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Pareto Optimal Partitions for a Particular Welfare Function

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Abstract

For a partition P of a set C we prove that if P maximizes the generalized utilitarian social welfare function then P is a Pareto optimal partition. We also give a condition under whic the partition that maximizes is unique.

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Introduction

During the middle of the Twentieth Century, economists and mathematicians started to develop a theoretical framework to approach existence regarding the problem of distributing an object. Mathematically speaking, this problem is associated to the cutting a cake. (Steinhaus, 1948 [13]). In a mathematics approach the problem have been studied leaving the normative aspects aside. However, even the most basic criterion used to distibute an object implies taking into account normative aspects. The criterion known as Pareto optimality is followintg Hervé Moulin [11] the most important tool in normative economics given the fact that it is considered the minimum requirement to be fullfill, to have a "good" distribution of an object. In general terms, a partition is said to be "Pareto optimal" if it is efficient. In this sense, the minimum requirement that can be expected of a "good" partition is to be efficient.

It was the pioneering work by Weller [14] that lead to the initial work by mathematicians like J. Barbanel, A. Taylor and others [1,2,3,4,6] in the last decade of the previous century. They constructed a wide theoretical framework related to the existence issues in the problem of "cutting a cake".

The context in which mathematicians and economists have studied the problem of partitioning an object differs in certain aspects. Particularly, economists deal with the allocation of a finite collection of divisible goods, while mathematicians regard it as the division of one single heterogeneous and divisible good, which naturally resembles to a "cake". In a pure exchange economy, before introducing utility functions over different amounts of goods, it is assumed that an agent is indifferent between two different units of the same good. However, this might not be the case always, as ilustrated by Debreu [8, p.14]:

> *...a commodity is therefore defined by a specification of all its physical characteristics, of its availability date, and of its availability location. As soon as one of these factors changes, a different commodity results.*

By considering the tools used by mathematicians to analyze a problem that is studied mostly by economics, the possibility of exploring the concepts of social welfare function in a more general commodity space comes up. In this manner, by considering the elements of welfare economics, we will reach a result that allows to identify unique Pareto optimal partitions by taking into account the welfare of all individuals to which we want to distribute the object.

Let's consider the problem of partitioning a set C in n parts. This problem can be intepreted as one of distributing a cake between n players. Lets assume that the "cake", C, is an arbitrary set and that we want to distribute this between players 1, 2, 3,..., n. We assume that each of the n players has the possibility of evaluating any piece of the "cake" using a specific criterion. We assume that each of these players uses a non-atomic countably additive probability measure to evaluate the pieces of C . This paper is closely relates to the work of Barbanel [3]. Following his study of the structure of Pareto optimal partitions, we extend his results for the case when a generalized social welfare functions is considered.

1 The structure of Pareto optimal partitions

We begin by formalizing individual preferences over the object that will be distributed and can be interpreted as a "cake". In the most general case, the "cake" can be represented by a set C, and the preferences of each of the players involved in the distribution by a reflexive, transitive and complete binary relation R_i, (where i refers to the i player) over set C. However, when dealing with the partition of a single heterogeneous good, an analogous and more adequate manner to represent players' preferences is by means of a finite countably additive non-atomic measure [4].

Suppose that the players are denoted by $1, 2, ..., n$, and that to each player i it correspond a countably additive non-atomic probability measure μ_i defined over a $\sigma-$algebra S of the set C. Each player i uses its own measure μ_i , to evaluate the size of the subset of C that he receives. We assume that each of the n measure is absolutely continous with respect to each of the other $n - 1$ measures. So it makes sense to say that a subset $A \subseteq C$ such that $A \in S$ has positive measure without specifying to which of the n measures we are refering. From this point on, when we refer to a subset of C, we will assume that it belongs to some $\sigma-$álgebra over which all of the n measures are well defined.

Let's consider a partition $P = \{P_1, P_2, ..., P_n\}$ of the cake such that the player i receives piece P_i. We consider the criterion known as Pareto optimality to evaluate this partition.

Definition 1 *A partition $P = \{P_1, P_2, ..., P_n\}$ of C is said to be Pareto optimal if there is no other partition $Q = \{Q_1, Q_2, ..., Q_n\}$ such that $\mu_i(Q_i) \geq \mu_i(P_i)$ for each $i \in \{1, 2, ..., n\}$ with strict inequality for at least one of the n inequalities.*

One of the geometric representations of Pareto optimal partitions is known as the Individual Piece Set (IPS) representation and is define as the set :

$$\{(\mu_1(P_1), \mu_2(P_2), ..., \mu_n(P_n)) : \{P_1, P_2, ..., P_n\} \text{ is a partition of } C\}$$

Each partition of C is associated to a point in the IPS. Dubins and Spanier [9] proved that the Individual Piece Set (IPS) representation is a closed convex subset of $\mathbb{R}^n$.

