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Abstract

In this paper, we introduce an iterative algorithm for finding a common element of the set of solutions of generalized mixed equilibrium problems and the set of common fixed points of a countable family of nonexpansive mappings in real Hilbert space. We prove that the proposed algorithm strongly converges to a common element of the above two sets. Furthermore, we apply our result to prove three new strong convergence theorems for fixed point problems, mixed equilibrium problems, generalized equilibrium problems and equilibrium problems.

Keywords: Generalized mixed equilibrium problem, Mixed equilibrium problem, Generalized equilibrium problems, Equilibrium problem, Countable family nonexpansive mappings, Fixed point, Inverse-strongly monotone mapping

1 Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let symbols $\rightharpoonup$ and $\rightarrow$ denote weak and strong convergence, respectively. Let $C$ be a nonempty closed convex subset of $H$, and $A : C \to H$ a nonlinear mapping. Recall that $A$ is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$
A is said to be $\alpha-$inverse strongly monotone if there exists a positive real number $\alpha$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$  

We can see that if $A$ is $\alpha-$inverse strongly monotone, then $A$ is a monotone mapping. Let $T : C \rightarrow C$ be a nonlinear mapping. In this paper, we use $F(T)$ to denote the fixed point set of $T$, i.e., $F(T) = \{x \in C : Tx = x\}$. Recall that the mapping $T$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C.$$  

A mapping $f : H \rightarrow H$ is said to be contraction if there exists a constant $\rho \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\| \quad \forall x, y \in H.$$  

Let $\varphi : C \rightarrow \mathbb{R}$ be a real value function, $A : C \rightarrow H$ an $\alpha-$inverse strongly monotone mapping and let $\Phi : C \times C \rightarrow \mathbb{R}$ be a bifunction. In this paper, we consider the following generalized mixed equilibrium problem:

Find $x^* \in C$ such that

$$(GMEP) : \quad \Phi(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$  

The set of solutions for problem (1.1) is denoted by $\Omega$, i.e.,

$$\Omega = \{x^* \in C : \Phi(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C\}. \tag{1.2}$$  

To study the generalized mixed equilibrium problems (1.1), we may assume that the bifunction $\Phi : C \times C \rightarrow \mathbb{R}$ satisfies the following conditions:

(A1) $\Phi(x, x) = 0$ for all $x \in C$;

(A2) $\Phi$ is monotone, i.e., $\Phi(x, y) + \Phi(y, x) \leq 0, \quad \forall x, y \in C$;

(A3) for all $x, y, z \in C$, $\lim_{t \to 0} \Phi(tz + (1 - t)x, y) \leq \Phi(x, y)$;

(A4) for all $x \in C$, $y \mapsto \Phi(x, y)$ is convex and lower semicontinuous;

Next, we give some special cases of problem (1.1).

(i) If $A \equiv 0$ in (1.1), then $(GMEP)$ (1.1) reduces to the following mixed equilibrium problem:
Find $x^* \in C$ such that

$$\Phi(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C. \quad (1.3)$$

In this paper, the set of such an $x^* \in C$ is denoted by $MEP(\Phi, \varphi)$, that is,

$$MEP(\Phi, \varphi) = \{x^* \in C : \Phi(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C\}. \quad (1.4)$$

(ii) If $\varphi \equiv 0$, then problem (1.1) is reduced to the generalized equilibrium problem:

Find $x^* \in C$ such that

$$\Phi(x^*, y) + \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \quad (1.5)$$

In this paper, the set of such an $x^* \in C$ is denoted by $EP$, that is,

$$EP = \{x^* \in C : \Phi(x^*, y) + \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C\}. \quad (1.6)$$

(iii) If $\varphi \equiv 0$ and $A \equiv 0$, then problem (1.1) is reduced to the classical equilibrium problem:

Find $x^* \in C$ such that

$$\Phi(x^*, y) \geq 0, \quad \forall y \in C. \quad (1.7)$$

In this paper, the set of such an $x^* \in C$ is denoted by $EP(\Phi)$, that is,

$$EP(\Phi) = \{x^* \in C : \Phi(x^*, y) \geq 0, \quad \forall y \in C\}. \quad (1.8)$$

(iv) If $\Phi \equiv 0$ and $\varphi \equiv 0$, then problem (1.1) is reduced to the classical variational inequality:

Find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \quad (1.9)$$

In this paper, the set of such an $x^* \in C$ is denoted by $VI(A, C)$, that is,

$$VI(A, C) = \{x^* \in C : \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C\}. \quad (1.10)$$

In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme for finding the best approximation to the initial data when $EP(\Phi) \neq \emptyset$ and proved a strong convergence theorem.
In 2006, Takahashi and Takahashi [12] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a strong convergence theorem.

In 2007, Tada and Takahashi [10] introduced two iterative schemes for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem. In 2008, Takahashi and Takahashi [11] introduced an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and then obtained that the iterative sequence converges strongly to a common element of the above two sets. Moreover, they proved three new strong convergence theorems for fixed point problems, variational inequalities and equilibrium problems.

