A Coupled Fixed Point Theorem for
Contraction Type Maps in
Quasi-ordered Metric Spaces

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Abstract

In this paper, we prove a coupled fixed point theorem for mixed monotone mappings endowed with generalized contraction in quasi-ordered complete metric spaces.

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1 Introduction

The Banach contraction principle is the most celebrated fixed point theorems, many authors extended the Banach contraction principle to the case of nonlinear contraction mappings. Existence of a fixed point in partially ordered metric spaces has been considered recently in [1-3,5-10], where some applications to matrix equations, ordinary differential equations and integral equations are presented.

In this paper, we prove the existence of a coupled fixed point for a mixed monotone mapping \( T : X \times X \rightarrow X \) under a generalized contraction and establish the uniqueness under an additional assumption on the quasi-ordered complete
metric space.

Let $X$ be a nonempty set and “$\preceq$” a quasi-order (preorder or pseudo-order; that is, a reflexive and transitive relation) on $X$ Then $(X, \preceq)$ is called a quasi-order set. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called $\preceq$-nondecreasing (resp. $\preceq$-nonincreasing) if $x_n \preceq x_{n+1}$ (resp. $x_{n+1} \preceq x_n$) for each $n \in \mathbb{N}$. Let $(X, d)$ be a metric space with a quasi-order $\preceq$ ($(X, d, \preceq)$ for short). We endow the product space $X \times X$ with the metric $\rho$ defined by

$$\rho((x, y), (u, v)) = \frac{d(x, u) + d(y, v)}{2}$$

for any $(x, y), (u, v) \in X \times X$.

A map $T : X \times X \to X$ is said to be continuous at $(x^*, y^*) \in X \times X$ if any sequence $\{(x_n, y_n)\} \subset X \times X$ with $(x_n, y_n) \to \rho (x^*, y^*)$ implies $T(x_n, y_n) \to_d T(x^*, y^*)$.

In this paper, we also endow the product space $X \times X$ with the following quasi-order $\preceq$:

$$(u, v) \preceq (x, y) \iff u \preceq x \text{ and } y \preceq v \text{ for any } (x, y), (u, v) \in X \times X.$$

2 Preliminary Notes

**Definition 1** [2] Let $(X, \preceq)$ be a quasi-ordered set and $T : X \times X \to X$ be a map. The mapping $T$ is said to has the mixed monotone property on $X$ if $T$ is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \ x_1 \preceq x_2 \Rightarrow T(x_1, y) \preceq T(x_2, y) \tag{2.1}$$

and

$$y_1, y_2 \in X, \ y_1 \preceq y_2 \Rightarrow T(x, y_1) \succeq T(x, y_2). \tag{2.2}$$

It is obvious that if $T : X \times X \to X$ has the mixed monotone property on $X$, then for any $(x, y), (u, v) \in X \times X$ with $(u, v) \preceq (x, y)$ (i.e. $u \preceq x$ and $y \preceq v$), $T(u, v) \preceq T(x, y)$.

**Definition 2** [2] Let $X$ be a nonempty set and $T : X \times X \to X$ be a map. we call an element $(x, y) \in X \times X$ a coupled fixed point of $T$ if

$$T(x, y) = x \text{ and } T(y, x) = y.$$

**Definition 3** Let $(X, d)$ be a metric space with a quasi-order $\preceq$. A nonempty subset $M$ of $X$ is said to be
Coupled fixed point

(i) sequentially $\preceq \uparrow$-complete if every $\preceq$-nondecreasing Cauchy sequence in $M$ converges;

(ii) sequentially $\preceq \downarrow$-complete if every $\preceq$-nonincreasing Cauchy sequence in $M$ converges;

(iii) sequentially $\preceq \uparrow \downarrow$-complete if it is both $\preceq \uparrow$-complete and $\preceq \downarrow$-complete.

Let $B$ denote the class of those functions $\beta : [0, \infty) \rightarrow [0, 1)$ which satisfies

\[ \beta(w_n) \rightarrow 1 \implies w_n \rightarrow 0. \]  

The following generalization of Banach’s contraction is due to Geraghty [4].

