Inequalities for the Polar Derivative of a Polynomial

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Abstract

Let

\[ D_\alpha p(z) = np(z) + (\alpha - z)p'(z) \]

be a polar derivative of a polynomial \( p(z) \) of degree \( n \) with respect to the point \( \alpha \), then the polynomial \( D_\alpha p(z) \) is of degree at most \( n - 1 \).

Now corresponding to a given \( n^{th} \) degree polynomial \( p(z) \), we construct a sequence of polar derivatives

\[ D_{\alpha_t}D_{\alpha_{t-1}}...D_{\alpha_1}p(z) = p_t(z) = (n - t + 1)p_{t-1}(z) + (\alpha_t - z)p'_{t-1}(z), \]

\[ t = 1, ..., n - 1, p_0(z) = p(z). \]

The points \( \alpha_1, ..., \alpha_t, t = 1, ..., n - 1, \) may be equal or unequal. Like the \( t^{th} \) ordinary derivative \( p^{(t)}(z) \) of \( p(z) \), the \( t^{th} \) polar derivative \( D_{\alpha_t}...D_{\alpha_1}p(z) \) of \( p(z) \) is a polynomial of degree \( n - t \).

In this paper we shall obtain inequalities for the \( t^{th} \) polar derivatives of the polynomial having all its zeros inside or outside a circle. Our results generalizes as well as improves upon some well known results.

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1. INTRODUCTION AND STATEMENT OF RESULTS

If \( p(z) = \sum_{m=0}^{n} a_m z^m \) is a polynomial of degree \( n \), then

\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.
\] (1)

The above inequality, which is an immediate consequence of Bernstein’s inequality on the derivative of a trigonometric polynomial is best possible with equality holding for the polynomial \( p(z) = \lambda z^n \), \( \lambda \) being a complex number. It is noted that in (1) equality hold if and only if \( p(z) \) has all its zeros at the origin and so it is natural to seek improvements under appropriate assumptions on the zeros of \( p(z) \). If \( p(z) \) having no zeros in \( |z| < 1 \), then the above inequality can be replaced by

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.
\] (2)

Inequality (2) was conjectured by Erdös and later proved by Lax [7]. On the other hand, it was shown by Turán [10] that if all the zeros of \( p(z) \) lie in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|.
\] (3)

Inequality (3) refined by Aziz and Dawood [1] who under the same hypothesis proved that

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \{ \max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)| \}.
\] (4)

As an extension of (2) and (3), Malik [9] proved that if \( p(z) \neq 0 \) in \( |z| < k \), \( k \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |p(z)|,
\] (5)

whereas if \( p(z) \) has all zeros in \( |z| \leq k \), \( k \leq 1 \), then

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{1 + k} \max_{|z|=1} |p(z)|.
\] (6)

Theorem A. If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq k, k \leq 1\), then for \(|z| = 1\)

\[
|p^{(m)}(z)| \geq \frac{n(n-1) \cdots (n-m+1)}{(1+k)^m} |p(z)|. \tag{7}
\]

As a refinement of (6) it was shown by Aziz and Shah [3] for the polynomial \( p(z) \) of degree \( n \) having all its zeros in \(|z| \leq k, k \leq 1\)

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \{ \max_{|z|=1} |p(z)| + \frac{1}{k} \min_{|z|=k} |p(z)| \}. \tag{8}
\]

On the other hand, Dewan, Singh and Lal [5] obtained a generalization of an equality (8) to polar derivative of a polynomial by proving the following result.

Theorem B. Let \( p(z) \) be a polynomial of degree \( n \) having all its zeros in \(|z| \leq k, k \leq 1\), then for every real or complex number \( \alpha \) with \(|\alpha| \geq k\), we have

\[
\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n(|\alpha|-k)}{(1+k)} \max_{|z|=1} |p(z)| + \frac{n(|\alpha|+1)}{k^{n-1}(1+k)} \min_{|z|=k} |p(z)|. \tag{9}
\]

In this paper, we obtain the following result which is a generalization of Theorem B.

