, m_X^2)\) is not a pairwise \(m-T_{\frac{1}{2}}\) space, since it is not a \(m^{(1,2)}-T_{\frac{1}{2}}\) space.

We now introduce the following definition.

Definition 4.15. A biminimal structure space \((X, m_X^1, m_X^2)\) is said to be an \(m^{(i,j)}-T_{\frac{\xi}{2}}\) space if every \(gm_X^{(i,j)}\)-closed set is \(\xi m_X^{(i,j)}\)-closed, where \(i, j = 1, 2\) and \(i \neq j\).

Proposition 4.16. Every \(m^{(i,j)}-T_{\frac{\xi}{2}}\) space is a \(m^{(i,j)}-T_{\frac{1}{2}}\) space but not conversely.

Example 4.17. Let \(X = \{a, b, c\}, m_X^1 = \{\emptyset, \{a\}, \{a, c\}, X\}\) and \(m_X^2 = \{\emptyset, \{a, b\}, X\}\). Then \((X, m_X^1, m_X^2)\) is a \(m^{(1,2)}-T_{\frac{1}{2}}\) space but not a \(m^{(1,2)}-T_{\frac{1}{2}}\) space.

Remark 6. \(m^{(i,j)}-T_{\frac{\xi}{2}}\) and \(m^{(i,j)}-T_{\frac{\xi}{2}}\) spaces are independent as seen from the following examples.

Example 4.18. Let \(X, m_X^1\) and \(m_X^2\) be as in Example 4.8. Then \((X, m_X^1, m_X^2)\) is not a \(m^{(1,2)}-T_{\frac{\xi}{2}}\) space, but it is \(m^{(1,2)}-T_{\frac{\xi}{2}}\) space.

Example 4.19. Let \(X = \{a, b, c\}, m_X^1 = \{\emptyset, \{a\}, X\}\) and \(m_X^2 = \{\emptyset, \{b, c\}, X\}\). Then \((X, m_X^1, m_X^2)\) is a \(m^{(1,2)}-T_{\frac{1}{2}}\) space, but it is not \(m^{(1,2)}-T_{\frac{1}{2}}\) space.

Theorem 4.20. A biminimal structure space \((X, m_X^1, m_X^2)\) is \(m^{(i,j)}-T_{\frac{\xi}{2}}\) space if and only if it is both \(m^{(i,j)}-T_{\frac{\xi}{2}}\) space and \(m^{(i,j)}-T_{\frac{\xi}{2}}\) space, where \(i, j = 1, 2\) and \(i \neq j\).
Proof. Suppose that \((X, m_X^1, m_X^2)\) is a \(m^{(i,j)}\)-\(T_{1\frac{1}{2}}\) space. Then by Proposition 4.16 and Proposition 4.7, \((X, m_X^1, m_X^2)\) is \(m^{(i,j)}\)-\(T_{1\frac{1}{2}}\) space and \(m^{(i,j)}\)-\(T_{1\frac{1}{2}}\) space.

Conversely, suppose that \((X, m_X^1, m_X^2)\) is both \(m^{(i,j)}\)-\(T_{1\frac{1}{2}}\) and \(m^{(i,j)}\)-\(T_{1\frac{1}{2}}\). Let \(A\) be a \(gm^{(i,j)}\)-closed set of \((X, m_X^1, m_X^2)\). Since \((X, m_X^1, m_X^2)\) is a \(m^{(i,j)}\)-\(T_{1\frac{1}{2}}\) space, \(A\) is a \(\xi m^{(i,j)}\)-closed set. Since \((X, m_X^1, m_X^2)\) is a \(m^{(i,j)}\)-\(T_{1\frac{1}{2}}\) space, \(A\) is \(m_X^j\)-closed set of \((X, m_X^1, m_X^2)\). Therefore, \((X, m_X^1, m_X^2)\) is a \(m^{(i,j)}\)-\(T_{1\frac{1}{2}}\).

\[\square\]

5 \ \(gM^{(i,j)}\)-continuous functions

In this section, we introduce \(gM^{(i,j)}\)-continuous functions and investigate some of their characterizations.

\textbf{Definition 5.1.} A function \(f : (X, m_X^1, m_X^2) \rightarrow (Y, m_Y^1, m_Y^2)\) is said to be \(gM^{(i,j)}\)-continuous if \(f^{-1}(F)\) is \(gm^{(i,j)}\)-closed in \(X\) for every \(m_Y^i\)-closed \(F\) of \(Y\), where \(i, j = 1, 2\) and \(i \neq j\).

A function \(f : (X, m_X^1, m_X^2) \rightarrow (Y, m_Y^1, m_Y^2)\) is \(gM^{(i,j)}\)-continuous if and only if \(f^{-1}(U)\) is \(gm^{(i,j)}\)-open in \(X\) for every \(m_Y^i\)-open \(U\) of \(Y\), where \(i, j = 1, 2\) and \(i \neq j\).