Recently, Ceng and Yao [2] introduced a hybrid iterative scheme for finding a common element of the set of solutions of mixed equilibrium problem (1.3) and the set of common fixed points of finitely many nonexpansive mappings and they proved that the sequences generated by the hybrid iterative scheme converges strongly to a common element of the set of solutions of mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings.

In 2008, Peng and Yao [7] obtained some strong convergence theorems for iterative schemes based on the hybrid method and the extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality.

Very recently, Klin-eam and Suantai [5] introduced an iterative scheme for a countable families of nonexpansive mappings \( \{T_n\} \) as follows:

\[
x_0 = x \in C, \quad \text{arbitrarily}; \nonumber \\
y_n = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n, \quad n \geq 0 \nonumber \\
x_{n+1} = (1 - \beta_n)y_n + \beta_n T_n y_n, \quad n \geq 0 \tag{1.11}
\]

where \( C \) is a nonempty closed subset of a Banach space, \( f \) is a contraction on \( C \), \( \{\alpha_n\}, \{\beta_n\} \) are sequences in \([0,1]\) and \( \{T_n\} \) is a sequence of nonexpansive mappings with some conditions. They, proved that the sequence \( \{x_n\} \) defined by (1.11) converges strongly to a common fixed point of \( \{T_n\} \).
In this paper, we introduce another iterative method, modified from (1.11), for finding an element of the set of solutions of generalized mixed equilibrium problems and the set of common fixed points of countable family of nonexpansive mappings in real Hilbert spaces, where $A : C \to H$ is also an $\alpha-$inverse strongly monotone mapping and then obtain a strong convergence theorem. Moreover, we use our main result to the problem for finding an element of $\cap_{i=1}^{\infty} F(T_i) \cap MEP(\Phi, \varphi) \cap \cap_{i=1}^{\infty} F(T_i) \cap EP$ and $\cap_{i=1}^{\infty} F(T_i) \cap EP(\Phi)$.

2 Preliminary Notes

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Then, for any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\|$, $\forall y \in C$. The mapping $P_C : x \to P_C(x)$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is nonexpansive.

The following characterizes the projection $P_C$.

**Lemma 2.1.** (See [9]) Given $x \in H$ and $y \in C$. Then $P_C(x) = y$ if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0, \ \forall z \in C.$$

The following lemmas will be useful for proving our main results.

**Lemma 2.2.** (See [9]) For all $x, y \in H$, there holds the inequality

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

**Lemma 2.3.** (See [14]) Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$, $n \geq 0$, where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$, (ii) $\limsup_{n \to \infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then $\lim_{n \to \infty} a_n = 0$.

**Lemma 2.4.** (See [1]) Let $C$ be a nonempty closed convex subset of $H$, and let $\{T_n\}$ be a family of mappings of $C$ into itself. Suppose that

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in C\} < \infty.$$  

Then, for each $y \in C$, $\{T_ny\}$ converges strongly to some point of $C$. Moreover, let $T$ be a mapping of $C$ into itself defined by

$$Ty = \lim_{n \to \infty} T_ny, \ \forall y \in C.$$  

Then

$$\limsup_{n \to \infty} \{\|Tz - T_nz\| : z \in C\} = 0.$$
Lemma 2.5. (see [6]) Let $H$ be a real Hilbert space, $C$ a nonempty closed convex subset of $H$, and $T : C \to C$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed at zero, that is, if ${x_n}$ is a sequence in $C$ such that $x_n \to \overline{x}$ and $x_n - Tx_n \to 0$, then $\overline{x} \in F(T)$.

Lemma 2.6. (See [13]) Let $C$ be a nonempty closed convex subset of $H$, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function and let $\Phi$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfy (A1)-(A4). Assume that either

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Phi(z, y_x) + \varphi(y_x) + \frac{1}{r} \langle y_x - z, z - x \rangle < \varphi(z).$$

or

(B2) $C$ is bounded set

holds. For $x \in C$ and $r > 0$, define a mapping $T_r : H \to C$ as follows.

$$T_r(x) := \{z \in C : \Phi(z, y) + \varphi(y) + \frac{1}{r} \langle y - z, z - x \rangle \geq \varphi(z), \, \forall y \in C\}$$

for all $x \in H$. Then, the following conditions hold:

(i) For each $x \in H$, $T_r(x) \neq \emptyset$;

(ii) $T_r$ is single-valued;

(iii) $T_r$ is firmly nonexpansive, i.e.,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \, \forall x, y \in H;$$

(iv) $F(T_r) = \text{MEP}(\Phi, \varphi)$;

(v) $\text{MEP}(\Phi, \varphi)$ is closed and convex.

