A Note on Hikami’s Mock Theta Functions

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Abstract

In the present paper, we obtain certain properties of Hikami’s Mock theta functions. We also obtain integral representation of the generalized form of their Mock theta functions and also provide certain representation for Hikami’s Mock theta functions.

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1 Introduction

The Mock theta functions in the absence of variable parameters except q could not allow us to the analytical study of these functions through an application of the operator of q-differentiation and q-integration. To explain the properties of Mock theta functions R.P. Agarwal [12] introduced two new parameters x and z in the definition of all the Mock theta functions and defined twenty general functions with a unifying character that all of them satisfy the simple q-difference equation

\[ D_{q,x}F(x, z) = F(x, z + 1) \text{, where } xD_{q,x}F(x, z) = F(x, z) - F(xq, z). \]

Later on G.E. Andrews [5] given certain general functions in solving a single parameter \(\alpha\), which gives certain third order Mock theta functions by taking three parameters function \(\sum_{n} \frac{(a)}{m} t^n\).
The generalized form and alternative form of Mock theta functions developed by the mathematicians namely G.E. Andrews [6, 7], Anju Gupta [1], Sneh D. Prasad [14], Bhaskar Srivastava [2, 3], R.Y. Denis and S.N. Singh [13], Pankaj Srivastava [11] and etc. These new forms of Mock theta functions provided a new platform for further analyzing the properties of Mock theta functions. Recently, Bhaskar Srivastava [2, 3] discussed some properties of Mock theta function of order two and order ten.

Recently, K. Hikami [9, 10] in 2005 and 2006 introduced some new members of Mock theta functions family namely of order four, order eight and order two respectively but Hikami remained silent about their properties. In order to investigate properties of these new members of Mock theta functions family, this article provide a systematic approach to study these functions via their generalized form and subsequently these generalized functions is expressed as $F_q$- function and their respective integral representations.

2 Definitions and Notations

We shall use the following usual basic hypergeometric notations. The q-shifted factorial is defined by

$$ (a; q)_n = \prod_{s=0}^{n-1} (1 - aq^s), \quad n \geq 1. $$

$$ (a; q^r)_n = \prod_{s=0}^{n-1} (1 - aq^{rs}), \quad n \geq 1. $$

$$ (a; q)_0 = 1, \quad (a; q^r)_0 = 1. $$

$$ (a; q^r)_\infty = \prod_{s=0}^{\infty} (1 - aq^{rs}). $$

Basic hypergeometric series is defined by

$$ \phi_{A-1}[a_1, a_2, ..., a_A; b_1, b_2, ..., b_{A-1}; q, z] = \sum_{n=0}^{\infty} \frac{(a_1; q)_n ... (a_A; q)_n}{(b_1; q)_n ... (b_{A-1}; q)_n (q; q)_n} z^n, \quad |z| < 1 $$

Hikami [10] introduced Mock theta function of order two as:

$$ D_5(q) = \sum_{n=0}^{\infty} \frac{q^n(-q; q)_n}{(q; q^2)_{n+1}}. \quad (1) $$
Hikami [9] introduced Mock theta functions of order four and eight as:

\begin{align*}
D_6(q) &= \sum_{n=0}^{\infty} \frac{q^n(-q^2;q^2)_n}{(q^{n+1};q)_{n+1}}. \\
I_{12}(q) &= \sum_{n=0}^{\infty} \frac{q^{2n}(-q;q^2)_n}{(q^{n+1};q)_{n+1}}, \\
I_{13}(q) &= \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q^{n+1};q)_{n+1}}.
\end{align*}

\(D_6(q)\) is of order four, \(I_{12}(q)\) and \(I_{13}(q)\) are of order eight.

### 3 Main Results

We have classified main results in four sections.

(i) Generalized Functions

In this section, we have made an attempt to develop generalized form of Hikami’s Mock theta functions. Each Hikami’s Mock theta function can be written in the form of

\[ F(x, z) = \sum_{n=0}^{\infty} \frac{q^{nz}\lambda_n}{(xq^n; q)_{\infty}} \]

where \(\lambda_n\) is an arbitrary sequence.