Theorem. If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq k, k \leq 1\), then for every \(|\alpha_j| \geq k, i = 1, \ldots, t, (t < n)\), and for \( rR \geq k^2 \) and \( r \leq R \), we have

\[
\max_{|z|=R} |D_{\alpha_1} \cdots D_{\alpha_t} p(z)| \geq \frac{n(n-1) \cdots (n-t+1)}{(R+k)^t} \{ (R+k)^n(|\alpha_t|-k) \cdots (|\alpha_1|-k) \max_{|z|=r} |p(z)|
\]
\[
+ \frac{R^{n-t}}{k^n} (R+k)^t |\alpha_t| \cdots |\alpha_1| - R^t (|\alpha_t|-k) \cdots (|\alpha_1|-k)m \}.
\tag{10}
\]

where \( m = \min_{|z|=k} |p(z)| \).

For \( t = R = r = 1 \), it reduces to the theorem B.

If we take \( r = k = 1 \) in theorem 1, we get following result.
Corollary 1. If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq 1\), then for every \(|\alpha_j| \geq 1, i = 1, \ldots, t, (t < n)\), and for \( 1 \leq R \), we have

\[
\max_{|z|=R} |D_{\alpha_1} \cdots D_{\alpha_t} p(z)| \geq \frac{n(n-1) \cdots (n-t+1)}{(R+1)^t} \left\{ (\frac{R+1}{2})^n (|\alpha_t| - 1) \cdots (|\alpha_1| - 1) \max_{|z|=1} |p(z)| \right.
+ R^{n-t} ((R+1)^t |\alpha_1| \cdots |\alpha_t| - R^t (|\alpha_t| - 1) \cdots (|\alpha_1| - 1))m \}
\]

(11)

where \( m = \min_{|z|=k} |p(z)| \).

We divide both sides of inequality (11) by \(|\alpha_t \cdots \alpha_1|\) and let \(|\alpha_t \cdots \alpha_1| \to \infty\), we have the following corollary 2.

Corollary 2. If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq k, k \leq 1\), then for \( rR \geq k^2, r \leq R \) and \( 0 \leq t < n \), we have

\[
\max_{|z|=R} |p^{(t)}(z)| \geq \frac{n(n-1) \cdots (n-t+1)}{(R+k)^t} \left\{ (\frac{R+k}{r+k})^n \max_{|z|=r} |p(z)| \right.
+ \frac{R^{n-t}}{k^n} ((R+k)^t - R^t)m \}
\]

(12)

where \( m = \min_{|z|=k} |p(z)| \).

This improves the Theorem A .

Remark 1. If we take \( t = R = r = 1 \), we have a result of Govil [6].

2. LEMMAS

For the proofs of these theorems we needs the following lemmas.

Lemma 2.1. If all the zeros of an \( n^{th} \) degree polynomial \( p(z) \) lie in a circular region \( C \) and if none of the points \( \alpha_1, \alpha_2, \ldots, \alpha_t, (t < n) \) lies in the region \( C \) then each of the polar derivatives \( D_{\alpha_1}D_{\alpha_2} \cdots D_{\alpha_t}p(z) \), has all of its zeros in region \( C \).

This lemma is due to laguerre[8].

Lemma 2.2. If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \(|z| \leq k, k \leq 1\), then for every real or complex number \( \alpha \) with \(|\alpha| \geq k\),

\[
\max_{|z|=1} |D_{\alpha} p(z)| \geq n(\frac{|\alpha| - k}{1 + k}) \max_{|z|=1} |p(z)|.
\]

(13)
The result is sharp and equality holds for \( p(z) = (z - k)^n \), with \( \alpha \geq 1 \).

This lemma is due to Aziz and Rather [2].

As an application of above lemma and using mathematical induction, we can have the following lemma.