Finally, a sequence $\{x_n\}$ in $H$ is said to be asymptotically regular if

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$

3 Main Results

Theorem 3.1. Let $H$ be a real Hilbert space, $C$ a closed convex nonempty subset of $H$, $\varphi : C \to \mathbb{R}$ a proper lower semicontinuous and convex functional, $A$ an $\alpha$-inverse strongly monotone mapping of $C$ into $H$, $\Phi : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)-(A4), $\{T_n\}$ a sequence of nonexpansive mappings of $C$ into itself such that $\cap_{n=1}^{\infty} F(T_n) \cap \Omega \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ and $\{r_n\}$ a sequence in $[0, 2\alpha]$. Assume that:
(i) either (B1) or (B2) holds;

(ii) the sequence \( \{r_n\} \) satisfies

\[
(C1) \quad 0 < c \leq r_n \leq d < 2\alpha;
\]

(iii) \( \lim_{n \to \infty} \alpha_n = 0; \)

(iv) the sequence \( \{\beta_n\} \) satisfies

\[
(E1) \quad \beta_n \in [0, b) \text{ for some } b \in (0, 1).
\]

Let \( f \) be a \( \rho \)-contraction of \( C \) into itself and let \( \{x_n\}, \{y_n\} \) and \( \{u_n\} \) be sequences generated by \( x_0 \in C \) and

\[
\begin{align*}
\Phi(u_n, x) + \varphi(x) - \varphi(u_n) + \langle Ax_n, x - u_n \rangle + \frac{1}{r_n} \langle x - u_n, u_n - x_n \rangle & \geq 0, \quad \forall x \in C \\
y_n &= \alpha_n f(u_n) + (1 - \alpha_n) T_n u_n, \\
x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0.
\end{align*}
\]

(3.1)

Suppose that \( \sum_{n=1}^{\infty} \sup \{\|T_{n+1} z - T_n z\|; z \in B\} < \infty \) for any bounded subset \( B \) of \( C \). Let \( T \) be a mapping of \( C \) into itself defined by \( Tz = \lim_{n \to \infty} T_n z \) for all \( z \in C \) and suppose that \( F(T) = \bigcap_{n=1}^{\infty} F(T_n) \). If \( \{x_n\} \) is asymptotic regular, then \( \{x_n\} \) converges strongly to \( x^* = P_{F(T) \cap \Omega}(x^*) \).

Proof. Let \( x, y \in C \). Since \( A \) is \( \alpha \)-inverse strongly monotone and \( r_n \in (0, 2\alpha) \) \( \forall n \in \mathbb{N} \), we have

\[
\|(I - r_n A)x - (I - r_n A)y\|^2 = \|x - y - r_n (Ax - Ay)\|^2
\]

\[
= \|x - y\|^2 - 2r_n \langle x - y, Ax - Ay \rangle + r_n^2 \|Ax - Ay\|^2
\]

\[
\leq \|x - y\|^2 - 2r_n\alpha\|Ax - Ay\|^2 + r_n^2 \|Ax - Ay\|^2
\]

\[
= \|x - y\|^2 + r_n (r_n - 2\alpha) \|Ax - Ay\|^2
\]

\[
\leq \|x - y\|^2,
\]

which implies that \( I - r_n A \) is nonexpansive.

Next, we prove that the sequences \( \{x_n\}, \{y_n\}, \{u_n\}, \{Ax_n\}, \{f(u_n)\} \) and \( \{T_n u_n\} \) are bounded. Since

\[
\Phi(u_n, x) + \varphi(x) - \varphi(u_n) + \langle Ax_n, x - u_n \rangle + \frac{1}{r_n} \langle x - u_n, u_n - x_n \rangle \geq 0, \quad \forall x \in C,
\]

we have

\[
\Phi(u_n, x) + \varphi(x) - \varphi(u_n) + \frac{1}{r_n} \langle x - u_n, u_n - (x_n - r_n Ax_n) \rangle \geq 0, \quad \forall x \in C.
\]
It follows from Lemma (2.6) that \( u_n = T_{r_n}(x_n - r_n Ax_n), \quad \forall n \in \mathbb{N}. \)
Let \( p \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega. \) Then we have
\[
\Phi(p, y) + \varphi(y) - \varphi(p) + \langle Ap, y - p \rangle \geq 0, \quad \forall y \in C;
\]
so
\[
\Phi(p, y) + \varphi(y) - \varphi(p) + \frac{1}{r_n} \langle y - p, p - (p - r_n Ap) \rangle \geq 0, \quad \forall y \in C.
\]
By Lemma (2.6), we have \( p = T_{r_n}(p - r_n Ap). \)
Since \( T_{r_n} \) and \( (I - r_n A) \) are nonexpansive, we have
\[
\|u_n - p\| = \|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(p - r_n Ap)\|
\leq \|(x_n - r_n Ax_n) - (p - r_n Ap)\|
\leq \|x_n - p\|. \tag{3.2}
\]
Notice that
\[
\|y_n - p\| = \|\alpha_n f(u_n) + (1 - \alpha_n) T_n u_n - p\|
= \|\alpha_n (f(u_n) - f(p)) + \alpha_n (f(p) - p) + (1 - \alpha_n) (T_n u_n - p)\|
\leq \rho \alpha_n \|u_n - p\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|u_n - p\|
= (1 - (1 - \rho) \alpha_n) \|u_n - p\| + \alpha_n \|f(p) - p\|
\leq (1 - (1 - \rho) \alpha_n) \|u_n - p\| + \alpha_n (1 - \rho) \cdot \frac{1}{(1 - \rho)} \|f(p) - p\|
\leq \max\{\|u_n - p\|, \frac{1}{1 - \rho} \|f(p) - p\|\}. \tag{3.3}
\]
From (3.1), (3.2) and (3.3), we deduce that
\[
\|x_{n+1} - p\| = \|(1 - \beta_n) y_n + \beta_n T_n y_n - p\|
= \|(1 - \beta_n) y_n + \beta_n p - \beta_n p + \beta_n T_n y_n - p\|
= \|(1 - \beta_n) y_n - (1 - \beta_n) p + \beta_n T_n y_n - \beta_n p\|
\leq (1 - \beta_n) \|y_n - p\| + \beta_n \|T_n y_n - p\|
\leq (1 - \beta_n) \|y_n - p\| + \beta_n \|y_n - p\|
= \|y_n - p\|
\leq \max\{\|u_n - p\|, \frac{1}{1 - \rho} \|f(p) - p\|\}
\leq \max\{\|x_n - p\|, \frac{1}{1 - \rho} \|f(p) - p\|\}. \tag{3.4}
\]
It follows from (3.4) and induction that
\[
\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1 - \rho} \|f(p) - p\|\} \quad \text{for all} \quad n \geq 0
\]
and all \( p \in \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega \). Hence \( \{x_n\} \) is bounded and so \( \{y_n\}, \{u_n\}, \{Ax_n\}, \{f(u_n)\} \) and \( \{T_nu_n\} \) are bounded. Moreover, it follows that

