A Novel Computational Technique for Finding Simple Roots of Nonlinear Equations

F. Soleymani\textsuperscript{1} and B. S. Mousavi\textsuperscript{2}

Young Researchers Club, Islamic Azad University
Zahedan Branch, Zahedan, Iran

Abstract

In this paper, a new optimal eighth-order class of methods is developed by using the approach of weight function. The developed class includes three evaluations of the function and one of its first derivative per iteration. The analytical proof of the main theorem reveals that any method from the proposed class reaches the optimal efficiency index $1.682$. Finally, the underlying theory developed in this paper is accompanied by numerical experiments.

Mathematics Subject Classification: 65H05, 65K05

Keywords: Nonlinear equations, multi-point methods, optimal order, efficiency index, simple root

1 Preliminary Notes

A method for finding the (simple) roots of an "arbitrary" (nonlinear) function that uses the derivative was first circulated by Isaac Newton in 1669. John Wallis published Newton’s scheme in 1685, and in 1690 Joseph Raphson (1648-1715) published an improved version, essentially the form in which we use it today. Newton had no great interest in the numerical solution of equations (his only numerical example was a cubic one). And there was a long history of efficient numerical solution of cubics, going back at least to Leonardo of Pisa ("Fibonacci," early thirteenth century). At first sight, the Newton’s method does not look like the Newton’s iteration that we know today. Since, the derivative was not even mentioned.

Historical Note. Newton’s work was done in 1669 but published much later.

\textsuperscript{1}Corresponding author. e-mail: fazlsoleymibsb@yahoo.com
\textsuperscript{2}E-mail: bbmoosavi@gmail.com
Numerical methods related to the Newton’s iteration were used by al-Kashi (Persian Mathematician), Viete, Briggs, and Oughtred, all many years before Newton.

Long time after these first developments, Kung and Traub in [5] provided two classes of $n$-step without memory iterations in which the total number of evaluation per full iteration is $n + 1$ (for each of the techniques). As an instance, for the case of $n = 3$ their contributions reach the eighth-order of convergence with high efficiency index. They moreover conjectured that an iterative without memory scheme including $n + 1$ evaluations per full cycle will reach $2^n$ as the optimal order of convergence. It is worth mentioning that methods which satisfy in this conjecture are called optimal techniques for solving nonlinear equations. Note that for obtaining a solid background on this active topic of study, we refer the readers to [4, 11, 13].

In this paper, we provide a wide class of three-step three-point methods which are consistent with the Kung and Traub conjecture for building optimal methods without memory. Each method from the developed class includes three evaluations of the function and one of its first derivative per full cycle. As a consequence, the efficiency of our class will be $8^{1/4} \approx 1.682$ which is greater than that of the optimal one-point method of Newton, i.e. $2^{1/2} \approx 1.414$. The remaining contents of the paper are organized as comes next. Section 2 includes the main contribution of this paper by applying the approach of weight function on the seventh-order method of Cordero et al. [2]. Section 3 evaluates the performance of the new methods from our class with some of the existing well-cited papers in literature. And ultimately, the conclusion of the paper has been drawn in Section 4.

2 Main Result

Consider the three-step three-point method of Cordero et al. [2] as comes next

$$\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}; \\
z_n &= x_n + \frac{f(x_n) + f(y_n)}{f'(x_n)} - 2 \frac{f(x_n)}{f'(x_n)} \frac{f(x_n)}{f(x_n) - f(y_n)}; \\
x_{n+1} &= z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n] - f(y_n)},
\end{aligned}$$

