A Equivalent Condition for Uniform Domains

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Abstract

Suppose that $D$ is a domain in $\mathbb{R}^n$, in this paper, we prove that $D$ is a uniform domain if and only if for any $\alpha \in (0, 1]$ there exists positive constant $M(\alpha)$ such that, each pair of points $z_1, z_2 \in D \setminus \{\infty\}$ can be joined by a rectifiable arc $\gamma \subset D$ such that for any positive integer $n$ and any $0 \leq c_1 < c_2 < \cdots < c_n \leq \frac{1}{2}$ yields

$$\sum_{i=1}^{n-1} \frac{1}{c_{i+1}^\alpha - c_i^\alpha} \int_{\gamma_j[c_i, c_{i+1}]} d(z, \partial D)^{\alpha-1} ds \leq (n - 1)M(\alpha)|z_1 - z_2|^\alpha,$$

where $\gamma_j[c_i, c_{i+1}] = \{\gamma(s) : c_i l \leq s \leq c_{i+1} l\}$ denotes the subcurve of $\gamma$ which starting from $z_j$ and with arc length $s$ of $\gamma$ as parameter, $j = 1, 2, i = 1, 2, \cdots, n-1, l = l(\gamma)$ is the euclidean length of $\gamma$.

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1. Introduction

Suppose that $D$ is a domain in $\mathbb{R}^n$, we say that $D$ is a uniform domain if there are constants $a$ and $b$ such that each pair of points $z_1, z_2 \in D \setminus \{\infty\}$ can be joined by a rectifiable arc $\gamma$ such that for any $0 \leq c_1 < c_2 < \cdots < c_n \leq \frac{1}{2}$ yields
joined by a rectifiable arc $\gamma \subset D$ for which

\[
\begin{aligned}
    l(\gamma) &\leq a|z_1 - z_2|, \\
    \min_{j=1,2} l(\gamma(z_j, z)) &\leq b d(z, \partial D) \quad \text{for all } z \in \gamma.
\end{aligned}
\]  

(1.1)

Here $l(\gamma)$ denotes the euclidean length of $\gamma$, $\gamma(z_j, z)$ the part of $\gamma$ between $z_j$ and $z$, and $d(z, \partial D)$ the euclidean distance from $z$ to $\partial D$.

Uniform domain was first introduced by O. Martio and J. Sarvas [9] when they studied the approximation and injectivity theory. Later, F. W. Gehring and B. G. Osgood [5], P. W. Jones [7], M. Vuorinen [10], V. Lappalainen and A. Lehtonen [8], L. Capogna and P. Q. Tang [2], A. V. Greshnov [6], H. Aikawa and T. Lundh [1], T. Futamura [3] et al. studied the uniform domains extensively, and many interesting and useful results are obtained.

In 1985, F. W. Gehring and O. Martio [4, Theorems 2.2 and 2.24] obtained the following results:

**Theorem A** If $D$ is a uniform domain in $\mathbb{R}^n$, then for any $\alpha \in (0, 1]$ there exists constant $M = M(\alpha)$ such that for all $z_1, z_2 \in D$ there is a rectifiable curve $\gamma$ joining $z_1$ to $z_2$ in $D$ with

\[
\int_{\gamma} d(z, \partial D)^{\alpha - 1} ds \leq M|z_1 - z_2|^\alpha.
\]

The main aim of this paper is to prove the following equivalent condition for uniform domain:

**Theorem 1.1.** If $D$ is a domain in $\mathbb{R}^n$, then $D$ is a uniform domain if and only if for any $\alpha \in (0, 1]$ there exists positive constant $M(\alpha)$ such that, each pair of points $z_1, z_2 \in D \setminus \{\infty\}$ can be joined by a rectifiable curve $\gamma \subset D$ such that for any positive integer $n$ and any $0 \leq c_1 < c_2 < \cdots < c_n \leq \frac{1}{2}$ yields

\[
\sum_{i=1}^{n-1} \frac{1}{c_{i+1}^\alpha - c_i^\alpha} \int_{\gamma_j[c_i, c_{i+1}]} d(z, \partial D)^{\alpha - 1} ds \leq (n - 1) M(\alpha)|z_1 - z_2|^\alpha,
\]

(1.2)

where $\gamma_j[c_i, c_{i+1}] = \{\gamma(s) : c_i l \leq s \leq c_{i+1} l\}$ denotes the subcurve of $\gamma$ which starting from $z_j$ and with arc length $s$ of $\gamma$ as parameter, $j = 1, 2, i = 1, 2, \cdots, n - 1, l = l(\gamma)$ is the euclidean length of $\gamma$.