By taking \(\lambda_n = C_n(-q; q)_n, C_n(-q^2;q^2)_n(q;q)_n\), \(C_n\frac{q^n(-q^2;q^2)_n(q;q)_n}{(q^2;q^2)_n}\) and \(C_n\frac{(-q^2;q^2)_n(q;q)_n}{(q^2;q^2)_n}\), where \(C_n = \frac{(q;q)_\infty}{(q^n;q^n)_n}\) in (5), we get generalized form of Hikami’s Mock theta functions \(D_5(x, z)\), \(D_6(x, z)\), \(I_{12}(x, z)\) and \(I_{13}(x, z)\) respectively.

\begin{align*}
D_5(x, z) &= (q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{nz}}{(xq^n; q)_{\infty}(q; q)_n(q; q^2)_{n+1}} \\
D_6(x, z) &= (q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^{nz}}{(xq^n; q)_{\infty}(q; q)_n(q^{n+1}; q)_{n+1}} \\
I_{12}(x, z) &= (q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_{n+1} q^{nz}}{(xq^n; q)_{\infty}(q; q)_n(q^{n+1}; q)_{n+1}} \\
I_{13}(x, z) &= (q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-q^2 q^z)_{n+1} q^{nz}}{(xq^n; q)_{\infty}(q; q)_n(q^{n+1}; q)_{n+1}}
\end{align*}
If we replace $x = q$ and $z = 1$ these functions reduce to Hikami’s Mock theta functions $D_5(q), D_6(q), I_{12}(q)$ and $I_{13}(q)$ respectively.

**(ii) Generalized function as $F_q$-function**

In this section, we have developed generalized function of the concerned mock theta functions as $F_q$-function.

As per C. Truesdel [4] concept, the functions which satisfy the functional equation

$$D_{q,x}F(x, z) = F(x, z + 1), \quad (10)$$

where

$$xD_{q,x}F(x, z) = F(x, z) - F(xq, z)$$

is termed as $F_q$-function.

Further, we establish that the function $F(x, z) = \sum_{n=0}^{\infty} \frac{q^{nz} \lambda_n}{(xq^n; q)_\infty}$ satisfies the $q$-difference equation.

$$xD_{q,x}F(x, z) = F(x, z) - F(xq, z)$$

$$xD_{q,x}F(x, z) = \sum_{n=0}^{\infty} \frac{q^{nz} \lambda_n}{(xq^n; q)_\infty} - \sum_{n=0}^{\infty} \frac{q^{nz} \lambda_n}{(xq^{n+1}; q)_\infty}$$

$$= \sum_{n=0}^{\infty} q^{nz} \lambda_n \left[ \frac{1}{(xq^n; q)_\infty} - \frac{1}{(xq^{n+1}; q)_\infty} \right]$$

$$= \sum_{n=0}^{\infty} \frac{q^{nz} \lambda_n}{(xq^n; q)_\infty} \left[ 1 - (1 - xq^n) \right]$$

$$= x \sum_{n=0}^{\infty} \frac{q^{(z+1)n} \lambda_n}{(xq^n; q)_\infty}$$

$$= xF(x, z + 1)$$

Hence,

$$D_{q,x}F(x, z) = F(x, z + 1)$$

As (5) contains only two variable parameters and satisfies the first order $q$-difference equation, so (5) is termed as a general class of $F_q$-Function.

**(iii) Integral Representation**

In this section, integral representation for the generalized form of Hikami’s
Mock theta functions have been developed. These integral representations of $D_5(x, z)$, $D_6(x, z)$, $I_{12}(x, z)$ and $I_{13}(x, z)$ are as follows:

\[
D_5(q, q^z) = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 D_5(0, zt)(tq; q)_\infty \, dq \, t. \tag{11}
\]

\[
D_6(q, q^z) = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 D_6(0, zt)(tq; q)_\infty \, dq \, t. \tag{12}
\]

\[
I_{12}(q, q^z) = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 I_{12}(0, zt)(tq; q)_\infty \, dq \, t. \tag{13}
\]

\[
I_{13}(q, q^z) = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 I_{13}(0, zt)(tq; q)_\infty \, dq \, t. \tag{14}
\]

where

\[
D_5(0, zt) = (q; q)_\infty \sum_{n=0}^{\infty} \frac{(-q^n; q)_{n}(zt)^n}{(q; q)_n(q^n+1; q)_{n+1}}, \quad D_6(0, zt) = (q; q)_\infty \sum_{n=0}^{\infty} \frac{(-q^n; q^2)_{n}(zt)^n}{(q; q)_n(q^n+1; q)_{n+1}},
\]

\[
I_{12}(0, zt) = (q; q)_\infty \sum_{n=0}^{\infty} \frac{(-q^2; q)_{n}(ztq)^n}{(q; q)_n(q^n+1; q)_{n+1}}, \quad I_{13}(0, zt) = (q; q)_\infty \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_{n}(zt)^n}{(q; q)_n(q^n+1; q)_{n+1}}.
\]

