Three-Point Boundary Value Problems for Second-Order Impulsive Integro-Differential Equations

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Abstract

In this paper, by using the method of lower and upper solutions coupled with monotone iterative technique, we investigate the existence of extreme solutions of the three-point boundary value problem for second-order impulsive integro-differential equations. Some comparison results are also formulated.

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1 Introduction

Impulsive differential equations appear as a description of many real world applications which have a short-term rapid change of their states at certain moments (see [1]). In this area, there are many publications that study two-point
conditions for boundary value problems of second-order impulsive differential equations by using the method of upper and lower solutions coupled with the monotone iterative techniques (see, [2-9] etc). However only a few papers, [10-12], have appeared where this technique is applied to second-order three-point boundary value problems without impulses.

In this paper, we study three-point boundary value problem for second-order impulsive integro-differential equations:

\[
\begin{aligned}
&x''(t) = f(t, x(t), (Tx)(t), (Sx)(t)) \equiv Fx(t), \quad t \neq t_k, \quad t \in J = [a, b], \\
&\Delta x(t_k) = I_k(x(t_k), x'(t_k)), \quad k = 1, \ldots, m, \\
&\Delta x'(t_k) = I_k^*(x(t_k)), \quad k = 1, \ldots, m, \\
&\omega x(a) = x(\eta), \quad x'(b) = 0,
\end{aligned}
\]

where \( a = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b \), \( \eta \in (a, b) \), \( \omega > 1 \), \( f \in C(J \times R^3, R) \), \( I_k \in C(R^2, R) \), \( I_k^* \in C(R, R) \),

\[
(Tx)(t) = \int_a^t k(t, s)x(s)ds,
\]

\[
(Sx)(t) = \int_a^b h(t, s)x(s)ds,
\]

\( k \in C(D, R^+) \), \( D = \{(t, s) \in J \times J : t \geq s\} \), \( h \in C(J \times J, R^+) \), \( \Delta x(t_k) = x(t_k^+) - x(t_k^-) \), \( \Delta x'(t_k) = x'(t_k^+) - x'(t_k^-) \), \( x(t_k^+) \) and \( x(t_k^-) \) denote the right and left limits of \( x \) at \( t_k \), respectively. Similarly, \( x'(t_k^+) \) and \( x'(t_k^-) \) denote the right and left limits of \( x' \) at \( t_k \), respectively.

\section{Preliminaries}

Let \( J' = J \setminus \{t_1, t_2, \ldots, t_m\} \), \( J_0 = [t_0, t_1] \), \( J_k = (t_k, t_{k+1}] \), \( k = 1, 2, \ldots, m \), \( PC(J, R) = \{x : J \rightarrow R \mid x(t) \text{ be continuous everywhere except for some} \ t_k \text{ at which} \ x(t_k^+) \text{ and} x(t_k^-) \text{ exist and} x(t_k^-) = x(t_k)\} \), \( PC^1(J, R) = \{x \in PC(J, R) \mid x(t) \text{ be continuous everywhere except for some} \ t_k \text{ at which} \ x'(t_k^+) \text{ and} x'(t_k^-) \text{ exist and} x'(t_k^-) = x'(t_k)\} \). Let \( PC(J, R) \) and \( PC^1(J, R) \) be Banach spaces with the respective norms,

\[
\|x\|_{PC} = \sup_{t \in J} |x(t)|, \quad \|x\|_{PC^1} = \max_{t \in J} \{\|x\|_{PC}, \|x'\|_{PC}\}.
\]

A function \( x \in PC^1(J, R) \cap C^2(J', R) \) is called a solution of BVP (1) if it satisfies (1).