\[
\|y_n - T_nu_n\| = \|\alpha_n f(u_n) + (1 - \alpha_n)T_nu_n - T_nu_n\|
= \alpha_n\|f(u_n) - T_nu_n\| \to 0 \text{ as } n \to \infty. \tag{3.5}
\]

On the other hand, we have

\[
\|u_n - p\|^2 = \|T_n(x_n - r_nAx_n) - T_n(p - r_nAp)\|^2
\leq \|(x_n - r_nAx_n) - (p - r_nAp)\|^2
= \|(x_n - p) - r_n(Ax_n - Ap)\|^2
= \|x_n - p\|^2 + r_n^2\|Ax_n - Ap\|^2 - 2r_n\langle x_n - p, Ax_n - Ap \rangle
\leq \|x_n - p\|^2 + r_n^2\|Ax_n - Ap\|^2 - 2r_n\alpha\|Ax_n - Ap\|^2
= \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2. \tag{3.6}
\]

Note that

\[
\|x_{n+1} - p\|^2 \leq (1 - \beta_n}\|y_n - p\|^2 + \beta_n\|T_ny_n - p\|^2
\leq (1 - \beta_n}\|y_n - p\|^2 + \beta_n\|y_n - p\|^2
= \|y_n - p\|^2 \tag{3.7}
\]

and

\[
\|y_n - p\|^2 \leq \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)\|T_nu_n - p\|^2
\leq \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2. \tag{3.8}
\]

From (3.6), (3.7) and (3.8) we have

\[
\|x_{n+1} - p\|^2 \leq \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2
\leq \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2
= \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - r_n(2\alpha - r_n)(1 - \alpha_n)\|Ax_n - Ap\|^2.
\]

This implies that

\[
r_n(1 - \alpha_n)(2\alpha - r_n)\|Ax_n - Ap\|^2 \leq \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{3.9}
\]

Since \( 0 < c \leq r_n \leq d < 2\alpha \), we obtain that

\[
c(1 - \alpha_n)(2\alpha - d)\|Ax_n - Ap\|^2 \leq \alpha_n\|f(u_n) - p\|^2 - \alpha_n\|x_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2
\leq \alpha_n\|f(u_n) - p\|^2 - \alpha_n\|x_n - p\|^2 + (\|x_n - p\| - \|x_{n+1} - p\|)
\times \|x_n - x_{n+1}\|.
\]
From $\alpha_n \to 0$ and $\{x_n\}$ is bounded and asymptotic regular, we obtain
\[
\lim_{n \to \infty} \|Ax_n - Ap\| = 0. \tag{3.10}
\]
Since $u_n = T_{r_n}(x_n - r_nAx_n)$, $p = T_{r_n}(p - r_nAp)$ and $T_{r_n}$ is firmly nonexpansive, we have
\[
\|u_n - p\|^2 = \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2
\leq \langle T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp), (x_n - r_nAx_n) - (p - r_nAp) \rangle
= \langle u_n - p, (x_n - r_nAx_n) - (p - r_nAp) \rangle
= \frac{1}{2}(\|u_n - p\|^2 + \|(x_n - r_nAx_n) - (p - r_nAp)\|^2 - \|(u_n - p) - ((x_n - r_nAx_n) - (p - r_nAp))\|^2)
\leq \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|(x_n - u_n) - r_n(Ax_n - Ap)\|^2)
= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2 + r_n^2\|Ax_n - Ap\|^2 - 2r_n\langle x_n - u_n, Ax_n - Ap \rangle)
= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ap \rangle - r_n^2\|Ax_n - Ap\|^2).
\]
This implies that
\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\||Ax_n - Ap||. \tag{3.11}
\]
It follows from (3.7), (3.8) and (3.11), we have
\[
\|x_{n+1} - p\|^2 \leq \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)(\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\||Ax_n - Ap||)
= \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 + 2(1 - \alpha_n)r_n\|x_n - u_n\||Ax_n - Ap||.
\]
This implies that
\[
(1 - \alpha_n)\|x_n - u_n\|^2 \leq \alpha_n\|f(u_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2(1 - \alpha_n)r_n\|x_n - u_n\||Ax_n - Ap||
\leq \alpha_n\|f(u_n) - p\|^2 - \|x_n - p\|^2 + \|x_n - p\| - \|x_{n+1} - p\|)
\times \|x_n - x_{n+1}\| + 2(1 - \alpha_n)r_n\|x_n - u_n\||Ax_n - Ap||. \tag{3.12}
\]
Since $\lim_{n \to \infty} \alpha_n = 0$, $\lim_{n \to \infty} \|Ax_n - Ap\| = 0$ and the sequence $\{x_n\}$ is asymptotic regular, it follows from (3.12) that
\[
\lim_{n \to \infty} \|x_n - u_n\| = 0. \tag{3.13}
\]
Observing that