(1)

where $f[z_n, y_n]$, $f[z_n, x_n, x_n]$ are divided difference of the function $f(x)$ and could be defined as follows $f[z_n, y_n] = \frac{f(z_n) - f(y_n)}{z_n - y_n}$, and $f[z_n, x_n, x_n] = \frac{f(z_n, x_n) - f'(x_n)}{z_n - x_n}$. This technique reaches the seventh-order of convergence using four evaluations per cycle. Therefore, it is not optimal with high efficiency index; it reaches the efficiency index $7^{1/4} \approx 1.626$. Here, we make our class of optimal derivative-involved methods based on (1). To do this, we make use of weight function approach in the last step. For this reason, we suggest the following without
memory iteration

\[
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= x_n + \frac{f(x_n) + f'(y_n) - 2 f(x_n)}{f'(x_n)} - f(x_n) - f'(y_n), \\
x_{n+1} &= z_n - \frac{f(z_n)}{f(z_n) + f(x_n, x_n)(z_n - y_n)}(L(\nu) \times K(\mu) \times H(\lambda) \times Q(\kappa) \times P(\iota)),
\end{aligned}
\]

wherein \(L(\nu), K(\mu), H(\lambda), Q(\kappa)\) and \(P(\iota)\) are five real valued weight functions with \(\nu = \frac{f(z)}{f(y)}, \mu = \frac{f(z)}{f(x)}, \lambda = \frac{f(z)}{f(x)}\) and \(\iota = \frac{f(y)}{f(x)}\). Theorem 2.1. shows that the order of convergence will arrive at the optimal level eight under certain conditions on the weight functions.

**Theorem 2.1.** Assume that function \(f : D \subseteq \mathbb{R} \to \mathbb{R}\) has a simple root \(\alpha \in D\), where \(D\) is an open interval. Assume furthermore that \(f(x)\) is a sufficiently differentiable function in the neighborhood of \(\alpha\), i.e. \(D\). Then, the order of convergence of the iterative class defined by (2) is eight when \(L(0) = 1, L'(0) = 0, |L'(2)(0)| \leq \infty, K(0) = 1, K'(0) = 2\) and \(|K'(2)(0)| \leq \infty, H(0) = 1, H'(0) = H''(0) = 0, H'''(0) = -36, \) and \(|H'(4)(0)| \leq \infty, Q(0) = 1, |Q'(0)| \leq \infty, P(0) = 1, P'(0) = 0,\) and \(|P'(2)(0)| \leq \infty.

**Proof.** We write the Taylor’s series expansion of the function \(f\) and its first derivative around the simple root in the \(n\)th iterate. Note that for simplicity, we assume \(c_k = (\frac{1}{k!})f^{(k)}(\alpha), k \geq 2\). Also consider \(e_n = x_n - \alpha\). Therefore, \(f(x_n) = f(\alpha)|e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9)|\), furthermore, we obtain \(f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_2^2e_n^2 + 4c_3e_n^3 + 5c_4e_n^4 + 6c_5e_n^5 + 7c_7e_n^7 + 8c_8e_n^7 + O(e_n^9)]\). Dividing these two on each other gives us \(f(x_n) = e_n - c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9)\). And also \(f(y_n) = c_2f'(\alpha)e_n^2 + 2(\alpha - c_2) + c_3f'(\alpha)e_n^3 + (5\alpha^2 - 7c_2 + 3c_4)f'(\alpha)e_n^4 + \cdots + O(e_n^9)\). Thus, we have the following error equation at the end of the second step of (2)

\[z_n - \alpha = (3c_2^3 - c_2c_3)e_n^4 + \cdots + O(e_n^9).\] (3)

This shows that the second step of our class (2) arrives at the optimal order four. This moreover reveals that the second step of this cycle (or the Cordero et al. cycle (1)) is in fact an especial case of the King’s fourth-order family with \(\beta = 1\). In addition, the approximation in the denominator of the third step of (1) was previously given by Bi et al. in [1]! Now, by writing the Taylor’s series expanding at the third step for \(f(z_n)\), we attain:

\[f(z_n) = (3c_2^3 - c_2c_3)f'(\alpha)e_n^4 + 2(9c_2^4 - 10c_2^2c_3 + c_3^2 + 2c_2c_4)f'(\alpha)e_n^5 + (70c_2^5 - 130c_2^2c_3 + 42c_2c_3^2 + 30c_2^2c_4 - 7c_2c_3 - 3c_4)f'(\alpha)e_n^6 + \cdots + O(e_n^9).\]