Definition 2 *The outer boundary of the IPS consists of all points m_1, m_2, ..., m_n in the IPS such that $B^+(m_1, m_2, ..., m_n) \cap IPS = \{(m_1, m_2, ..., m_n)\}$, where $B^+(m_1, m_2, ..., m_n) = \{(q_1, q_2, ..., q_n) \in \mathbb{R}^n : q_i \geq m_i$, for every $i \in \{1, 2, ..., n\}\}$*

With this definition we can appreciate that the outer boundary of the IPS contains those points of the IPS for which it is not possible to improve the situation of all n players.In the literature, the "outer boundary" of the IPS is some times refers as to the Pareto boundary, given that it contains all the points of the IPS associated to Pareto optimal partitions of the set C.

2 An extension for a particular social welfare function

Consider the interior of the closed simplex in $\mathbb{R}^n$,denoted by

$$\vartheta = \{(x_1, x_2, ..., x_n) : x_i > 0 \text{ y } x_1 + x_2 + ... + x_n = 1\}.$$

From the analysis of Pareto optimal partitions, notice that depending on vector $\alpha \in \vartheta$ chosen, we can find different efficient partitions asociated to α. For a convex combinations of measures, we can imagine a way to construct a fictitious measure of preferences by a normative representative player, this is, an external player based on a normative judgement chooses a vector $\alpha \in \vartheta$ associated to the convex combination.

A social welfare function is a function $W : (\mu_1, \mu_2, ..., \mu_n) \longrightarrow \mathbb{R}$ that aggregates the way in which each player evaluates or measures its piece of the cake into a social valuation for each measure vector or profile $\mu = (\mu_1, \mu_2, ..., \mu_n)$ for which there exists a partition of the set C. We will denote $W(\mu_1, \mu_2, ..., \mu_n) = W(\mu)$. Intuitively, we can imagine $W(\mu)$ refelecting the way in which society represented by an external player, sometimes called socail planner, evaluates its own judgements about the desired distribution of the cake.

Even when the social planner chooses Pareto optimal partitions, we know that there can be more than one, which gives room for the planner to choose one in particular. Based on a certain criterion represented by a social welfare function, there exist some desired properties so that the planner does not have the freedom to do whatever he wants. We now present some of these properties.

(No Paternalism) This concept is implicit in the definition of a social welfare function. The expression for social welfare must exclusively take into account the vector $\mu = (\mu_1, \mu_2, ..., \mu_n)$. If there are two partitions $P = \{P_1, P_2, ..., P_n\}$ and $Q = \{Q_1, Q_2, ..., Q_n\}$ such that $\mu_i(P_i) = \mu_i(Q_i)$ for every $i \in \{1, 2, ..., n\}$, then $W(\mu_1(P_1), \mu_2(P_2), ..., \mu_n(P_n)) = W(\mu_1(Q_1), \mu_2(Q_2), ..., \mu_n(Q_n))$.

(Paretian Property) If $W(\mu)$ is monotone increasing, this is if $p_i = \mu_i(P_i) \geq \mu_i(Q_i) = q_i$ for every $i \in \{1, 2, ..., n\}$, then $W(p) \geq W(q)$, and if $p_i = \mu_i(P_i) > \mu_i(Q_i) = q_i$ for $i \in \{1, 2, ..., n\}$, then $W(p) > W(q)$, we say that $W(\mu)$ satisfies the paretian property. If $W(\mu)$ is strictly increasung, then we say that $W(\mu)$ is strictly paretian.

Notice that if $W(\mu)$ is strictly paretian, then the partition for which it attains its maximum is necessarily Pareto optimal.

(Symmetry) $W(\mu)$ is symmetric if $W(\mu) = W(\mu')$ where μ' is a permutation of the entries of μ.

Intuitively, a social welfare function will be symmetric if the index of each player is irrelevant.

(Concavity) The concavity of $W(\mu)$ can be interpreted as inequality aversion by the social planner. Notice that if $W(\mu)$ is concave and $W(\mu) = W(\mu')$, then $W(\frac{1}{2}\mu + \frac{1}{2}\mu') \geq W(\mu)$.

Now lets consider the following social welfare function:

Definition 3 *A social welfare function is a generalized utilitarian if it is of the form* $W(\mu) = \sum_{i=1}^{n} g(\mu_i)$, *where* $g(.)$ *is a monotone increasing and concave function.*

Intuitively, given the valuations that each player assigns to the subsets of a set C that we want to distribute, there is a socially deliberated decision to give a decreasing weight to the valuation of each agent for each additional unit that one receives. In some sense, by considering a generalized utilitarian social welfare function, we want the distribution of the set C to be "more equal". If player i already has a high valuation of the subset he gets relative to player $j's$ valuation, then if we transfer a part of the subset belonging to player i to player j, the social valuation, $W(\mu)$, would increase. Notice that this social welfare function satisfies all the properties stated earlier.