\[\|x_{n+1} - y_n\| = \|(1 - \beta_n)y_n + \beta_n T_n y_n - y_n\|\]

\[= \beta_n \|T_n y_n - y_n\|\]

\[\leq \beta_n (\|T_n y_n - T_n u_n\| + \|T_n u_n - y_n\|)\]

\[\leq b (\|y_n - u_n\| + \|T_n u_n - y_n\|)\]

\[\leq b (\|y_n - x_{n+1}\| + \|x_{n+1} - u_n\| + \|T_n u_n - y_n\|).\]

It follows that

\[\|x_{n+1} - y_n\| \leq \frac{b}{1 - b} (\|x_{n+1} - u_n\| + \|T_n u_n - y_n\|)\]

\[\leq \frac{b}{1 - b} (\|x_{n+1} - x_n\| + \|x_n - u_n\| + \|T_n u_n - y_n\|). \tag{3.14}\]

So, by (3.5), (3.13) and the asymptotic regularity of \(\{x_n\}\), we have

\[\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0. \tag{3.15}\]

This implies that

\[\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.16}\]

From (3.13) and (3.16), we have

\[\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.17}\]

Next, we will show that \(\lim_{n \to \infty} \|T x_n - x_n\| = 0\). By the asymptotic regularity of \(\{x_n\}\) and Lemma (2.4), we have

\[\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 = \lim_{n \to \infty} \sup \{\|T z - T_n z\| : z \in \{x_k\}\}. \tag{3.18}\]

By (3.14), we obtain that

\[\|T_n x_n - x_n\| \leq \|x_{n+1} - x_n\| + \|y_n - x_{n+1}\| + \|T_n x_n - y_n\|\]

\[\leq \|x_{n+1} - x_n\| + \frac{b}{1 - b} (\|x_{n+1} - x_n\| + \|x_n - u_n\| + \|T_n u_n - y_n\|)\]

\[+ \|T_n x_n - T_n u_n\| + \|T_n u_n - y_n\|\]

\[\leq \|x_{n+1} - x_n\| + \frac{b}{1 - b} (\|x_{n+1} - x_n\| + \|x_n - u_n\| + \|T_n u_n - y_n\|)\]

\[+ \|x_n - u_n\| + \|T_n u_n - y_n\|\]

\[\leq \frac{1}{1 - b} (\|x_{n+1} - x_n\| + \|x_n - u_n\| + \|T_n u_n - y_n\|). \tag{3.19}\]
It follows by (3.5), (3.13) and the asymptotic regularity of \( \{ x_n \} \) that

\[ \lim_{n \to \infty} \| T_n x_n - x_n \| = 0. \tag{3.20} \]

Hence, by (3.18) and (3.20), we obtain

\[ \| T x_n - x_n \| \leq \| T x_n - T_n x_n \| + \| T_n x_n - x_n \| \]
\[ \leq \sup \{ \| T z - T_n z \| : z \in \{ x_n \} \} + \| T_n x_n - x_n \| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.21} \]

Now, observe that

\[ \| y_n - T_n y_n \| \leq \| y_n - x_n \| + \| x_n - T_n x_n \| + \| T_n x_n - T_n y_n \| \]
\[ \leq \| y_n - x_n \| + \| x_n - T_n x_n \| + \| x_n - y_n \| \]
\[ = 2 \| x_n - y_n \| + \| x_n - T_n x_n \|. \tag{3.22} \]

It follows by (3.16) and (3.20) that

\[ \lim_{n \to \infty} \| y_n - T_n y_n \| = 0. \tag{3.23} \]

