On the Growth of Entire Functions
of Several Complex Variables

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Abstract

In this paper we generalize and improve the results of R. K. Sri-
vastava, Vinod Kumar [4] and S. S. Dalal [1]. Here we considered the
Taylor series expansion of an entire function in terms of homogeneous
polynomials of degree \((m + n)\) in two complex variables.

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lower type

1. Introduction

If \(\nu : \mathbb{C}^2 \rightarrow \mathbb{R}^+ = [0, \infty]\), be a real-valued function such that the following
conditions hold:

\[(i) \quad \nu(z + z') \leq \nu(z) + \nu(z') \quad \forall z, z' \in \mathbb{C}^2,

(ii) \quad \nu(\lambda z) \leq |\lambda|\nu(z) \quad \forall \lambda \in \mathbb{C}

(iii) \quad \nu(z) = 0 \iff z = 0, \quad \text{then } \nu \text{ is a norm.}\]

Let \(f(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}(z_1, z_2)\), the Taylor series expansion of \(f(z_1, z_2)\)
in terms of homogeneous polynomials \(P_{m,n}(z_1, z_2) : \mathbb{C}^2 \rightarrow \mathbb{C}\) of degree \((m+n)\).

We have

\[M(r_1, r_2) = \sup_{\nu(z \leq r)} |f(z_1, z_2)|, \quad t = 1, 2, \quad r = \max(r_1, r_2),\]
is the maximum modulus of $f(z_1, z_2) \forall r_1, r_2 \in R^+$ with respect to the norm $\nu$.

Define

$$C_{m,n} = \sup_{\nu(z_1) \leq 1} |P_{m,n}(z_1, z_2)|.$$ 

The order, lower order and type of the function are defined respectively by

$$\rho = \lim_{r_1, r_2 \to \infty} \sup \inf \frac{\log \log M(r_1, r_2)}{\log(r_1, r_2)}$$

$$\lambda = \lim_{r_1, r_2 \to \infty} \sup \inf \frac{\log M(r_1, r_2)}{(r_1^\rho + r_2^\rho)}.$$ 

In [2], we have proved the following results

$$\rho = \limsup_{m+n \to \infty} \frac{\log[(m+n)\alpha_{m+n}]}{\log(C_{m,n})^{-1/m+n}},$$

$$e^\rho T = \limsup_{m+n \to \infty} \frac{(m+n)\alpha_{m,n}}{(C_{m,n})^{-\rho/m+n}}.$$ (1.1)

where

$$\alpha_{m,n} = \begin{cases} \frac{[m^n n^m]^{1/m+n}}{(m+n)} & \text{if } m, n \geq 1 \\ 0 & \text{if } m, n = 0 \end{cases}$$

Analogously, the lower order and lower type are defined by

$$\lambda = \liminf_{m+n \to \infty} \frac{\log[(m+n)\alpha_{m,n}]}{\log(C_{m,n})^{-1/m+n}},$$

$$e^\lambda t = \liminf_{m+n \to \infty} \frac{(m+n)\alpha_{m,n}}{(C_{m,n})^{-\rho/m+n}}.$$ (1.3)

In this paper we have generalized and improved the results of R. K. Srivastava, Vinod Kumar [4] and S. S. Dalal [1]. Surprisingly, they have not mention the factor $\alpha_{m,n}$ in their results. Here I have defined the orders and types different from those of above author. To reduce the mechanical labour we have considered only two variables, though the results can easily be extended to several complex variables.
2. Main Results

**Theorem 1.** \( f_i(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}^{(i)}(z_1, z_2) \), where \( i = 1, \ldots, k \), be \( k \) entire functions of finite orders \( \rho_1, \rho_2, \ldots, \rho_k \), and nonzero lower orders \( \lambda_1, \lambda_2, \ldots, \lambda_k \), respectively. Then the function

\[
f(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}(z_1, z_2)
\]

where

\[
C_{m,n} \sim \prod_{i=1}^{k} (C_{m,n}^{(i)})^{m_i}, \text{m_i are constant}
\]

is an entire function such that

\[
\sum_{i=1}^{k-1} \frac{m_i}{\rho_i} \leq \left\{ \left( \frac{1}{\rho} - \frac{m_k}{\rho_k} \right), \left( \frac{1}{\lambda} - \frac{m_k}{\lambda_k} \right) \right\} \leq \sum_{i=1}^{k-1} \frac{m_i}{\lambda_i}
\]