\textbf{Definition 2.1} A function \( y_0 \in PC^1(J, R) \cap C^2(J', R) \) is called a lower solution of BVP (1) if

\[
\begin{aligned}
&y''_0(t) \geq f(t, y_0(t), (Ty_0)(t), (Sy_0)(t)) \equiv Fy_0(t), \quad t \in J', \\
&\Delta y_0(t_k) = I_k(y_0(t_k), y'_0(t_k)), \quad k = 1, \ldots, m, \\
&\Delta y'_0(t_k) \geq I_k^*(y_0(t_k)), \quad k = 1, \ldots, m, \\
&\omega y_0(a) \leq y_0(\eta), \quad y'_0(b) \leq 0.
\end{aligned}
\]
**Definition 2.2** A function \( z_0 \in PC^1(J, R) \cap C^2(J', R) \) is called an upper solution of BVP (1) if

\[
\begin{cases}
    z_0''(t) \leq f(t, z_0(t)), & t \in J', \\
    \Delta z_0(t_k) = I_k(z_0(t_k), z_0'(t_k)), & k = 1, ..., m, \\
    \Delta z_0'(t_k) \leq I_k^*(z_0(t_k)), & k = 1, ..., m, \\
    \omega z_0(a) \geq z_0(\eta), & z_0(b) \geq 0.
\end{cases}
\]

(3)

Consider the BVP

\[
\begin{cases}
    p''(t) = M p(t) + N(T p)(t) + N_1(S p)(t) + \sigma(t), & t \in J', \\
    \Delta p(t_k) = L_k p'(t_k) + \lambda_k, & k = 1, ..., m, \\
    \Delta p'(t_k) = L_k^* p(t_k) + \gamma_k, & k = 1, ..., m, \\
    \omega p(a) = p(\eta), & p(b) = 0.
\end{cases}
\]

(4)

where \( M > 0, N \geq 0, N_1 \geq 0, \omega > 1, \eta \in (a, b), \lambda_k, \gamma_k, k = 1, ..., m, \) are constants and \( \sigma(t) \in PC(J, R). \)

**Lemma 2.3** \( p \in PC^1(J, R) \cap C^2(J', R) \) is a solution of (4) if and only if \( p \in PC^1(J, R) \) is a solution of the impulsive integral equation

\[
p(t) = \int_a^b G_1(t, s) \sigma_1(s) ds + \sum_{k=1}^m G_1(t, t_k)(L_k p(t_k) + \gamma_k) + \sum_{k=1}^m G_2(t, t_k)(L_k^* p(t_k) + \lambda_k),
\]

(5)

where \( \sigma_1(t) = M p(t) + N(T p)(t) + N_1(S p)(t) + \sigma(t), \)

\[
G_1(t, s) = \frac{1}{\omega - 1} \begin{cases} 
-\omega s + \omega a, & a \leq s \leq t \leq \eta \leq b, \\
-\omega s + \omega a, & a \leq \eta \leq t \leq b, \\
-\eta - \omega s + s + \omega a, & a \leq \eta \leq s \leq t \leq b, \\
-\eta - \omega t + t + \omega a, & a \leq \eta \leq t \leq s \leq b, \\
-\eta - \omega t + t + \omega a, & a \leq t \leq \eta \leq s \leq b, \\
-s - \omega t + t + \omega a, & a \leq t \leq s \leq \eta \leq b,
\end{cases}
\]

and

\[
G_2(t, s) = \frac{1}{\omega - 1} \begin{cases} 
\omega, & a \leq s \leq t \leq \eta \leq b, \\
\omega, & a \leq s \leq \eta \leq t \leq b, \\
\omega - 1, & a \leq \eta \leq s \leq t \leq b, \\
0, & a \leq \eta \leq t \leq s \leq b, \\
0, & a \leq t \leq \eta \leq s \leq b, \\
1, & a \leq t \leq s \leq \eta \leq b.
\end{cases}
\]
Proof. Suppose that \( p(t) \) is a solution of (4) for \( t \in J \). Integrating (4) from \( a \) to \( t \), it follows that

\[
p'(t) = p'(a) + \int_a^t \sigma_1(s)ds - \sum_{a < t_k < t} \left( L_k^* p(t_k) + \gamma_k \right).
\] (6)

Again integrating (6) from \( a \) to \( t \), then

\[
p(t) = p(a) + p'(a)(t - a) + \int_a^t (t - s)\sigma_1(s)ds + \sum_{a < t_k < t} \left( L_k^* p(t_k) + \gamma_k \right)(t - t_k) + \sum_{a < t_k < t} \left( L_k p'(t_k) + \lambda_k \right).
\] (7)

By (6) and \( p'(b) = 0 \), we have

\[
0 = p'(b) = p'(a) + \int_a^b \sigma_1(s)ds + \sum_{a < t_k < b} \left( L_k^* p(t_k) + \gamma_k \right).
\]