Therefore

\[ \| T y_n - y_n \| \leq \| T y_n - T_n y_n \| + \| T_n y_n - y_n \| \]
\[ \leq \sup \{ \| T z - T_n z \| : z \in \{ y_n \} \} + \| T_n y_n - y_n \| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.24} \]

Next, we show that

\[ \limsup_{n \to \infty} \langle f(x^*) - x^*, y_n - x^* \rangle \leq 0, \tag{3.25} \]

where \( x^* = P_{F(T) \cap \Omega} f(x^*) \). To show this inequality, we can choose a subsequence \( \{ y_{n_i} \} \) of \( \{ y_n \} \) such that

\[ \lim_{i \to \infty} \langle f(x^*) - x^*, y_{n_i} - x^* \rangle = \limsup_{n \to \infty} \langle f(x^*) - x^*, y_n - x^* \rangle. \tag{3.26} \]

Since \( \{ y_{n_i} \} \) is bounded, there exists a subsequence \( \{ y_{n_{i_j}} \} \) of \( \{ y_{n_i} \} \) which converges weakly to \( \omega \). Without loss of generality, we can assume that \( y_{n_{i_j}} \rightharpoonup \omega \). From \( \| y_n - u_n \| \to 0 \) as \( n \to \infty \), so we have \( u_{n_i} \rightharpoonup \omega \). Let us show \( \omega \in F(T) \cap \Omega \). First, we show that \( \omega \in \Omega \). Since \( u_n = T_{r_n}(x_n - r_n Ax_n) \), for any \( z \in C \) we have

\[ \Phi(u_n, z) + \varphi(z) - \varphi(u_n) + \langle Ax_n, z - u_n \rangle + \frac{1}{r_n} \langle z - u_n, u_n - x_n \rangle \geq 0. \]

From (A2) we have

\[ \varphi(z) - \varphi(u_n) + \langle Ax_n, z - u_n \rangle + \frac{1}{r_n} \langle z - u_n, u_n - x_n \rangle \geq -\Phi(u_n, z) \geq \Phi(z, u_n), \]
and hence
\[ \varphi(z) - \varphi(u_n) + \langle Ax_n, z - u_n \rangle + \langle z - u_n, \frac{u_n - x_n}{r_n} \rangle \geq \Phi(z, u_n). \tag{3.27} \]

Put \( u_t = tz + (1 - t)\omega \) for all \( t \in (0, 1] \) and \( z \in C \). Then we have \( u_t \in C \). From (3.27) we have
\[
\varphi(u_t) - \varphi(u_n) + \langle u_t - u_n, Au_t \rangle \\
\geq \langle u_t - u_n, Au_t \rangle - \langle u_t - u_n, Ax_n \rangle - \langle u_t - u_n, \frac{u_n - x_n}{r_n} \rangle + \Phi(u_t, u_n) \\
= \langle u_t - u_n, Au_t - Au_n \rangle + \langle u_t - u_n, Au_n - Ax_n \rangle - \langle u_t - u_n, \frac{u_n - x_n}{r_n} \rangle \\
+ \Phi(u_t, u_n). 
\]

Since \( \|u_n - x_n\| \to 0 \) as \( n \to \infty \), we have \( \|Au_n - Ax_n\| \to 0 \) as \( n \to \infty \).
Further, from monotonicity of \( A \), we have \( \langle u_t - u_n, Au_t - Au_n \rangle \geq 0 \).
Thus from the weakly semicontinuity of \( \varphi \) and (A4), we have
\[ \varphi(u_t) - \varphi(\omega) + \langle u_t - \omega, Au_t \rangle \geq \Phi(u_t, \omega) \text{ as } i \to \infty. \tag{3.28} \]
From (A1), (A4), (3.28) and the convexity of \( \varphi \), we also have
\[
0 = \Phi(u_t, u_t) + \varphi(u_t) - \varphi(u_t) \\
= \Phi(u_t, (tz + (1 - t)\omega)) + \varphi(tz + (1 - t)\omega) - \varphi(u_t) \\
\leq t\Phi(u_t, z) + (1 - t)\Phi(u_t, \omega) + t\varphi(z) + (1 - t)\varphi(\omega) - \varphi(u_t) \\
\leq t\Phi(u_t, z) + (1 - t)(\varphi(u_t) - \varphi(\omega) + \langle u_t - \omega, Au_t \rangle) + t\varphi(z) + (1 - t)\varphi(\omega) - \varphi(u_t) \\
= t\Phi(u_t, z) - t\varphi(u_t) + (1 - t)(u_t - \omega, Au_t) + t\varphi(z) \\
= t[\Phi(u_t, z) - \varphi(u_t) + \varphi(z)] + (1 - t)t(z - \omega, Au_t). \tag{3.29} 
\]