**Theorem 1** Let $(X, d)$ be a complete metric space and let $f : X \rightarrow X$ be a map. Suppose there exists $\beta \in B$ such that for each $x, y \in X$,

\[ d(f(x), f(y)) \leq \beta(d(x, y))d(x, y). \]

Then $f$ has a unique fixed point $x_0 \in X$, and $\{f^n(x)\}$ converges to $x_0$ for each $x \in X$.

In next section, we prove a coupled fixed point theorem according to condition in Theorem 1 in the context of quasi-ordered complete metric spaces for mixed monotone mappings.

### 3 Main Results

**Theorem 2** Let $(X, d, \preceq)$ be a sequentially $\preceq \uparrow \downarrow$-complete metric space and $T : X \times X \rightarrow X$ be a continuous map having the mixed monotone property on $X$. Assume that there exists $\beta \in B$ such that for any $(x, y), (u, v) \in X \times X$ with $(u, v) \preceq (x, y),

\[ d(T(x, y), T(u, v)) \leq \beta(\rho((x, y), (u, v)))\rho((x, y), (u, v)). \]  

If there exists $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \preceq T(y_0, x_0)$, then there exist $x^*, y^* \in X$, such that $x^* = T(x^*, y^*)$ and $y^* = T(y^*, x^*)$.

**Proof.** Let $x_0, y_0 \in X$ be such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \preceq T(y_0, x_0)$. We can choose $x_1, y_1 \in X$ such that $x_1 = T(x_0, y_0)$ and $y_1 = T(y_0, x_0)$, again we can choose $x_2, y_2 \in X$ such that $x_2 = T(x_1, y_1)$ and $y_2 = T(y_1, x_1)$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

\[ x_{n+1} = T(x_n, y_n) \text{ and } y_{n+1} = T(y_n, x_n) \text{ for all } n \geq 0. \]
Further, for \( n = 1, 2, \ldots \), we let,

\[
x_{n+1} = T^{n+1}(x_0, y_0) = T(T^n(x_0, y_0), T^n(y_0, x_0))
\]

and

\[
y_{n+1} = T^{n+1}(y_0, x_0) = T(T^n(y_0, x_0), T^n(x_0, y_0)).
\]

We shall show that

\[
x_n \preceq x_{n+1} \quad \text{for all } n \geq 0 \tag{3.3}
\]

and

\[
y_n \succeq y_{n+1} \quad \text{for all } n \geq 0. \tag{3.4}
\]

We use the mathematical induction. Let \( n = 0 \), since \( x_0 \preceq T(x_0, y_0) \) and \( y_0 \succeq T(y_0, x_0) \), and as \( x_1 = T(x_0, y_0) \) and \( y_1 = T(y_0, x_0) \), we have \( x_0 \preceq x_1 \) and \( y_0 \succeq y_1 \). Thus (3.3) and (3.4) hold for \( n = 0 \). Suppose now that (3.3) and (3.4) hold for some fixed \( n \geq 0 \). Then, since \( x_n \preceq x_{n+1} \) and \( y_n \succeq y_{n+1} \), and as \( T \) has the mixed monotone property, from (2.1) and (2.2),

\[
x_{n+1} = T(x_n, y_n) \preceq T(x_{n+1}, y_n) \quad \text{and} \quad T(y_{n+1}, x_n) \preceq T(y_n, x_n) = y_{n+1}, \tag{3.5}
\]

and

\[
x_{n+2} = T(x_{n+1}, y_{n+1}) \succeq T(x_{n+1}, y_n) \quad \text{and} \quad T(y_{n+1}, x_{n+1}) \succeq T(y_{n+1}, x_n) = y_{n+2}. \tag{3.6}
\]

Now from (3.5) and (3.6) we get

\[
x_{n+1} \preceq x_{n+2} \quad \text{and} \quad y_{n+1} \succeq y_{n+2}.
\]

Thus by the mathematical induction we conclude that (3.3) and (3.4) hold for all \( n \geq 0 \). Therefore,

\[
x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots \tag{3.7}
\]

and

\[
y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \cdots \tag{3.8}
\]