Thus

\[
p'(a) = -\int_a^b \sigma_1(s)ds - \sum_{a < t_k < b} \left( L_k^* p(t_k) + \gamma_k \right).
\] (8)

Substituting \( t = \eta \) into (7), we get

\[
p(\eta) = p(a) + p'(a)(\eta - a) + \int_a^\eta (\eta - s)\sigma_1(s)ds + \sum_{a < t_k < \eta} \left( L_k^* p(t_k) + \gamma_k \right)(\eta - t_k) + \sum_{a < t_k < \eta} \left( L_k p'(t_k) + \lambda_k \right).
\]

Since \( \omega p(a) = p(\eta) \), we have

\[
p(a) = \frac{1}{\omega - 1} \left[ p'(a)(\eta - a) + \int_a^\eta (\eta - s)\sigma_1(s)ds + \sum_{a < t_k < \eta} \left( L_k^* p(t_k) + \gamma_k \right)(\eta - t_k) + \sum_{a < t_k < \eta} \left( L_k p'(t_k) + \lambda_k \right) \right].
\]

Then

\[
p(t) = \frac{1}{\omega - 1} \left[ \int_a^\eta (\eta - s)\sigma_1(s)ds + \sum_{a < t_k < \eta} \left( L_k^* p(t_k) + \gamma_k \right)(\eta - t_k) \right.
\]

\[
+ \sum_{a < t_k < \eta} \left( L_k p'(t_k) + \lambda_k \right) - \left( \frac{\eta + \omega t - t - \omega a}{\omega - 1} \right) \left[ \int_a^\eta \sigma_1(s)ds + \sum_{a < t_k < \eta} \left( L_k^* p(t_k) + \gamma_k \right) + \int_a^t (t - s)\sigma_1(s)ds \right.
\]

\[
+ \sum_{a < t_k < b} \left( L_k^* p(t_k) + \gamma_k \right) + \int_a^t (t - s)\sigma_1(s)ds + \sum_{a < t_k < t} \left( L_k^* p(t_k) + \gamma_k \right)(t - t_k) + \sum_{a < t_k < t} \left( L_k p'(t_k) + \lambda_k \right) \bigg].
\] (9)
Let $\gamma_k^* = L_k^* p(t_k) + \gamma_k$ and $\lambda_k^* = L_k p(t_k) + \lambda_k$, we obtain

$$p(t) = \int_a^b G_1(t, s)\sigma_1(s)ds + \sum_{a<t_k<b} G_1(t, t_k)\gamma_k^* + \sum_{a<t_k<b} G_2(t, t_k)\lambda_k^*,$$

i.e., $p(t)$ is also the solution of (4).

Conversely, assume that $p(t)$ is a solution of (5), then differentiating on (5) for $t \neq t_k$, we obtain

$$p'(t) = -\left(\int_a^b \sigma_1(s)ds + \sum_{a<t_k<b} \gamma_k^*\right) + \int_a^t \sigma_1(s)ds + \sum_{a<t_k<t} \gamma_k^*.$$ (10)

Again differentiating on (10) for $t \neq t_k$, we have

$$p''(t) = M p(t) + N(Tp)(t) + N_1(Sp)(t) + \sigma(t).$$

By computing directly, we have

$$\Delta p(t_k) = L_k p'(t_k) + \lambda_k \quad \text{and} \quad \Delta p'(t_k) = L_k^* p(t_k) + \gamma_k.$$ (11)

It is easy to see that $\omega p(a) = p(\eta)$ and $p'(b) = 0$, for $\eta \in (a, b)$. This completes the proof. \hfill \square

**Lemma 2.4** Assume that $p \in PC^1(J, R) \cap C^2(J', R)$ satisfies

$$\begin{cases}
p''(t) \geq M p(t) + N(Tp)(t) + N_1(Sp)(t), & t \in J', \\
\Delta p(t_k) = L_k p'(t_k), & k = 1, \ldots, m, \\
\Delta p'(t_k) \geq L_k^* p(t_k), & k = 1, \ldots, m, \\
\omega p(a) \leq p(\eta), & p'(b) \leq 0,
\end{cases}$$

where the constants $M > 0$, $N \geq 0$, $N_1 \geq 0$, $\omega > 1$, $L_k \geq 0$, $L_k^* \geq 0$, $(k = 1, 2, \ldots, m)$ also satisfy the inequality

$$\left(\int_a^b (M + N \int_a^s k(s, r)dr) + N_1 \int_a^b h(s, r)dr ds + \sum_{i=1}^m L_i^* \right) \left(b - a + \sum_{i=r}^v L_i \right) \leq 1.$$ (12)

Then $p(t) \leq 0$ for all $t \in J$.