Dividing by \( t \), we have
\[ \Phi(u_t, z) - \varphi(u_t) + \varphi(z) + (1 - t)\langle z - \omega, Au_t \rangle \geq 0, \quad \forall z \in C. \]
Letting \( t \to 0 \), it follows from (A3) and the weakly semicontinuity of \( \varphi \) that
\[ \Phi(\omega, z) - \varphi(\omega) + \varphi(z) + \langle z - \omega, A\omega \rangle \geq 0, \quad \forall z \in C. \tag{3.30} \]
Therefore \( \omega \in \Omega \). Next, we show that \( \omega \in F(T) \). Since \( y_n \to \omega \), by the demiclosed principle and the fact that \( \|Ty_n - y_n\| \to 0 \) as \( n \to \infty \), we obtain \( \omega \in F(T) \). So \( \omega \in \Omega \cap F(T) \).
Since \( x^* = P_{F(T)\cap \Omega}f(x^*) \), we have
\[
\limsup_{n \to \infty} (f(x^*) - x^*, y_n - x^*) = \lim_{i \to \infty} (f(x^*) - x^*, y_{n_i} - x^*) \\
\leq (f(x^*) - x^*, \omega - x^*) \leq 0.
\]
Finally, we show that \( \{x_n\} \) converges strongly to \( x^* \). Indeed, using (3.2), (3.3) and (3.7), we obtain that
\[
\|x_{n+1} - x^*\|^2 \leq \|y_n - x^*\|^2 = \|\alpha_n f(u_n) + (1 - \alpha_n)T_n u_n - x^*\|^2
\]
\[
= \|\alpha_n (f(u_n) - x^*) + (1 - \alpha_n) (T_n u_n - x^*)\|^2
\]
\[
\leq (1 - \alpha_n)^2 \|T_n u_n - x^*\|^2 + 2 \alpha_n \langle f(u_n) - x^*, \alpha_n (f(u_n) - x^*) \rangle
\]
\[
+ (1 - \alpha_n) (T_n u_n - x^*)
\]
\[
= (1 - \alpha_n)^2 \|T_n u_n - x^*\|^2 + 2 \alpha_n \langle f(u_n) - x^*, y_n - x^* \rangle
\]
\[
\leq (1 - \alpha_n)^2 \|u_n - x^*\|^2 + 2 \alpha_n \langle f(u_n) - x^*, y_n - x^* \rangle
\]
\[
\leq (1 - \alpha_n)^2 \|u_n - x^*\|^2 + 2 \alpha_n \|f(x^*) - x^*, y_n - x^*\|
\]
\[
= (1 - \alpha_n)^2 \|u_n - x^*\|^2 + 2 \alpha_n \|f(x^*) - x^*, y_n - x^*\|
\]
\[
\leq (1 - \alpha_n)^2 \|u_n - x^*\|^2 + 2 \alpha_n \|f(x^*) - x^*, y_n - x^*\|
\]
\[
= (1 - \alpha_n)^2 \|u_n - x^*\|^2 + 2 \alpha_n \|f(x^*) - x^*, y_n - x^*\|
\]
where \( \gamma_n = \alpha_n (2 - \alpha_n - 2 \rho (1 - (1 - \rho) \alpha_n)) \) and \( \delta_n = \alpha_n (2 \langle f(x^*) - x^*, y_n - x^* \rangle + 2 \rho \alpha_n \|u_n - x^*\| \|f(x^*) - x^*\|) \). It is easy to see that \( \lim_{n \to \infty} \gamma_n = 0 \) and \( \lim_{n \to \infty} \delta_n \leq 0 \). Applying Lemma (2.3) to (3.31), we conclude that \( x_n \to x^* \) as \( n \to \infty \). This completes the proof. \( \square \)

**Corollary 3.2.** Let \( H \) be a real Hilbert space, \( C \) a closed convex nonempty subset of \( H \), \( \varphi : C \to \mathbb{R} \) a proper lower semicontinuous and convex functional, \( \Phi : C \times C \to \mathbb{R} \) a bifunction satisfying (A1)-(A4), \( \{T_n\} \) a sequence of nonexpansive mappings of \( C \) into itself such that \( \cap_{n=1}^{\infty} F(T_n) \cap MEPU(\Phi, \varphi) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be real sequences in \([0, 1]\) and \( \{r_n\} \) a sequence in \([0, 2\alpha]\).

Assume that:

(i) either (B1) or (B2) holds;

(ii) the sequence \( \{r_n\} \) satisfies
\[
(C1) \quad 0 < c \leq r_n \leq d < 2\alpha;
\]

(iii) \( \lim_{n \to \infty} \alpha_n = 0; \)

(iv) the sequence \( \{\beta_n\} \) satisfies
\[
(E1) \quad \beta_n \in [0, b) \text{ for some } b \in (0, 1).\]
Let $f$ be a $\rho$-contraction of $C$ into itself and let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and

\[
\begin{cases}
\Phi(u_n, x) + \varphi(x) - \varphi(u_n) + \frac{1}{r_n}(x - u_n, u_n - x) \geq 0, & \forall x \in C \\
y_n = \alpha_n f(u_n) + (1 - \alpha_n)T_n u_n, \\
x_{n+1} = (1 - \beta_n)y_n + \beta_n T_n y_n, & n \geq 0.
\end{cases}
\]

Suppose that $\sum_{n=1}^{\infty} \sup \{\|T_{n+1} z - T_n z\| : z \in B\} < \infty$ for any bounded subset $B$ of $C$. Let $T$ be a mapping of $C$ into itself defined by $T z = \lim_{n \to \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to $x^* = P_{F(T) \cap MEP(\Phi, f)}(x^*)$.