\textbf{Theorem 5.2.} Let \((X, m_X^1, m_X^2)\) be a \(m^{(i,j)}\)-\(T_{1\frac{1}{2}}\) space. For an injective function \(f : (X, m_X^1, m_X^2) \rightarrow (Y, m_Y^1, m_Y^2)\), the following are equivalent:

1. \(f\) is \(gM^{(i,j)}\)-continuous;
2. For each \(x \in X\) and for each \(m_X^i\)-open set \(V\) containing \(f(x)\), there exists a \(gm^{(i,j)}\)-open set \(U\) containing \(x\) such that \(f(U) \subseteq V\);
3. \(f(m_X^i - Cl(A)) \subseteq m_Y^i - Cl(f(A))\) for each subset \(A \subseteq X\);
4. \(m_X^i - Cl(f^{-1}(B)) \subseteq f^{-1}(m_Y^i - Cl(B))\) for each subset \(B \subseteq Y\).

\textbf{Proof.} (1) \(\Rightarrow\) (2): Let \(x \in X\) and \(V\) be \(m_Y^i\)-open set containing \(f(x)\). Then by (1), \(f^{-1}(V)\) is \(gm^{(i,j)}\)-open set of \(X\) which containing \(x\). If \(U = f^{-1}(V)\), then \(f(U) \subseteq V\).

(2) \(\Rightarrow\) (3): Let \(A\) be a subset \(X\) and \(f(x) \notin m_Y^i - Cl(f(A))\). Then, there exists \(m_Y^i\)-open set \(V\) of \(Y\) containing \(f(x)\) such that \(V \cap f(A) = \emptyset\). Then by (2), there is a \(gm^{(i,j)}\)-open set \(U\) such that \(f(x) \in f(U) \subseteq V\). Hence, \(f(U) \cap f(A) = \emptyset\) implies \(U \cap A = \emptyset\). Consequently, \(x \notin m_X^i - Cl(A)\) and \(f(x) \notin f(m_X^i - Cl(A))\).

(3) \(\Rightarrow\) (4): Let \(B\) be a subset of \(Y\) and \(A = f^{-1}(B)\). By (3), \(f(m_X^i - Cl(f^{-1}(B))) \subseteq m_Y^i - Cl(f(f^{-1}(B)))\). Thus \(m_X^i - Cl(f^{-1}(B)) \subseteq f^{-1}(m_Y^i - Cl(B))\).
(4) $\Rightarrow$ (1): Let $F$ be a $m_1^i$-closed set of $Y$. Let $U$ be a $m_1^i$-open set of $X$ such that $f^{-1}(F) \subseteq U$. Since $m_1^i$-Cl$(F) = F$ and by (4), $m_1^i$-Cl$(f^{-1}(F)) \subseteq U$. Hence, $f$ is $gM^{(i,j)}$-continuous. 

**Definition 5.3.** A function $f : (X, m_1^X, m_2^X) \to (Y, m_1^Y, m_2^Y)$ is said to be $gM^i$-irresolute if $f^{-1}(F)$ is a $g_m^i$-closed set of $(X, m_1^X)$ for every $g_m^i$-closed $F$ of $(Y, m_1^Y)$. A function $f : (X, m_1^X, m_2^X) \to (Y, m_1^Y, m_2^Y)$ is said to be $gM^i$-irresolute if $f$ is $gM^i$-irresolute for each $i = 1, 2$.

**Definition 5.4.** A function $f : (X, m_1^X, m_2^X) \to (Y, m_1^Y, m_2^Y)$ is said to be $M^i$-closed (resp. $M^i$-open) if $f(F)$ is a $m_1^i$-closed (resp. $m_1^i$-open) set of $(Y, m_1^Y)$ for every $m_1^i$-closed ($m_1^i$-open) set $F$ of $(X, m_1^X)$. A function $f : (X, m_1^X, m_2^X) \to (Y, m_1^Y, m_2^Y)$ is said to be $M$-closed (resp. $M$-open) if $f$ is $M^i$-closed (resp. $M^i$-open) for each $i = 1, 2$.

**Proposition 5.5.** Let $(X, m_1^X, m_2^X)$ be a biminimal structure space, where $m_1^X, m_2^X$ have property $\mathcal{B}$. If $f : (X, m_1^X, m_2^X) \to (Y, m_1^Y, m_2^Y)$ is $gM^i$-irresolute and $M^i$-closed, then $f(A)$ is $\xi gm^{(i,j)}$-closed set of $Y$ for every $\xi gm^{(i,j)}$-closed set of $X$, where $i, j = 1, 2$ and $i \neq j$.