**Proof.** Firstly, we show that $\inf_{t \in J} \{p(t)\} \leq 0$. Let $\inf_{t \in J} \{p(t)\} = c$, where $c$ is a constant. There exists a point $t_\ast \in J$, $r \in \{0, 1, \ldots, m\}$, such that $p(t_\ast) = c$ or $p(t_\ast^+) = c$. We will only consider $p(t_\ast) = c$. For the case $p(t_\ast^+) = c$ the proof is similar. It is easy to verify that

$$p''(t) \geq M p(t) + N(Tp)(t) + N_1(Sp)(t) \geq c \left( M + N \int_a^t k(t, s)ds + N_1 \int_a^b h(t, s)ds \right).$$ (13)
Integrating the differential inequality (13) from $t \in J_h (h \in \{0, 1, \ldots, m\})$ to $b$, we have

$$p'(b) - p'(t) \geq c \int_t^b \left( M + N \int_a^s k(s, r)dr + N_1 \int_a^b h(s, r)dr \right) ds + c \sum_{i=h+1}^m L_i^*,$$

where $\sum_r^l = 0$, when $r > l$, and then

$$0 \geq p'(b) \geq p'(t) + c \int_t^b \left( M + N \int_a^s k(s, r)dr + N_1 \int_a^b h(s, r)dr \right) ds + c \sum_{i=h+1}^m L_i^*.$$

Hence

$$p'(t) \leq -c \left( \int_t^b \left( M + N \int_a^s k(s, r)dr + N_1 \int_a^b h(s, r)dr \right) ds + \sum_{i=h+1}^m L_i^* \right). \tag{14}$$

If $c > 0$, then $p'(t) < 0$ for all $t \in J$ and $\Delta p(t_k) \leq L_k p'(t_k) \leq 0$, $k = 1, 2, \ldots, m$. Therefore, $p(a) > p(\eta)$, a contradiction. Then $\inf_{t \in J} \{p(t)\} = c \leq 0$.

Next, we will show that $p(t) \leq 0$ for all $t \in J$. Suppose, to the contrary, that $p(t^*) > 0$ for $t^* \in J_v$, $v \in \{0, 1, \ldots, m\}$. Let $d = -c$. Then from (14), we have

$$p'(t) \leq d \left( \int_a^b \left( M + N \int_a^s k(s, r)dr + N_1 \int_a^b h(s, r)dr \right) ds + \sum_{i=1}^m L_i^* \right). \tag{15}$$

Assume that $t^* > t_v$. Then $v > r$. For the case $t^* \leq t_v$, the proof is similar and thus we omit it. By mean value theorem, we have

$$p(t^*) - p(t_{v}^-) \leq d \left( \int_a^b \left( M + N \int_a^s k(s, r)dr + N_1 \int_a^b h(s, r)dr \right) ds + \sum_{i=1}^m L_i^* \right) \times \left( t^* - t_v^+ \right) + L_v,$$

$$p(t_v^-) - p(t_{v-1}^-) \leq d \left( \int_a^b \left( M + N \int_a^s k(s, r)dr + N_1 \int_a^b h(s, r)dr \right) ds + \sum_{i=1}^m L_i^* \right) \times \left( t_v^- - t_{v-1}^+ \right) + L_{v-1},$$

$$\vdots$$

$$p(t_{r+1}^-) - p(t_r) \leq d \left( \int_a^b \left( M + N \int_a^s k(s, r)dr + N_1 \int_a^b h(s, r)dr \right) ds + \sum_{i=1}^m L_i^* \right) \times \left( t_{r+1}^- - t_r \right).$$
Summing, we get