**(2.1)**

**Proof.** In can be easily seen [3, p. 9] that necessary and sufficient condition for \( f(z_1, z_2) \) to represent an entire function of two complex variables \( z_1 \) and \( z_2 \) is

\[
\limsup_{m+n \to \infty} (C_{m,n})^{1/m+n} = 0.
\]

since \( f_i(z_1, z_2) \) are entire functions so it leads

\[
\limsup_{m+n \to \infty} (C_{m,n}^{(i)})^{1/m+n} = 0, \text{ for } i = 1, \ldots, k.
\]

Also

\[
C_{m,n} \sim \prod_{i=1}^{k} (C_{m,n}^{(i)})^{m_i},
\]

which gives

\[
\limsup_{m+n \to \infty} \left( \frac{C_{m,n}}{\alpha_{m,n}} \right)^{1/m+n} \leq \prod_{i=1}^{k} \limsup_{m+n \to \infty} \left[ \left( \frac{C_{m,n}^{(i)}}{\alpha_{m,n}^{(i)}} \right)^{m_i} \right]^{1/m+n}.
\]

**(2.2)**

Hence \( f(z_1, z_2) \) is an entire function.

Now applying (1.1) and (1.3) for functions \( f_i(z_1, z_2) \), we get

\[
\lim_{m+n \to \infty} \sup \frac{\log[(m+n)\alpha_{m,n}^{(i)}]}{\inf \log(C_{m,n}^{(i)})^{-1/m+n}} = \frac{\rho_i}{\lambda_i},
\]

\[
\log(C_{m,n}^{(i)})^{m_i} < -\frac{m_i(m+n)\log((m+n)\alpha_{m,n})}{(\rho_i + \varepsilon)} \text{ for } m+n > m_0 + n_0,
\]

**(2.3)**
and
\[
\log(C_{m,n}^{(i)})^m < \frac{-m_i(m+n)\log((m+n)\alpha_{m,n})}{(\lambda_i + \varepsilon)} \tag{2.4}
\]
for an infinite sequence of values of m,n. Taking \(i = 1, \ldots, k\), in (2.3) and adding, we get
\[
\log \Pi_{i=1}^k(C_{m,n}^{(i)})^m < -(m+n)\log((m+n)\alpha_{m,n}) \sum_{i=1}^k \frac{m_i}{(\rho_i + \varepsilon)}.\]
Using (2.2), we have
\[
\log C_{m,n} < -(m+n)\log((m+n)\alpha_{m,n}) \sum_{i=1}^k \frac{m_i}{(\rho_i + \varepsilon)},
\]
or
\[
\limsup_{m+n \to \infty} \frac{\log((m+n)\alpha_{m+n})}{\log(C_{m,n})} \cdot \log((m+n)\alpha_{m,n})^{-1} < \sum_{i=1}^k \left( \frac{m_i}{\rho_i} \right)^{-1},
\]
or
\[
\rho \leq \left( \sum_{i=1}^k \frac{m_i}{\rho_i} \right)^{-1}
\]
or
\[
\sum_{i=0}^{k-1} \frac{m_i}{\rho_i} \leq \frac{1}{\rho} - \frac{m_k}{\rho_k},
\]
which proves a part of the left hand side of (2.1).

In the same way by taking \(i = 1, \ldots, k - 1\) in (2.3) and \(i = k\) in (2.4) and adding, we obtain
\[
\sum_{i=0}^{k-1} \frac{m_i}{\rho_i} \leq \frac{1}{\lambda} - \frac{m_k}{\lambda_k}
\]
which proves the second part of the left hand side of (2.1).

Similarly the right hand side can be proved.

**Theorem 2.** \(f_i(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}^{(i)}(z_1, z_2)\), where \(i = 1, \ldots, k\), be \(k\) entire functions of finite nonzero orders \(\rho_1, \rho_2, \ldots, \rho_k\), and types \(T_1, T_2, \ldots, T_k\), and lower type \(t_1, t_2, \ldots, t_k\) respectively \((0 < t_i < T_i < \infty)(i = 1, \ldots, k)\). Then the function
\[
f(z_1, z_2) = \sum_{m,n=0}^{\infty} P_{m,n}(z_1, z_2),
\]
is an entire function such that

\[
\Pi_{i=1}^{k-1}(\rho_i t_i)^{m_i/\rho_i} \leq \left\{ \frac{(\rho t)^{1/\rho}}{(\rho k t_k)^{m_k/\rho_k}}, \frac{(\rho T)^{1/\rho}}{(\rho k T_k)^{m_k/\rho_k}} \right\} \leq \Pi_{i=1}^{k-1}(\rho_i T_i)^{m_i/\rho_i}
\tag{2.5}
\]

where \(\rho, T\) and \(t\) are order, type and lower type of \(f(z_1, z_2)\) respectively.