In the Economics' literature, social welfare functions play an important role given their characteristics. However, in the mathematical literature related to Pareto optimal partitions, there is no prior reference to this kind of functions. Next proposition shows a way in which the results related to convex combinations of measures can be extended to other types of social welfare functions. The possibility of refining the set of possible Pareto optimal partitions also provides a more powerful tool in case of applications.

Theorem 1 *Let P be a partition of C. If P maximizes the generalized utilitarian social welfare function $W(\mu)$,then P is a Pareto optimal partition. If the function $g(.)$ associated to $W(\mu)$ is strictly concave then the partition that maximizes it, is unique.*

Proof. Let P be a partition of C, and suppose that $P = \{P_1, P_2, ..., P_n\}$ is not a Pareto optimal partition. Then, there exists a partition $Q = \{Q_1, Q_2, ..., Q_n\}$ of C such that $\mu_i(P_i) \leq \mu_i(Q_i)$ for every $i \ , j \neq i$ and $\mu_j(P_j) < \mu_j(Q_j)$ for some j. Since g is a monotone increasing function we have that $g(\mu_i(P_i)) \leq g(\mu_i(Q_i))$ for every $i \ , \ j \neq i$ and $g(\mu_j(P_j)) < g(\mu_j(Q_j))$ for j. Hence

$$\begin{aligned} & g(\mu_1(P_1)) + g(\mu_2(P_2)) + ... + g(\mu_n(P_n)) \\ < \ & g(\mu_1(Q_1)) + g(\mu_2(Q_2)) + ... + g(\mu_n(Q_n)). \end{aligned}$$

Hence, P does not maximize the generalized utilitarian social welfare function $W(\mu)$.Now suppose that partition P is not unique and that $g(.)$ is strictly concave. Then, there exists another partition Q of C such that Q is Pareto optimal and

$$\begin{aligned} & g(\mu_1(P_1)) + g(\mu_2(P_2)) + ... + g(\mu_j(P_j)) + ... + g(\mu_n(P_n)) \\ = \ & g(\mu_1(Q_1)) + g(\mu_2(Q_2)) + ... + g(\mu_j(Q_j)) + ... + g(\mu_n(Q_n)). \end{aligned}$$

Lets consider a convex combination of $(\mu_1(P_1), \mu_2(P_2), ..., \mu_n(P_n)) \in \mathbb{R}^n$ and $(\mu_1(Q_1), \mu_2(Q_2), ..., \mu_n(Q_n)) \in \mathbb{R}^n$, both being elements of the IPS. Then, by Lyapounov's Theorem, we know that the IPS is a convex set, so there exists a partition $R = \{R_1, R_2, ..., R_n\}$ of C such that $(\mu_1(R_1), \mu_2(R_2), ..., \mu_n(R_n)) = \lambda(\mu_1(P_1), \mu_2(P_2), ..., \mu_n(P_n)) + (1-\lambda)(\mu_1(Q_1), \mu_2(Q_2), ..., \mu_n(Q_n)) \in CPI$ and $\lambda \in (0,1)$. Hence, given $\lambda \in (0,1)$

$$\begin{array}{rcl} \mu_1(R_1) & = & \lambda\mu_1(P_1) + (1-\lambda)\mu_1(Q_1) \\ \mu_2(R_2) & = & \lambda\mu_2(P_2) + (1-\lambda)\mu_2(Q_2) \\ & & . \\ & & . \\ \mu_n(R_n) & = & \lambda\mu_n(P_n) + (1-\lambda)\mu_n(Q_n) \end{array}$$

Since g is strictly concave, then, for every $i \in \{1, 2, ..., n\}$ and $\lambda \in (0,1)$, $g(\mu_i(R_i)) > \lambda g(\mu_i(P_i)) + (1-\lambda)g(\mu_i(Q_i))$. So we get that

$$\begin{array}{rcl} \sum_{i=1}^{n} g(\mu_i(R_i)) & > & \lambda \sum_{i=1}^{n} g(\mu_i(P_i)) + (1-\lambda) \sum_{i=1}^{n} g(\mu_i(Q_i)) \\ & \Longrightarrow & \sum_{i=1}^{n} g(\mu_i(R_i)) > \sum_{i=1}^{n} g(\mu_i(P_i)) \end{array}$$

since

$$\begin{array}{rl} & g(\mu_1(P_1)) + g(\mu_2(P_2)) + ... + g(\mu_n(P_n)) \\ = & g(\mu_1(Q_1)) + g(\mu_2(Q_2)) + ... + g(\mu_n(Q_n)). \end{array}$$

However, this contradicts that partition $P = \{P_1, P_2, ..., P_n\}$ maximizes the social welfare function $W(\mu)$. ■

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