First of all, we would like to express the integral representation of the generalized function $F(x, z)$. Thomas and Jackson [8, p.19] defined q- integral as

\[
\int_0^1 f(t) dq \, t = (1-q) \sum_{n=0}^{\infty} f(q^n)q^n.
\]

The limiting case of q-beta integral [8, p.19(1.11.7)] is as follows:

\[
\frac{1}{(q^x; q)_\infty} = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 t^{x-1}(tq; q)_\infty \, dq \, t. \tag{15}
\]

Replacing $x$ by $n+1$ in (15), we get

\[
\frac{1}{(q^{n+1}; q)_\infty} = \frac{(1-q)^{-1}}{(q; q)_\infty} \int_0^1 t^n(tq; q)_\infty \, dq \, t. \tag{16}
\]

We have

\[
F(x, z) = \sum_{n=0}^{\infty} \frac{q^{nz} \lambda_n}{(xq^n; q)_\infty} \tag{17}
\]
writing \( q^z \) for \( z \) and \( x = q \) in (17), we get

\[
F(q, q^z) = \sum_{n=0}^{\infty} \frac{z^n \lambda_n}{(q^{n+1}; q)_{\infty}}
\]  
(18)

Making use of (16) and after simplification, we get

\[
= \frac{(1 - q)^{-1}}{(q; q)_{\infty}} \int_{0}^{1} (tq; q)_{\infty} \sum_{n=0}^{\infty} (zt)^n \lambda_n d_q t.
\]

(19)

writing \( q^z \) for \( z \) after that replacing \( z \) by \( zt \) and taking \( x = 0 \) in (17), we have

\[
F(0, zt) = \sum_{n=0}^{\infty} (zt)^n \lambda_n
\]

(20)

Making use of (20), (19) can be written as

\[
F(q, q^z) = \frac{(1 - q)^{-1}}{(q; q)_{\infty}} \int_{0}^{1} F(0, zt)(tq; q)_{\infty} d_q t.
\]

(21)

which represent the integral representation for \( F(q, z) \).

Following the process, one can establish the integral representation of \( D_5(q) \), \( D_6(q) \), \( I_{12}(q) \) and \( I_{13}(q) \).

(iv) Certain representation of Hikami’s Mock theta functions:

In this section, we show Hikami’s Mock theta functions can be express in the form of basic hypergeometric series. These are as follows:

\[
D_5(q) = \frac{1}{(1 - q)_{3} \phi_2} \left[ \begin{array}{c}
-q, q, 0 \\
q^{3/2}, -q^{3/2}
\end{array} \right]_{\phi_2}
\]

(22)

\[
D_6(q) = \frac{1}{(1 - q)_{4} \phi_3} \left[ \begin{array}{c}
iq, -iq, q, 0 \\
-q, q^{3/2}, -q^{3/2}
\end{array} \right]_{\phi_3}
\]

(23)

\[
I_{12}(q) = \frac{1}{(1 - q)_{4} \phi_3} \left[ \begin{array}{c}
i\sqrt{q}, -i\sqrt{q}, q, 0 \\
-q, q^{3/2}, -q^{3/2}
\end{array} \right]_{\phi_3}
\]

(24)

\[
I_{13}(q) = \frac{1}{(1 - q)_{4} \phi_3} \left[ \begin{array}{c}
i\sqrt{q}, -i\sqrt{q}, q, 0 \\
-q, q^{3/2}, -q^{3/2}
\end{array} \right]_{\phi_3}
\]

(25)

Following the concept of basic hypergeometric functions, representation of \( D_5(q) \), \( D_6(q) \), \( I_{12}(q) \) and \( I_{13}(q) \) can be established.
References


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