In addition, we find that

\[z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n][z_n - y_n]} = 2c_2c_3(-3c_2 + c_3)e_n^7 + c_2(9c_2^6 + 45c_2^4c_3 - 56c_2^2c_3^2 + 8c_3^3 - 9c_2^3c_4 + 7c_2c_3c_4)e_n^8 + O(e_n^9).\] (4)
And finally, by considering $L(0) = 1$, $L'(0) = 0$, and $|L^{(2)}(0)| \leq \infty$, $K(0) = 1$, $K'(0) = 2$ and $|K^{(2)}(0)| \leq \infty$, $H(0) = 1$, $H'(0) = H''(0) = 0$, $H'''(0) = -36$, and $|H^{(4)}(0)| \leq \infty$, $Q(0) = 1$, and $|Q'(0)| \leq \infty$, $P(0) = 1$, $P'(0) = 0$, and $|P^{(2)}(0)| \leq \infty$, we obtain the following error equation for the contributed class (2)

$$e_{n+1} = -\frac{1}{24}(c_2(3c_2^2 - c_3)(12(2c_2^2Q'(0) - 2c_2(c_4 + c_3Q'(0)) + c_2L''(0)) + 9c_2^2(2 + L''(0)) + c_2^2(c_3(4 - 6L''(0)) + P''(0))) + c_2^2H^{(4)}(0)))e_n^8 + O(e_n^9).$$

(5)

This reveals that our class of derivative-involved methods (2) reaches the convergence order eight by using only three evaluations of the function and one of its first derivative. This completes the proof.

We now give a comparison on the index of efficiency with some of the available methods in literature. Each method from our class (2) (by considering the suitable weight functions as discussed in Theorem 2.1.) achieves the optimal efficiency index $8^{1/4} \approx 1.682$ which is greater than $2^{1/2} \approx 1.414$ of Newton’s method and the method in [3], $8^{1/5} \approx 1.515$ of the method given in [6], $6^{1/4} \approx 1.565$ of the scheme in [10, 12], $7^{1/4} \approx 1.626$ of the method (1), and is equal to 1.682 of the three-step families given by Sharma and Sharma in [9], Neta and Petkovic in [7] and Kung-Traub in [5]. However, this index is lower than 1.695 of the class of methods given by Sargolzaei and Soleymani in [8]. In what follows, we give a simple but efficient case of our class (2) in the form of a three-step three-point without memory iteration

$$\begin{align*}
y_n &= x_n - \frac{f(x_n)}{f(x_n)}, \\
z_n &= x_n + \frac{f(x_n) + f(y_n)}{f(x_n) - f(y_n)} - 2f(x_n) \frac{f(x_n)}{f(x_n) - f(y_n)}, \\
x_{n+1} &= z_n - \frac{f(z_n) \times (1 + (\frac{f(x_n)}{f(z_n)})^2)(1 + 2\frac{f(x_n)}{f(z_n)}(1 - 6\frac{f(x_n)}{f(z_n)})^3 - 9(\frac{f(x_n)}{f(z_n)})^4)(1 + (\frac{f(x_n)}{f(z_n)})^2)(1 + (\frac{f(x_n)}{f(z_n)})^3))}{f(z_n, y_n) + f[z_n, x_n, x_n](z_n - y_n)},
\end{align*}$$

(6)

where it reads the following very simple error equation

$$e_{n+1} = -c_2(3c_2^2 - c_3)(9c_2^4 - 4c_2^2c_3 + c_3^2 - c_2c_4)e_n^8 + O(e_n^9).$$

(7)