\[ p(t^*) - p(t_*) \leq d \left( \int_a^b \left( M + N \int_a^s k(s, r) dr + N_1 \int_a^b h(s, r) dr \right) ds + \sum_{i=1}^m L_i^* \right) \times \left( (t^* - t_*) + \sum_{i=r+1}^v L_i \right), \]

then

\[ 0 < p(t^*) \leq p(t_*) + d \left( \int_a^b \left( M + N \int_a^s k(s, r) dr + N_1 \int_a^b h(s, r) dr \right) ds + \sum_{i=1}^m L_i^* \right) \times \left( (t^* - t_*) + \sum_{i=r+1}^v L_i \right). \]

Thus

\[ \left( \int_a^b \left( M + N \int_a^s k(s, r) dr + N_1 \int_a^b h(s, r) dr \right) ds + \sum_{i=1}^m L_i^* \right) \left( (b - a) + \sum_{i=1}^m L_i \right) > 1, \]

which contradicts (12). This completes the proof. \( \square \)

**Lemma 2.5** Let \( M > 0, N, N_1 \geq 0, 0 \leq L_k < 1, 0 \leq L_k^* < 1, \omega > 1, \eta \in (a, b), \) and assume that

\[ \psi \equiv \int_a^b \left( M + N \int_a^s k(s, r) dr + N_1 \int_a^b h(s, r) dr \right) ds + \sum_{k=1}^m L_k^* < 1. \quad (16) \]

and

\[ \xi \equiv \psi \left( \frac{(b - a)\omega - b + \eta}{\omega - 1} \right) + \frac{\omega}{\omega - 1} \sum_{k=1}^m L_k < 1. \quad (17) \]

Then (4) has an unique solution.

**Proof.** For any \( p \in PC^1(J, R) \cap C^2(J', R), \) define an operator \( A \) by

\[ (Ap)(t) = \int_a^b G_1(t, s) \left[ M p(s) + N(Tp)(s) + N_1(Sp)(s) + \sigma(s) \right] ds + \sum_{k=1}^m G_1(t, t_k) \times \left( L_k^* p(t_k) + \gamma_k \right) + \sum_{k=1}^m G_2(t, t_k) \left( L_k p'(t_k) + \lambda_k \right). \]
where $G_1, G_2$ are given by Lemma 2.3. Since
\[
\max_{t \in J} \{|G_1(t, s)|\} = \frac{(b-a)\omega - b + \eta}{\omega - 1},
\]
\[
\max_{t \in J} \{|G_2(t, s)|\} = \frac{\omega}{\omega - 1},
\]
then for any $x, y \in PC^1(J, R)$, we have
\[
\|Ax - Ay\|_{PC} = \sup_{t \in J} |Ax - Ay|
\leq \sup_{t \in J} \left\{ \int_a^b |G_1(t, s)| \left( M + N \int_a^s k(s, r)dr 
+ N_1 \int_a^b h(s, r)dr \right) ds \right\} \|x - y\|_{PC}
+ \sup_{t \in J} \left\{ \sum_{k=1}^m |G_1(t, s)| L_k^* + |G_2(t, t_k)| L_k \right\} \|x - y\|_{PC^1}
\leq \xi \|x - y\|_{PC^1}.
\]
Similarly,
\[
\|Ax' - Ay'\|_{PC} = \sup_{t \in J} |Ax' - Ay'|
\leq \sup_{t \in J} \left\{ \int_a^b |G_1(t, s)| \left( M + N \int_a^s k(s, r)dr 
+ N_1 \int_a^b h(s, r)dr \right) ds \right\} \|x - y\|_{PC}
+ \sup_{t \in J} \left\{ \sum_{k=1}^m |G_1(t, s)| L_k^* + |G_2(t, t_k)| L_k \right\} \|x - y\|_{PC^1}
\leq \psi \|x - y\|_{PC^1}.
\]
Hence
\[
\|Ax - Ay\|_{PC^1} \leq \max\{\xi, \psi\} \|x - y\|_{PC^1}.
\]

By (16), (17) and Banach fixed point theorem, $A$ has an unique fixed point $y^* \in PC^1$. By Lemma 2.3, $y^*$ is also the unique solution of (4), which completes the proof. \qed
3 Main Results

We are now in a position to prove that the problem (1) has extremal solutions.