Denote

\[
\delta_n = \rho((x_n, y_n), (x_{n+1}, y_{n+1})) = \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2}.
\]

We show that \( \delta_n < \delta_{n-1} \). Since \( x_{n-1} \preceq x_n \) and \( y_{n-1} \succeq y_n \), from (3.1) we have

\[
d(x_n, x_{n+1}) = d(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \leq \beta(\rho((x_{n-1}, y_{n-1}), (x_n, y_n))) \rho((x_{n-1}, y_{n-1}), (x_n, y_n)) = \beta(\delta_{n-1})\delta_{n-1}.
\]
Similarly,
\[ d(y_{n+1}, y_n) = d(T(y_n, x_n), T(y_{n-1}, x_{n-1})) \]
\[ \leq \beta(\rho((y_n, x_n), (y_{n-1}, x_{n-1}))) \rho((y_n, x_n), (y_{n-1}, x_{n-1})) \]
\[ = \beta(\delta_{n-1})\delta_{n-1}. \]

Adding two above inequalities we have
\[ \delta_n \leq \beta(\delta_{n-1})\delta_{n-1} \leq \delta_{n-1}. \] (3.9)

It follows that a sequence \( \{\delta_n\} \) is a decreasing sequence and bounded below. Therefore, there is some \( \delta \geq 0 \) such that
\[ \lim_{n \to \infty} \delta_n = \delta. \]

We show that \( \delta = 0 \). Suppose to the contrary that \( \delta > 0 \). Then from (3.1) we have
\[ \frac{\delta_{n+1}}{\delta_n} \leq \beta(\delta_n) < 1, \]
the above inequality yields \( \lim_{n \to \infty} \beta(\delta_n) = 1 \), and since \( \beta \in B \) this implies \( \delta = 0 \) Therefore,
\[ \lim_{n \to \infty} \rho((x_n, y_n), (x_{n+1}, y_{n+1})) = \lim_{n \to \infty} \frac{1}{2}[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = 0. \] (3.10)

So,
\[ \lim n \to \infty[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = 0. \] (3.11)

Now, we prove that \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences. Suppose to the contrary that, at least one of \( \{x_n\} \) or \( \{y_n\} \) is not Cauchy sequence. Then there exists an \( \epsilon > 0 \) and two subsequences of integers \( \{l(k)\}, \{m(k)\}, m(k) > l(k) \geq k \) with
\[ d(m(k), l(k)) = d(x_{l(k)}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)}) \geq \epsilon \quad \text{for} \quad k \in \{1, 2, \ldots \}. \] (3.12)