This was just one case from our wide class of derivative-involved methods (2). Choosing different weight functions as discussed in Theorem 2.1. will result in different accurate optimal eighth-order methods with 1.682 as the efficiency index. As another example from our class, we can provide the following three-step method

$$\begin{align*}
y_n &= x_n - \frac{f(x_n)}{f(x_n)}, \\
z_n &= x_n + \frac{f(x_n) + f(y_n)}{f(x_n) - f(y_n)} - 2f(x_n) \frac{f(x_n)}{f(x_n) - f(y_n)}, \\
x_{n+1} &= z_n - \frac{f(z_n) \times (1 + (\frac{f(x_n)}{f(z_n)})^2)(1 + 2\frac{f(x_n)}{f(z_n)}(1 - 6\frac{f(x_n)}{f(z_n)})^3 - 9(\frac{f(x_n)}{f(z_n)})^4)(1 + (\frac{f(x_n)}{f(z_n)})^2)(1 + (\frac{f(x_n)}{f(z_n)})^3))}{f(z_n, y_n) + f[z_n, x_n, x_n](z_n - y_n)},
\end{align*}$$

(8)
where its error equation is 
\[ e_{n+1} = -c_2(3c_2^3 - c_3)(18c_2^4 - 4c_2^2c_3 + c_3^2 - c_2c_4)e_n^8 + O(e_n^9). \]

### 3 Numerical Results

In this section, the obtained theoretical results are confirmed by numerical experiments. We present numerical test results for our eighth-order class by comparison with famous iterative methods of different orders. All computations were done with MATLAB 7.6 using 750 digit floating point arithmetic (VPA=750). We use the following stopping criterion in our calculations:

\[ |f(x_n)| < \varepsilon \]

where \( \varepsilon \) is the exact solution of the considered nonlinear equations.

For numerical illustrations in this section, we have used the fixed stopping criterion \( \varepsilon = 10^{-750} \). We present the numerical test results for the iterative methods in Tables 1 and 2. The test functions and their simple roots are as follows:

\[
\begin{align*}
 f_1(x) &= \sqrt{x^4 + 8\sin(\frac{x}{\sqrt{x+3}}) + \frac{x^3}{x+3}} - \sqrt{6 + \frac{1}{x}}, \quad \alpha = -2, \quad x_0 = -1.9, \\
 f_2(x) &= x^8 - 20x + \sin x - 10, \quad \alpha \approx -0.524762865170188, \quad x_0 = 0.9, \\
 f_3(x) &= \sin^{-1}(x^2 - 1) - \frac{x}{2} + 1, \quad \alpha \approx 0.594810968398369, \quad x_0 = 0.3, \\
 f_4(x) &= x^2 - \sin x - 20, \quad \alpha \approx 4.365717705159766, \quad x_0 = 4, \\
 f_5(x) &= (x - 2)(x^{10} + x + 1)e^{-x-1}, \quad \alpha = 2, \quad x_0 = 2.1.
\end{align*}
\]

**Table 1.** Comparison of different methods after two full iterations.

<table>
<thead>
<tr>
<th>Test Functions</th>
<th>(1)</th>
<th>(KT8)</th>
<th>(NP8)</th>
<th>(6)</th>
<th>(8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 )</td>
<td>0.5e-51</td>
<td>0.4e-64</td>
<td>0.3e-65</td>
<td>0.6e-62</td>
<td>0.3e-60</td>
</tr>
<tr>
<td>( f_2 )</td>
<td>0.5e-25</td>
<td>0.9e-24</td>
<td>0.1e-24</td>
<td>0.1e-22</td>
<td>0.9e-24</td>
</tr>
<tr>
<td>( f_3 )</td>
<td>0.1e-49</td>
<td>0.1e-66</td>
<td>0.1e-65</td>
<td>0.1e-70</td>
<td>0.3e-66</td>
</tr>
<tr>
<td>( f_4 )</td>
<td>0.1e-66</td>
<td>0.1e-82</td>
<td>0.1e-83</td>
<td>0.5e-86</td>
<td>0.3e-83</td>
</tr>
<tr>
<td>( f_5 )</td>
<td>0.6e-24</td>
<td>0.3e-26</td>
<td>0.5e-29</td>
<td>0.1e-24</td>
<td>0.4e-22</td>
</tr>
</tbody>
</table>