**Lemma 2.3.** If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k, k \leq 1 \), then for every \( |\alpha_i| \geq k, i = 1, ..., t (t < n) \)

\[
\max_{|z|=1} |D_{\alpha_1}...D_{\alpha_t}p(z)| \geq \frac{n(n-1)...(n-t+1)}{(1+k)^t}(|\alpha_1| - k)...(|\alpha_t| - k) \max_{|z|=1} |p(z)|.
\]  

(14)

Equality hold for \( p(z) = (z - k)^n \) with \( \alpha_1 \geq 1, ..., \alpha_t \geq 1 \).

**Lemma 2.4.** If \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k, k \leq 1 \), then for every \( |\alpha_i| \geq k, i = 1, ..., t (t < n) \)

\[
\max_{|z|=1} |D_{\alpha_1}...D_{\alpha_t}p(z)| \geq \frac{n(n-1)...(n-t+1)}{(1+k)^t}(|\alpha_1| - k)...(|\alpha_t| - k) \max_{|z|=1} |p(z)|
\]

\[
+ \frac{1}{k^n} \{ (1+k)^t|\alpha_1|...|\alpha_t| - (|\alpha_1| - k)...(|\alpha_t| - k) \} \min_{|z|=k} |p(z)|
\]

(15)

where \( m = \min_{|z|=k} |p(z)| \).

**Proof of lemma 2.4.** If \( p(z) \) has a zero on \( |z| = k \), then \( \min_{|z|=k} |p(z)| = m = 0 \) and the result follows from lemma 2.3. Henceforth, we suppose that all the zeros of \( p(z) \) lie in \( |z| < k, k \leq 1 \), so that \( m > 0 \). Now \( m \leq |p(z)| \) for \( |z| = k \), therefore if \( \lambda \) is any real or complex number such that \( |\lambda| < 1 \), then \( \frac{m\lambda z^n}{k^n} < |p(z)| \) for \( |z| = k \).

Since all zeros of \( p(z) \) lie in \( |z| < k \), it follows by Rouch’s theorem that all the zeros of \( F(z) = p(z) - \frac{m\lambda z^n}{k^n} \) also lie in \( |z| < k \). Hence by lemma 2.1, the polynomial \( \{D_{\alpha_1}...D_{\alpha_t}F(z)\} \) for \( |\alpha_1| \geq k, ..., |\alpha_t| \geq k, |\lambda| < 1 \) has all its zeros in \( |z| < k, k \leq 1 \).

Thus we can apply lemma 2.3 to \( p(z) - \frac{m\lambda z^n}{k^n} \), for \( |\alpha_i| \geq k \) & \( i = 1, ..., t \) and obtain

\[
|D_{\alpha_1}...D_{\alpha_t}\{p(z) - \frac{m\lambda z^n}{k^n}\}| \geq \frac{n(n-1)...(n-t+1)}{(1+k)^t}(|\alpha_1| - k)+...+(|\alpha_t| - k)|p(z) - \frac{m\lambda z^n}{k^n}|}
\]
for \(|z| = 1\), \(|\alpha_1| \geq k, \ldots, |\alpha_t| \geq k\),

i.e.

\[
|D_{\alpha_1} \ldots D_{\alpha_t} p(z) - \frac{m\lambda}{k^n} n(n-1) \ldots (n-t+1) \alpha_1 \ldots \alpha_t z^{n-t}| \geq \frac{n(n-1) \ldots (n-t+1)}{(1+k)^t} (|\alpha_1| - k + \ldots + |\alpha_t| - k) |p(z)| - \frac{m\lambda z^n}{k^n}
\]  \hspace{1cm} (16)

for \(|z| = 1\) & \(|\alpha_1| \geq k, \ldots, |\alpha_t| \geq k, |\lambda| < 1\). Now we choose the argument of \(\lambda\), such that, we obtain

\[
|D_{\alpha_1} \ldots D_{\alpha_t} p(z)| - \frac{m|\lambda|}{k^n} n(n-1) \ldots (n-t+1) |\alpha_1| \ldots |\alpha_t|
\geq \frac{n(n-1) \ldots (n-t+1)}{(1+k)^t} (|\alpha_1| - k + \ldots + |\alpha_t| - k) |p(z)| - \frac{m|\lambda|}{k^n}
\]  \hspace{1cm} (17)

for \(|z| = 1\), \(|\lambda| < 1\) & \(|\alpha_1| \geq k, \ldots, |\alpha_t| \geq k\).