We may also assume
\[ d(x_{l(k)}, x_{m(k)-1}) + d(y_{l(k)}, y_{m(k)-1}) < \epsilon \] (3.13)
by choosing \( m(k) \) to be the smallest number exceeding \( l(k) \) for which (3.12) holds. From (3.12), (3.13) and by the triangle inequality,
\[ \epsilon \leq d(m(k), l(k)) \leq d(x_{l(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) + d(y_{l(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) \]
\[ = d(x_{l(k)}, x_{m(k)-1}) + d(y_{l(k)}, y_{m(k)-1}) + 2\delta_{m(k)-1} \]
\[ < \epsilon + 2\delta_{m(k)-1}. \]
Taking the limit as $k \to \infty$ we get by (3.11),
\[
\lim_{k \to \infty} d_{m(k),l(k)} = \epsilon. \tag{3.14}
\]
Again, the triangle inequality give us
\[
d(x_{l(k)}, x_{m(k)}) \leq d(x_{l(k)}, x_{l(k)-1}) + d(x_{l(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}),
\]
\[
d(x_{l(k)-1}, x_{m(k)-1}) \leq d(x_{l(k)-1}, x_{l(k)}) + d(x_{l(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}),
\]
\[
d(y_{l(k)}, y_{m(k)}) \leq d(y_{l(k)}, y_{l(k)-1}) + d(y_{l(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}),
\]
\[
d(y_{l(k)-1}, y_{m(k)-1}) \leq d(y_{l(k)-1}, y_{l(k)}) + d(y_{l(k)}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)-1}).
\]
Letting $k \to \infty$ in the above four inequalities and using (3.11) and (3.14), we have
\[
\lim_{k \to \infty} d_{m(k)-1,l(k)-1} = \epsilon. \tag{3.15}
\]
As $m(k) > l(k)$ and $x_{l(k)-1}$ and $x_{m(k)-1}$, $y_{l(k)-1}$ and $y_{m(k)-1}$ are comparable, from (3.1) we have
\[
\frac{d_{m(k),l(k)}}{d_{m(k)-1,l(k)-1}} \leq \beta(d_{m(k)-1,l(k)-1}) < 1,
\]
So this yields $\lim_{k \to \infty} \beta(d_{m(k)-1,l(k)-1}) = 1$, and since $\beta \in \mathcal{B}$ this implies $\epsilon = 0$ which is a contradiction. This show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since $X$ is complete, there exist $x^*, y^* \in X$ such that
\[
\lim_{n \to \infty} x_n = x^* \quad \text{and} \quad \lim_{n \to \infty} y_n = y^*. \tag{3.16}
\]
Moreover, the continuity of $T$ implies that
\[
x^* = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} T(x_n, y_n) = T(x^*, y^*),
\]
Also,
\[
y^* = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} T(y_n, x_n) = T(y^*, x^*).
\]
These prove that $(x^*, y^*)$ is a coupled fixed point. □

**Remark 1** Theorem 2 generalizes and improves Bhaskar-Lakshmikantham’s coupled fixed point theorem [2, Theorem 2.1] and some results in [7, 8].

Following a similar argument as in the proof of [2, Theorem 2.2] and applying Theorem 2, one can verify the following result which $T$ is not necessarily continuous.
**Theorem 3** Let \((X,d,\preceq)\) be a sequentially \(\preceq\)-complete metric space and \(T:X \times X \to X\) be a map having the mixed monotone property on \(X\). Assume that

(i) any \(\preceq\)-nondecreasing sequence \(\{x_n\}\) with \(x_n \to x^*\) implies \(x_n \preceq x^*\) for each \(n \in \mathbb{N}\);

(ii) any \(\preceq\)-nonincreasing sequence \(\{y_n\}\) with \(y_n \to y^*\) implies \(y^* \preceq y_n\) for each \(n \in \mathbb{N}\).

Assume that there exists \(\beta \in \mathcal{B}\) such that for any \((x,y),(u,v)\in X \times X\) with \((u,v) \preceq (x,y)\),

\[
d(T(x,y),T(u,v)) \leq \beta(\rho((x,y),(u,v)))\rho((x,y),(u,v)). \tag{3.17}
\]

If there exists \(x_0,y_0 \in X\) such that \(x_0 \preceq T(x_0,y_0)\) and \(y_0 \preceq T(y_0,x_0)\), then there exist \(x^*,y^* \in X\), such that \(x^* = T(x^*,y^*)\) and \(y^* = T(y^*,x^*)\).

**Proof.** According to Theorem 2 there exist a nondecreasing sequence \(\{x_n\}\) and a nonincreasing sequence \(\{y_n\}\) such that \(x_n \to x^*\) and \(y_n \to y^*\), from (i) and (ii) we have \(x_n \preceq x^*\) and \(y_n \preceq y^*\) for all \(n \in \mathbb{N}\). Then by the triangle inequality and (3.17) we get

\[
d(x^*,T(x^*,y^*)) \leq d(x^*,x_{n+1}) + d(x_{n+1},T(x^*,y^*))
\]

\[
= d(x^*,x_{n+1}) + d(T(x_n,y_n),T(x^*,y^*))
\]

\[
\leq d(x^*,x_{n+1}) + \beta\left(\frac{1}{2}[d(x_n,x^*) + d(y_n,y^*)]\right)^\frac{1}{2}[d(x_n,x^*) + d(y_n,y^*)].
\]