Theorem 3.1 Let the following assumptions hold:

(H1): The function \( y_0, z_0 \in PC^1(J, R) \cap C^2(J', R) \) are lower and upper solutions of problem (1), respectively, with \( y_0(t) \leq z_0(t) \) on \( J \).

(H2): The function \( f \) satisfies

\[
f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w}) \geq M[u - \bar{u}] + N[v - \bar{v}] + N_1[w - \bar{w}]
\]

for \( y_0(t) \leq u \leq \bar{u} \leq z_0(t), (Ty_0)(t) \leq v \leq \bar{v} \leq (Tz_0)(t), (Sy_0)(t) \leq w \leq \bar{w} \leq (Sz_0)(t), t \in J \).

(H3): There exist constants \( L_k, L_k^* \), \( k = 1, 2, \ldots, m \), such that

\[
I_k(r(t_k), r'(t_k)) - I_k(\bar{r}(t_k), \bar{r}'(t_k)) = L_k[r'(t_k) - \bar{r}'(t_k)],
\]

\[
I_k^*(r(t_k)) - I_k^*(\bar{r}(t_k)) \geq L_k^*[r(t_k) - \bar{r}(t_k)],
\]

for \( y_0(t) \leq r(t) \leq \bar{r}(t) \leq z_0(t), k = 1, 2, \ldots, m \).

(H4): Constant \( \eta \in (a, b), \omega > 1, M > 0, N, N_1 \geq 0, L_k \geq 0, L_k^* \geq 0, k = 1, 2, \ldots, m \), and satisfy (12), (16) and (17).

Then problem (1) has extremal solutions \( y_0(t) \leq w(t) \leq z_0(t), t \in J \).

Proof. Consider the following sequence:

\[
\begin{align*}
y_n(t) &= Fy_{n-1}(t) + M(y_n(t) - y_{n-1}(t)) \\
&\quad + N(T(y_n - y_{n-1}))(t) + N_1(S(y_n - y_{n-1}))(t), \quad t \in J', \\
\Delta y_n(t_k) &= I_k(y_{n-1}(t_k), y'_{n-1}(t_k)) + L_k[y_n(t_k) - y_{n-1}(t_k)], \quad k = 1, 2, \ldots, m, \\
\Delta y'_n(t_k) &= I_k^*(y_{n-1}(t_k)) + L_k^*[y_n(t_k) - y_{n-1}(t_k)], \quad k = 1, 2, \ldots, m, \\
\omega y_n(a) &= y_n(\eta), \quad y'_n(b) = 0,
\end{align*}
\]

and

\[
\begin{align*}
z_n(t) &= Fz_{n-1}(t) + M[z_n(t) - z_{n-1}(t)] \\
&\quad + N(T(z_n - z_{n-1}))(t) + N_1(S(z_n - z_{n-1}))(t), \quad t \in J', \\
\Delta z_n(t_k) &= I_k(z_{n-1}(t_k), z'_{n-1}(t_k)) + L_k[z_n(t_k) - z_{n-1}(t_k)], \quad k = 1, 2, \ldots, m, \\
\Delta z'_n(t_k) &= I_k^*(z_{n-1}(t_k)) + L_k^*[z_n(t_k) - z_{n-1}(t_k)], \quad k = 1, 2, \ldots, m, \\
\omega z_n(a) &= z_n(\eta), \quad z'_n(b) = 0,
\end{align*}
\]

for \( n = 1, 2, \ldots \) Moreover, by Lemma 2.5, we have \( y_1 \) and \( z_1 \) are well defined. Firstly, we show that

\[
y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J. \quad (18)
\]
Let \( v(t) = y_0(t) - y_1(t) \). By definition 2.1 of a lower solution of (1), we have
\[
v''(t) = y''_0(t) - y''_1(t) \geq Mv(t) + N(Tv(t)) + N_1(Sv(t)),
\]
and
\[
\begin{align*}
\Delta v(t_k) &= \Delta y_0(t_k) - \Delta y_1(t_k) = L_kv'(t_k), \quad k = 1, 2, \ldots, m, \\
\Delta v'(t_k) &= \Delta y'_0(t_k) - \Delta y'_1(t_k) \geq L_k^* v(t_k), \quad k = 1, 2, \ldots, m, \\
\omega v(a) &= \omega y_0(a) - \omega y_1(a) \leq v(\eta), \\
v'(b) &= y'_0(b) - y'_1(b) \leq 0.
\end{align*}
\]

Then, by Lemma 2.4, \( v(t) \leq 0 \), which implies \( y_0(t) \leq y_1(t) \), \( t \in J \). In similar way, we can show that \( z_0(t) \geq z_1(t) \), \( t \in J \).