**Table 2.** Comparison of different methods after three full iterations.

<table>
<thead>
<tr>
<th>Test Functions</th>
<th>(1)</th>
<th>(KT8)</th>
<th>(NP8)</th>
<th>(6)</th>
<th>(8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 )</td>
<td>0.3e-356</td>
<td>0.1e-511</td>
<td>0.8e-521</td>
<td>0.3e-494</td>
<td>0.2e-480</td>
</tr>
<tr>
<td>( f_2 )</td>
<td>0.1e-188</td>
<td>0.3e-205</td>
<td>0.1e-211</td>
<td>0.4e-197</td>
<td>0.5e-206</td>
</tr>
<tr>
<td>( f_3 )</td>
<td>0.1e-351</td>
<td>0.4e-537</td>
<td>0.6e-529</td>
<td>0.1e-570</td>
<td>0.4e-535</td>
</tr>
<tr>
<td>( f_4 )</td>
<td>0.6e-480</td>
<td>0.5e-675</td>
<td>0.2e-684</td>
<td>0.6e-704</td>
<td>0.3e-681</td>
</tr>
<tr>
<td>( f_5 )</td>
<td>0.1e-175</td>
<td>0.3e-218</td>
<td>0.2e-241</td>
<td>0.2e-204</td>
<td>0.5e-185</td>
</tr>
</tbody>
</table>

In the numerical examples, the seventh-order method by Cordero et al. (1), derivative-involved eighth-order scheme of Kung and Traub [5] (KT8), the optimal eighth-order scheme of Neta and Petkovic (NP8), and our optimal eighth-order methods (6) and (8) from the class (2) are used for comparison. It can be observed from Tables 1 and 2, that there is no clear winner among
the methods of eighth-order in the sense that in different situations different methods may be the winner. In case of same-order methods the convergence behavior is almost similar because of similar character. Numerical experiments have been performed with the minimum number of precision digits chosen as 750, being large enough to minimize round-off errors as well as to clearly observe the computed asymptotic error constants. Note that we choose "free parameter=0" in (NP8). The formulas have been employed on several other nonlinear equations and results were found at par with those presented here. A reasonably close starting value is necessary for the method to converge. This condition, however, practically applies to all iterative methods for solving equations.

Nonlinear equations solving by numerical optimal methods have a lot of applications in science and engineering. For instance, we can apply (6) or (8) to solve the following application-oriented problem; a loan of $A$ dollars is repaid by making $n$ equal monthly payments of $M$ dollars, starting a month after the loan is made. It can be shown that if the monthly interest rate is $r$, then

\[ Ar = M \left(1 - \frac{1}{(1+r)^n}\right). \]  

(9)

For example, consider a car loan of 10000 was repaid in 60 monthly payments of 250. Now we can use (6) or (8) to find the monthly interest rate correct to hundreds of significant figures.

4 Concluding Remarks

We have demonstrated the performance of a new class of eighth-order derivative-involved methods. Convergence analysis proved that the new methods from the proposed class preserve their order of convergence. There are two major merits of the eighth-order without memory methods in this paper. Firstly, we have the highest efficiency index by using the smallest amount of number of evaluations; and secondly, we have established a higher order of convergence method than the existing method (1). We have examined the effectiveness of the new derivative-involved methods by showing the accuracy of the simple root of a nonlinear equation. Hence, the contribution given in this research is reliable to solve nonlinear equations.

References


Received: October, 2010