So letting \(n \to \infty\) yields \(d(x^*,T(x^*,y^*)) \leq 0\), hence \(x^* = T(x^*,y^*)\). Similarly one can show that \(y^* = T(y^*,x^*)\). \(\square\)

One can prove that the coupled fixed point is unique, provided that the product space \(X \times X\) endowed with the partial order mentioned earlier has the following property:

**Every pair of elements has either a lower bound or upper bound.** \hspace{1cm} (3.18)

**Remark 2** It is known (see [7]) that this condition is equivalent to:

For every \((x,y),(\bar{x},\bar{y})\in X \times X\), there exists a \((z_1,z_2)\in X \times X\) that is comparable to \((x,y)\) and \((\bar{x},\bar{y})\).

**Theorem 4** Adding condition (3.18) to the hypothesis of Theorem 2, we obtain the uniqueness of the coupled fixed point of \(T\).
Proof. If \((\bar{x}, \bar{y})\) is another coupled fixed point of \(T\), then we show that
\[
\rho((x^*, y^*), (\bar{x}, \bar{y})) = 0,
\]
where
\[
\lim_{n \to \infty} T^n(x_0, y_0) = x^*, \quad \text{and} \quad \lim_{m \to \infty} T^m(y_0, x_0) = y^*.
\]
We consider two cases:
Case 1: If \((x^*, y^*)\) is comparable to \((\bar{x}, \bar{y})\) with respect to the ordering in \(X \times X\) then,
\[
d(x^*, \bar{x}) = d(T(x^*, y^*), T(\bar{x}, \bar{y})) \\
\leq \beta(\rho((x^*, y^*), (\bar{x}, \bar{y})))\rho((x^*, y^*), (\bar{x}, \bar{y})),
\]
similarly,
\[
d(y^*, \bar{y}) = d(T(y^*, x^*), T(\bar{y}, \bar{x})) \\
\leq \beta(\rho((x^*, y^*), (\bar{x}, \bar{y})))\rho((x^*, y^*), (\bar{x}, \bar{y})).
\]
By adding two above inequalities we have,
\[
d(x^*, \bar{x}) + d(y^*, \bar{y}) \leq 2[\beta(\rho((x^*, y^*), (\bar{x}, \bar{y})))\rho((x^*, y^*), (\bar{x}, \bar{y}))]
\leq \beta\left(\frac{d(x^*, \bar{x}) + d(y^*, \bar{y})}{2}\right)[d(x^*, \bar{x}) + d(y^*, \bar{y})]
< d(x^*, \bar{x}) + d(y^*, \bar{y}).
\]
Which implies that \(x^* = \bar{x}\) and \(y^* = \bar{y}\).
Case 2: If \((x^*, y^*)\) is not comparable to \((\bar{x}, \bar{y})\), then there exists an upper bound or lower bound \((z_1, z_2)\) \(X \times X\). Then, for all \(n = 0, 1, 2, \ldots\), \((T^n(z_1, z_2), T^n(z_2, z_1))\) is comparable to \((T^n(x^*, y^*), T^n(y^*, x^*)) = (x^*, y^*)\) and \((T^n(z_1, z_2), T^n(\bar{z}_1, \bar{z}_2)) = (\bar{x}, \bar{y})\). Denote
\[
d_n = d(x^*, T^n(z_1, z_2)) + d(y^*, T^n(z_2, z_1)). \quad (3.19)
\]
\[
d'_n = d(\bar{x}, T^n(\bar{z}_1, \bar{z}_2)) + d(\bar{y}, T^n(\bar{z}_2, \bar{z}_1)). \quad (3.20)
\]
We have
\[
d(x^*, T^n(z_1, z_2)) = d(T^n(x^*, y^*), T^n(z_1, z_2)),
\leq \beta\left(\frac{1}{2}\right)[d(T^{n-1}(x^*, y^*, T^{n-1}(z_1, z_2)), T(T^{n-1}(z_1, z_2), T(T^{n-1}(z_2, z_1)))]
\leq \beta\left(\frac{1}{2}\right)[d(T^{n-1}(x^*, y^*), T^{n-1}(z_1, z_2)) + d(T^{n-1}(y^*, x^*), T^{n-1}(z_2, z_1)))],
\]
\[
\left(\frac{1}{2}\right)[d(T^{n-1}(x^*, y^*), T^{n-1}(z_1, z_2)) + d(T^{n-1}(y^*, x^*), T^{n-1}(z_2, z_1))].
\]
Also,
\[ d(y^*, T^n(z_2, z_1)) = d(T^n(\bar{x}, \bar{y})), \]
\[ = d(T(T^{n-1}(y^*, x^*), T^{n-1}(x^*, y^*))) = T(T^n(z_2, z_1), T^{n-1}(z_1, z_2))), \]
\[ \leq \beta\left(\frac{1}{2}[d(T^{n-1}(y^*, x^*), T^{n-1}(z_2, z_1)) + d(T^{n-1}(x^*, y^*), T^{n-1}(z_1, z_2))]\right), \]
\[ \left(\frac{1}{2}[d(T^{n-1}(y^*, x^*), T^{n-1}(z_2, z_1)) + d(T^{n-1}(x^*, y^*), T^{n-1}(z_1, z_2))]\right). \]