**Proof.** Using (1.2) and (1.4) for functions \(f_i(z_1, z_2)\), we have

\[
(C_{m,n}^{(i)})^{m_i} < \left[ \frac{[e \rho_i(T_i + \varepsilon)]^{m+n}}{((m+n)(\alpha_{m,n}))^{m+n}} \right]^{m_i/\rho_i}
\quad \text{for } m+n > m_0 + n_0,
\tag{2.6}
\]

and

\[
(C_{m,n}^{(i)})^{m_i} < \left[ \frac{[e \rho_i(t_i + \varepsilon)]^{m+n}}{((m+n)(\alpha_{m,n}))^{m+n}} \right]^{m_i/\rho_i}
\tag{2.7}
\]

for an infinite sequence of values of \(m, n\).

Taking \(i = 1, \ldots, k\) in (2.6) and multiplying, we have

\[
\prod_{i=1}^{k} (C_{m,n}^{(i)})^{m_i} \leq \prod_{i=1}^{k} \left[ \frac{[e \rho_i(T_i + \varepsilon)]^{m+n}}{((m+n)(\alpha_{m,n}))^{m+n}} \right]^{m_i/\rho_i}.
\]

Using (2.2), we get

\[
C_{m,n} < \left\{ \left[ \frac{e \rho_k(T_k + \varepsilon)}{(m+n)\alpha_{m,n}} \right]^{m+n} \right\}^{m_k/\rho_k} \prod_{i=1}^{k-1} \left\{ \left[ \frac{e \rho_i(T_i + \varepsilon)}{(m+n)\alpha_{m,n}} \right]^{m+n} \right\}^{m_i/\rho_i}
\]

or

\[
(C_{m,n})^{\rho/m+n} < \left\{ \left[ \frac{e \rho_k(T_k + \varepsilon)}{(m+n)\alpha_{m,n}} \right]^{\rho} \right\}^{m_k/\rho_k} \prod_{i=1}^{k-1} \left\{ \left[ \frac{e \rho_i(T_i + \varepsilon)}{(m+n)\alpha_{m,n}} \right]^{\rho} \right\}^{m_i/\rho_i}.
\]

Using the fact that if any \(k\) \(L\)'s out of \(L, L_1, \ldots, L_k\), are equal to one, then all the \((k+1)\) \(L\)'s are equal to one and

\[
\frac{1}{\rho} = \sum_{i=1}^{k-1} \frac{m_i}{\rho_i}, \quad L = \frac{\rho}{\lambda}, \quad L_i = \frac{\rho_i}{\lambda_i}.
\]

We get

\[
\frac{(m+n)\alpha_{m,n}}{(C_{m,n})^{-\rho/m+n}} < e \left\{ [\rho_k(T_k + \varepsilon)]^{\rho} \right\}^{m_k/\rho_k} \prod_{i=1}^{k-1} \left\{ [\rho_i(T_i + \varepsilon)]^{\rho} \right\}^{m_i/\rho_i}.
\]
In view of (1.2), we get
\[
\frac{(\rho T)^{1/\rho}}{\rho_k T_k m_k/\rho_k} \leq \prod_{i=1}^{k-1} (\rho_i T_i)^{m_i/\rho_i}.
\]  
(2.8)

Now taking \(i = 1, \ldots, k-1\) in (2.6) and \(i = k\) in (2.7) and multiplying, we get
\[
\frac{(\rho t)^{1/\rho}}{(\rho_k t_k)^{m_k/\rho_k}} \leq \prod_{i=1}^{k-1} (\rho_i T_i)^{m_i/\rho_i}.
\]  
(2.9)

(2.8) and (2.9) together proves that left hand side of (2.5). Similarly the right hand side can be proved. Hence the proof of Theorem 2 is complete.

References


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