**Proof.** Let $A$ be a $\xi gm^{(i,j)}$-closed set of $X$. Let $U$ be a $g_{m_1}^i$-open set of $Y$ such that $f(A) \subseteq U$. Then $A \subseteq f^{-1}(U)$ and $f^{-1}(U)$ is $g_m^i$-open in $X$, since $f$ is $gM^i$-irresolute. Since $A$ is $\xi gm^{(i,j)}$-closed, $m_1^i$-$\text{Cl}(A) \subseteq f^{-1}(U)$ and hence $f(m_1^i$-$\text{Cl}(A)) \subseteq U$. Therefore, we have $m_1^i$-$\text{Cl}(f(A)) \subseteq m_1^i$-$\text{Cl}(f(m_1^i$-$\text{Cl}(A))) = f(m_1^i$-$\text{Cl}(A)) \subseteq U$, since $f$ is $M^i$-closed. Hence, $f(A)$ is $\xi gm^{(i,j)}$-closed in $Y$. 

**Lemma 5.6.** If $f : (X, m_1^X, m_2^X) \to (Y, m_1^Y, m_2^Y)$ is $M^i$-closed, then for each subset $S \subseteq Y$ and each $m_1^X$-open set $U$ of $X$ containing $f^{-1}(S)$, there exists a $m_1^Y$-open set $V$ of $Y$ containing $S$ such that $f^{-1}(V) \subseteq U$.

**Proposition 5.7.** If $f : (X, m_1^X, m_2^X) \to (Y, m_1^Y, m_2^Y)$ is $M^i$-closed and $gM^{(i,j)}$-continuous, then $f^{-1}(B)$ is $gm^{(i,j)}$-closed set of $X$ for every $gm^{(i,j)}$-closed set $B$ of $Y$, where $i, j = 1, 2$ and $i \neq j$.

**Proof.** Let $B$ be a $gm^{(i,j)}$-closed set of $Y$. Let $U$ be a $m_1^i$-open set of $X$ such that $f^{-1}(B) \subseteq U$. Since $f$ is $M^i$-closed and by Lemma 5.6, there exists a $m_1^i$-open set $V$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$. Since $B$ is $gm^{(i,j)}$-closed and $B \subseteq V$, then $m_1^i$-$\text{Cl}(B) \subseteq V$. Hence, $f^{-1}(m_1^i$-$\text{Cl}(B)) \subseteq f^{-1}(V) \subseteq U$. By Theorem 5.2, $m_1^i$-$\text{Cl}(f^{-1}(B)) \subseteq U$ and hence $f^{-1}(B)$ is $gm^{(i,j)}$-closed set in $X$. 

**Proposition 5.8.** Let $(Y, m_1^Y, m_2^Y)$ be a $m^{(i,j)}$-$T_\mathcal{U}$. Let $f : (X, m_1^X, m_2^X) \to (Y, m_1^Y, m_2^Y)$ and $g : (Y, m_1^Y, m_2^Y) \to (Z, m_1^Z, m_2^Z)$ be functions. If $f$ and $g$ are $gM^{(i,j)}$-continuous, then $g \circ f$ is $gM^{(i,j)}$-continuous.
Proof. Let $F$ be a $m^i_Y$-closed set of $Z$. Since $g$ is $gM^{(i,j)}$-continuous, then $g^{-1}(F)$ is a $gm^{(i,j)}$-closed set of $Y$. Since $(Y,m^1_Y,m^2_Y)$ is a $m^{(i,j)}-T_{\frac{1}{2}}$ space, then $g^{-1}(F)$ is a $m^i_Y$-closed set of $Y$. Since $f$ is $gM^{(i,j)}$-continuous, then $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $gm^{(i,j)}$-closed set of $X$. Therefore, $g \circ f$ is $gM^{(i,j)}$-continuous.

Proposition 5.9. Let $(X,m^1_X,m^2_X)$ be a $m^{(i,j)}-T_{\frac{1}{2}}$ space. If $f : (X,m^1_X,m^2_X) \rightarrow (Y,m^1_Y,m^2_Y)$ is surjective, $M$-closed and $gM^{(i,j)}$-continuous, then $(Y,m^1_Y,m^2_Y)$ is a $m^{(i,j)}-T_{\frac{1}{2}}$ space, where $i, j = 1, 2$ and $i \neq j$.

Proof. Let $F$ be a $gm^{(i,j)}$-closed set of $Y$. By Proposition 5.7, $f^{-1}(F)$ is a $gm^{(i,j)}$-closed set of $X$. Since $(X,m^1_X,m^2_X)$ is a $m^{(i,j)}-T_{\frac{1}{2}}$ space, $f^{-1}(F)$ is $m^i_Y$-closed. It follows by assumptions that, $F$ is $m^i_Y$-closed. Therefore, $(Y,m^1_Y,m^2_Y)$ is a $m^{(i,j)}-T_{\frac{1}{2}}$ space.

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