Then, we obtain
\[ d_n \leq \beta\left(\frac{1}{2}d_{n-1}\right)d_{n-1} \leq d_{n-1}. \quad (3.21) \]

Hence \( d_n \) is a nonnegative decreasing sequence and, consequently there exists \( d \) such that \( \lim_{n \to \infty} d_n = d \). Letting \( n \to \infty \) in (3.21) implies \( d = 0 \), that is \( \lim_{n \to \infty} d_n = 0 \).

Similarly, we can prove \( \lim_{n \to \infty} d'_n = 0 \). Then by triangle inequalities we obtain \( x^* = \bar{x} \) and \( y^* = \bar{y} \) that is \( (x^*, y^*) = (\bar{x}, \bar{y}) \). \( \square \)

Finally, we discuss the following coupled fixed point theorem in (usual) complete metric spaces.

**Theorem 5** Let \( (X, d) \) be a complete metric space and \( T : X \times X \to X \) be a map having the mixed monotone property on \( X \). Assume that there exists \( \beta \in \mathcal{B} \) such that for any \( (x, y), (u, v) \in X \times X \),
\[ d(T(x, y), T(u, v)) \leq \beta(\rho((x, y), (u, v)))\rho((x, y), (u, v)). \]

Then \( T \) has a unique coupled fixed point in \( X \times X \), that is, there exists unique \( (x^*, y^*) \in X \times X \), such that \( x^* = T(x^*, y^*) \) and \( y^* = T(y^*, x^*) \).

**Proof.** Let \( x_0, y_0 \in X \) be given. For any \( n \in \mathbb{N} \), define \( x_n = T(x_{n-1}, y_{n-1}) \) and \( y_n = T(y_{n-1}, x_{n-1}) \). By our hypothesis, we know that \( T \) is continuous. Following the same argument as in the proof Theorem 2, there exists \( (x^*, y^*) \in X \times X \), such that \( x^* = T(x^*, y^*) \) and \( y^* = T(y^*, x^*) \). We prove the uniqueness of the coupled fixed point of \( T \). On the contrary, suppose that there exists \( (u^*, v^*) \in X \times X \), such that \( u^* = T(u^*, v^*) \) and \( v^* = T(v^*, u^*) \). Then we obtain
\[ d(x^*, u^*) = d(T(x^*, y^*), T(u^*, v^*)) < \frac{1}{2}[d(x^*, u^*) + d(y^*, v^*)] \quad (3.22) \]
and
\[ d(y^*, v^*) = d(T(y^*, x^*), T(v^*, u^*)) < \frac{1}{2}[d(x^*, u^*) + d(y^*, v^*)]. \quad (3.23) \]

It follows from (3.22) and (3.23) that
\[ d(x^*, u^*) + d(y^*, v^*) < d(x^*, u^*) + d(y^*, v^*), \]
a contradiction. The proof is completed. \( \square \)
References


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