**Proof.** Put $A \equiv 0$. Then, for all $\alpha \in (0, \infty)$, we have

\[
\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.
\]

Hence all the conditions of Theorem 3.1 are satisfied. Therefore the corollary is obtained by Theorem 3.1. 

**Corollary 3.3.** Let $H$ be a real Hilbert space, $C$ a closed convex nonempty subset of $H$, $A$ an $\alpha$-inverse strongly monotone mapping of $C$ into $H$, $\Phi : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)-(A4), $\{T_n\}$ a sequence of nonexpansive mappings of $C$ into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \cap EP \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ and $\{r_n\}$ a sequence in $[0, 2\alpha]$. Assume that:

(i) either (B1) or (B2) holds;

(ii) the sequence $\{r_n\}$ satisfies

\[C1 \quad 0 < c \leq r_n \leq d < 2\alpha;\]

(iii) $\lim_{n \to \infty} \alpha_n = 0$;

(iv) the sequence $\{\beta_n\}$ satisfies

\[E1 \quad \beta_n \in [0, b) \text{ for some } b \in (0, 1).\]

Let $f$ be a $\rho$-contraction of $C$ into itself and let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and

\[
\begin{cases}
\Phi(u_n, x) + \langle Ax_n, x - u_n \rangle + \frac{1}{r_n}(x - u_n, u_n - x) \geq 0, & \forall x \in C \\
y_n = \alpha_n f(u_n) + (1 - \alpha_n)T_n u_n, \\
x_{n+1} = (1 - \beta_n)y_n + \beta_n T_n y_n, & n \geq 0.
\end{cases}
\]

Suppose that $\sum_{n=1}^{\infty} \sup \{\|T_{n+1} z - T_n z\| : z \in B\} < \infty$ for any bounded subset $B$ of $C$. Let $T$ be a mapping of $C$ into itself defined by $T z = \lim_{n \to \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. If $\{x_n\}$ is asymptotic regular, then $\{x_n\}$ converges strongly to $x^* = P_{F(T) \cap EP}(x^*)$. 

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Proof. Put \( \varphi \equiv 0 \) in Theorem 3.1. Hence all the conditions of Theorem 3.1 are satisfied. Therefore the corollary is obtained by Theorem 3.1. \( \square \)

**Corollary 3.4.** Let \( H \) be a real Hilbert space, \( C \) a closed convex nonempty subset of \( H \), \( \Phi : C \times C \to \mathbb{R} \) a bifunction satisfying (A1)-(A4), \( \{T_n\} \) a sequence of nonexpansive mappings of \( C \) into itself such that \( \bigcap_{n=1}^{\infty} F(T_n) \cap EP(\Phi) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be real sequences in \([0, 1]\) and \( \{r_n\} \) a sequence in \([0, 2\alpha]\). Assume that:

(i) either (B1) or (B2) holds;

(ii) the sequence \( \{r_n\} \) satisfies

\[
(C1) \quad 0 < c \leq r_n \leq d < 2\alpha;
\]

(iii) \( \lim_{n \to \infty} \alpha_n = 0; \)

(iv) the sequence \( \{\beta_n\} \) satisfies

\[
(E1) \quad \beta_n \in [0, b) \quad \text{for some} \quad b \in (0, 1).
\]

Let \( f \) be a \( \rho \)-contraction of \( C \) into itself and let \( \{x_n\}, \{y_n\} \) and \( \{u_n\} \) be sequences generated by \( x_0 \in C \) and

\[
\begin{align*}
\Phi(u_n, x) + \frac{1}{r_n} \langle x - u_n, u_n - x_n \rangle & \geq 0, \quad \forall x \in C \\
y_n &= \alpha_n f(u_n) + (1 - \alpha_n) T_n u_n, \\
x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \geq 0.
\end{align*}
\]

Suppose that \( \sum_{n=1}^{\infty} \sup \{\|T_{n+1} z - T_n z\|; z \in B\} < \infty \) for any bounded subset \( B \) of \( C \). Let \( T \) be a mapping of \( C \) into itself defined by \( T z = \lim_{n \to \infty} T_n z \) for all \( z \in C \) and suppose that \( F(T) = \bigcap_{n=1}^{\infty} F(T_n) \). If \( \{x_n\} \) is asymptotic regular, then \( \{x_n\} \) converges strongly to \( x^* = P_{F(T) \cap EP(\Phi)} f(x^*) \).

**Proof.** Put \( \varphi \equiv 0 \) and \( A \equiv 0 \) in Theorem 3.1. Hence the corollary is obtained by Theorem 3.1. \( \square \)

**Remark 3.5.** If we put \( \Phi \equiv 0, \ \varphi \equiv 0 \) and \( A \equiv 0 \), then we obtain the result of Klin-eam and Suantai [5].

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