Now if we make \(|\lambda| \to 1\), we obtain inequality (15).

This completes the proof of lemma 2.4.

**Lemma 2.5.** If \(p(z)\) is a polynomial of degree \(n\) having all its zeros in \(|z| < k\), \(k > 0\), then for \(rR \geq k^2\) and \(r \leq R\), we have for \(|z| = 1\),

\[
|p(Rz)| \geq (\frac{R+k}{r+k})^n |p(rz)|.
\]  \hspace{1cm} (18)

Equality in (20) holds for the polynomial \(p(z) = (z+k)^n\).

Lemma 5 is due to Aziz and Zargar [?].

3. PROOF OF THE THEOREM

**Proof of theorem.** By hypothesis the polynomial \(p(z)\) has all its zeros in \(|z| \leq k\), where \(k \geq 1\), therefore it follows that \(F(z) = p(Rz)\) has all its zeros in \(|z| \leq \frac{k}{R}\), where \(\frac{k}{R} \geq 1\) and \(|\alpha_1| > \frac{k}{R}, \ldots, |\alpha_t| > \frac{k}{R}\). Applying lemma (2.4) to the polynomial \(F(z)\), we get

\[
\max_{|z|=1} |D_{\alpha_1} \ldots D_{\alpha_t} F(z)| \geq \frac{n(n-1) \ldots (n-t+1)}{(1+k)^t} \min_{|z|=\frac{k}{R}} |F(z)|
\]

\[
+ \frac{R^n}{k^n} \left((1+k)^t |\alpha_1| \ldots |\alpha_t| - (|\alpha_1| - \frac{k}{R}) \ldots (|\alpha_t| - \frac{k}{R})\right) \min_{|z|=\frac{k}{R}} |F(z)|, \hspace{1cm} (19)
\]
which gives

\[
\max_{|z|=R} |D_{R\alpha_1}...D_{R\alpha_t} p(z)| \geq \frac{n(n-1)...(n-t+1)}{(R+k)^t} \left( ((|R\alpha_t|-k)...(|R\alpha_1|-k) \max_{|z|=R} |p(z)|) \right.
\]
\[
+ \frac{R^n}{k^n} \left\{ (R+k)^t - (|R\alpha_1|-|R\alpha_t|) - ((|R\alpha_1|-k)...(|R\alpha_t|-k)) \right\} \left( \max_{|z|=R} |p(z)| \right).
\]

(20)

Since \(|R\alpha_i| \geq k\), replacing \(R\alpha_i\) by \(\alpha_i\) for \(i = 1, ..., t\), we get

\[
\max_{|z|=R} |D_{\alpha_1}...D_{\alpha_t} p(z)| \geq \frac{n(n-1)...(n-t+1)}{(R+k)^t} \left( ((|\alpha_t|-k)...(|\alpha_1|-k) \max_{|z|=R} |p(z)|) \right.
\]
\[
+ \frac{R^n}{k^n} \left\{ (R+k)^t - (|\alpha_1|-|\alpha_t|) - ((|\alpha_1|-k)...(|\alpha_t|-k)) \right\} \min_{|z|=k} |p(z)|\] \]

\[
(21)
\]

Now if \(0 \leq r \leq R \leq k\), then by lemma 2.5, we have

\[
\max_{|z|=R} |p(z)| \geq \left( \frac{R+k}{r+k} \right)^n \max_{|z|=r} |p(z)|. \]

(22)

From (21) and (22), we obtain inequality (11).

This complete, the proof theorem.

References


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