**2. Proof of Theorem 1.1**

**Proof of Theorem 1.1.** The necessary. If $D$ is a uniform domain, then there exist constants $a$ and $b$ such that each pair of points $z_1, z_2 \in D \setminus \{\infty\}$ can be joined by a rectifiable curve $\gamma \subset D$ for which (1.1) holds. Making use
of (1.1) and straightforward computations reveal that

\[
\frac{1}{c_{i+1}^\alpha - c_i^\alpha} \int_{\gamma_j[c_i, c_{i+1}]} d(z, \partial D)^{\alpha-1} ds \\
\leq \frac{b^{1-\alpha}}{c_{i+1}^\alpha - c_i^\alpha} \int_{\gamma_j[c_i, c_{i+1}]} \left[ \min_{j=1,2} l(\gamma(z_j, z)) \right]^{\alpha-1} ds \\
= \frac{b^{1-\alpha}}{c_{i+1}^\alpha - c_i^\alpha} \int_{c_i}^{c_{i+1}} s^{\alpha-1} ds \\
= \frac{b^{1-\alpha}}{\alpha} l^\alpha \leq \frac{a_0 b^{1-\alpha}}{\alpha} |z_1 - z_2|^\alpha, \quad i = 1, 2, \ldots, n - 1.
\]

This gives

\[
\sum_{i=1}^{n-1} \frac{1}{c_{i+1}^\alpha - c_i^\alpha} \int_{\gamma_j[c_i, c_{i+1}]} d(z, \partial D)^{\alpha-1} ds \leq (n-1) \frac{a_0 b^{1-\alpha}}{\alpha} |z_1 - z_2|^\alpha.
\]

Hence (1.2) holds with \( M(\alpha) = \frac{a_0 b^{1-\alpha}}{\alpha} \).

The sufficiency. Firstly, taking \( n = 2, \alpha = 1, c_1 = 0 \) and \( c_2 = \frac{1}{2} \) in (1.2), straightforward computations reveal that

\[
l(\gamma) \leq M(1) |z_1 - z_2|.
\]

Secondly, taking \( n = 2 \), for any \( 0 \leq c_1 < c_2 \leq \frac{1}{2} \), inequality (1.2) implies that

\[
\frac{1}{c_2^\alpha - c_1^\alpha} \int_{\gamma_j[c_1, c_2]} d(z, \partial D)^{\alpha-1} ds \\
\leq M(\alpha) |z_1 - z_2|^\alpha \leq M(\alpha) l^\alpha \\
= \frac{\alpha M(\alpha)}{c_2^\alpha - c_1^\alpha} \int_{\gamma_j[c_1, c_2]} \left[ \min_{j=1,2} l(\gamma(z_j, z)) \right]^{\alpha-1} ds.
\]

Because of the randomicity of \( 0 \leq c_1 < c_2 \leq \frac{1}{2} \) and the continuity of \( d(z, \partial D)^{\alpha-1} \) and \( \left[ \min_{j=1,2} l(\gamma(z_j, z)) \right]^{\alpha-1} \) on \( \gamma_j[c_1, c_2] \) \((j = 1, 2)\), from (2.2) we can get

\[
d(z, \partial D)^{\alpha-1} \leq \alpha M(\alpha) \left[ \min_{j=1,2} l(\gamma(z_j, z)) \right]^{\alpha-1}
\]

for all \( z \in \gamma \).
In particular, if we take $\alpha = \frac{1}{2}$ in (2.3), then we have

$$\min_{j=1,2} l(\gamma(z_j, z)) \leq \left[ \frac{1}{2} M(\frac{1}{2}) \right]^{\frac{1}{2}} d(z, \partial D).$$

(2.4)

Therefore, $D$ is a uniform domain follows from (2.1) and (2.4).

References


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