Next, we will show that \( y_1(t) \leq z_1(t) \), \( t \in J \). Let \( v(t) = y_1(t) - z_1(t) \) then, we have
\[
v''(t) = y''_1(t) - z''_1(t) \\
= Fy_0(t) + M[y_1(t) - y_0(t) + N(T(y_1 - y_0))(t) + N_1(S(y_1 - y_0))(t)] \\
- Fz_0(t) - M[z_1(t) - z_0(t)] - N(T(z_1 - z_0))(t) - N_1(S(z_1 - z_0))(t) \\
\geq Mv(t) + N(Tv(t)) + N_1(Sv(t)),
\]
and
\[
\begin{align*}
\Delta v(t_k) &= \Delta y_1(t_k) - \Delta z_1(t_k) = L_kv'(t_k), \quad k = 1, 2, \ldots, m, \\
\Delta v'(t_k) &= \Delta y'_1(t_k) - \Delta z'_1(t_k) \geq L_k^* v(t_k), \quad k = 1, 2, \ldots, m, \\
\omega v(a) &= \omega y_1(a) - \omega z_1(a) = v(\eta), \\
v'(b) &= y'_1(b) - z'_1(b) = 0.
\end{align*}
\]

Still by Lemma 2.4, \( v(t) \leq 0 \), which implies \( y_1(t) \leq z_1(t) \), \( t \in J \).

Using mathematical induction, we can show that
\[
y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t) \quad \text{for all} \quad t \in J,
\]
n = 1, 2, \ldots. Employing standard argument, we have
\[
\lim_{n \to \infty} y_n(t) = y(t), \quad \lim_{n \to \infty} z_n(t) = z(t),
\]
uniformly on \( t \in J \) and the limit functions \( y(t) \), \( z(t) \) satisfy problem (1). Moreover \( y(t), z(t) \in [y_0(t), z_0(t)] \).

Finally, we will show that \( y \) is the minimal solution and \( z \) is the maximal solution of (1), respectively. To prove it we assume that \( u \) is any solution of
problem (1) such that \( u \in [y_0, z_0] \). Let \( y_{n-1}(t) \leq u(t) \leq z_{n-1}(t), \ t \in J \), for some positive integer \( n \). Put \( v(t) = y_n(t) - u(t) \). Then

\[
v''(t) = Fy_{n-1}(t) + M[y_n(t) - y_{n-1}(t)] \\
+ N(T(y_n - y_{n-1}))(t) + N_1(S(y_n - y_{n-1}))(t) - Fu(t) \\
\geq Mv(t) + N(Tv(t) + N_1(Sv(t)),
\]

and

\[
\Delta v(t_k) = \Delta y_n(t_k) - \Delta u(t_k) = L_k v'(t_k), \quad k = 1, 2, \ldots, m,
\]

\[
\Delta v'(t_k) = \Delta y_n'(t_k) - \Delta u'(t_k) \geq L_k v(t_k), \quad k = 1, 2, \ldots, m,
\]

\[
\omega v(a) = \omega y_n(a) - \omega u(a) = v(\eta),
\]

\[
v'(b) = y_n'(b) - u'(b) = 0.
\]

Hence, \( y_n \leq u, \ t \in J \), by Lemma 2.4. Similarly, we can show that \( u(t) \leq z_n(t), \ t \in J \). This yields \( y_n(t) \leq u(t) \leq z_n(t), \ t \in J \). If \( n \to \infty \), then \( y_0(t) \leq y(t) \leq u(t) \leq z(t) \leq z_0(t), \ t \in J \). The proof is complete. \( \square \)

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**References**


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