A General Iterative Method for
Generalized Equilibrium Problems and
Fixed Point Problems in Hilbert Spaces

Kiattisak Rattanaseeha

Division of Mathematics, Department of Science
Faculty of Science and Technology
Loei Rajabhat University
Loei 42000, Thailand
kiattisakrat@live.com

Abstract. In this paper, the researcher found a general iterative method for finding a common element of the set of generalized equilibrium problems, of the set of a system of variational inequalities and of the set of fixed point of a strictly pseudocontractive mappings in Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by many others.

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1. Introduction

Let $H$ be a real Hilbert space with the inner product $\langle \cdot , \cdot \rangle$ and the norm $\| \cdot \|$. Let $C$ be a nonemtpy closed and convex subset of $H$ and let $S : C \rightarrow C$ be a nonlinear mapping. In this paper, we use $F(S)$ to denote the fixed point set of $S$.

Recollect the following definitions.

(1) The mappings $S$ is said to be nonexpansive if

$$
\| Sx - Sy \| \leq \| x - y \|, \ \forall x, y \in C.
$$
The mappings $S$ is said to be strictly pseudo-contractive with the coefficient $k \in [0,1)$ if
\[
\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.
\]
In such a case, $S$ is said to be a $k$-strictly pseudo-contraction too.

The mappings $S$ is said to be pseudo-contractive if
\[
\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.
\]
Obviously, the class of strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions. Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example,[11, 14, 26, 27, 28, 29]

Following, let $F$ be a bifunction from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The so-called equilibrium problem for $F$: $C \times C \rightarrow \mathbb{R}$ is to find $y \in C$ such that
\[
F(y, u) \geq 0, \forall u \in C.
\]
The set of solution of (1.1.4) is denoted by $EP(F)$. Give a mapping $G: C \rightarrow H$, let $F(y, u) = \langle Gy, u - y \rangle$ for all $y, u \in C$. Then $z \in EP(F)$ if and only if $(Gz, u - z) \geq 0$ for all $u \in C$. Numerous problem in physics, optimization and economics reduce to find a solution of (1.1.4).

In order to solve the equilibrium problems for a bifunction $F: C \times C \rightarrow \mathbb{R}$, let us assume that $F$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) $F$ is monotone, i.e, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, \( \lim_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y) \);

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Recently, many authors considered the iterative methods for finding a common element of the set of solutions to the problem (1.1.4) and of the set of fixed points of nonexpansive mappings; see, for example,[14, 19] and the references therein.

In 2007, Takahashi and Takahashi [19] studied the problem (1.1.4), they proved the following result.

**Theorem 1.1.** Let $C$ be a closed convex subset of $H$. Let $F: C \times C \rightarrow \mathbb{R}$ satisfies (A1)-(A4) and let $S$ be a nonexpansive mapping of $C$ into $H$ such that $F(S) \cap EP(F) \neq \emptyset$. Let $f$ be a contraction of $H$ into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

\[
\begin{aligned}
F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle & \geq 0, \quad \forall y \in C, \\
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)S u_n, \quad \forall n \geq 1,
\end{aligned}
\]
A general iterative method for generalized equilibrium problems

where \( \{\alpha_n\} \subset [0, 1] \), and \( \{r_n\} \subset (0, \infty) \) satisfying

(i) \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \),

(ii) \( \lim \inf_{n \to \infty} r_n > 0 \),

(iii) \( \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty \).

Then \( \{x_n\} \) and \( \{u_n\} \) converges strongly to \( z \in F(S) \cap EP(F) \), where

\[ z = P_{F(S) \cap EP(F)} f(z). \]

Next, let \( T : C \to H \) be a nonlinear mapping. We recall the following definitions:

(4) \( T \) is said to be monotone, if

\[ \langle Tx - Ty, x - y \rangle \geq 0, \ \forall x, y \in C. \]

(5) \( T \) is said to be strongly monotone, if there exists a constant \( \theta > 0 \) such that

\[ \langle Tx - Ty, x - y \rangle \geq \theta \|x - y\|^2, \ \forall x, y \in C. \]

In such a case, \( T \) is said to be \( \theta \)-strongly monotone.

(6) \( T \) is said to be inverse-strongly monotone, if there exists a constant \( \theta > 0 \) such that

\[ \langle Tx - Ty, x - y \rangle \geq \theta \|Tx - Ty\|^2, \ \forall x, y \in C. \]

In such a case, \( T \) is said to be \( \theta \)-inverse-strongly monotone.

The classical variational inequality is to find \( u \in C \) such that

(1.1.5) \[ \langle Tu, v - u \rangle \geq 0, \forall v \in C. \]

In this paper, we use \( VI(C, T) \) to denote the set of solutions to the problem (1.1.5). One can easily see that the variational inequality problem is equivalent to a fixed point problem. \( u \in C \) is a solution to the problem (1.1.5) if and only if \( u \) is a fixed point of the mapping \( PC(I - \lambda T) \), where \( \lambda > 0 \) is a constant. The variational inequality has been widely studied in the literature; see, for example, the work of Plubtieng and Punpaeng [25], Wangkeeree and Kamraksa [26] and the references therein.

Let \( A, B : C \to H \) be two nonlinear mappings. In 2008, Ceng et al. [4] considered the following problem of finding \( (x^*, y^*) \in C \times C \) such that

(1.1.6) \[ \begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C, \\ \langle \eta Bx^* + y^* - x^*, x - y^* \rangle \geq 0, \forall x \in C, \end{cases} \]

which is called a general system of variational inequalities, where \( \lambda > 0 \) and \( \mu > 0 \) are two constants. In particular, if \( A = B \), then problem (1.1.6) reduces to finding \( (x^*, y^*) \in C \times C \) such that

(1.1.7) \[ \begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C, \\ \langle \eta Ax^* + y^* - x^*, x - y^* \rangle \geq 0, \forall x \in C, \end{cases} \]
which is defined by Verma [24], and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem (1.1.7) reduces to the classical variational inequality (1.1.5).

Recently, Ceng, Wang and Yao [4] considered an iterative methods for the system of variational inequalities (1.1.5). They got a strongly convergence theorem for the problem (1.1.5) and a fixed point problem for a single nonexpansive mapping; see [4] for more details.

Next, let $F$ be a bifunction from $C \times C$ into $\mathbb{R}$ and $T : C \rightarrow H$ be a $\theta$-inverse-strongly monotone mapping. The generalized equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Tz, y - z \rangle \geq 0, \forall y \in C.$$  

(1.1.8)

In this paper, the set of solutions of the generalized equilibrium problem is denoted by $EP(F, T)$, i.e.,

$$EP(F, T) = \{ z \in C : F(z, y) + \langle Tz, y - z \rangle \geq 0, \forall y \in C \}.$$

In the case of $T \equiv 0$ the problem (1.1.8) is reduced to the problem (1.1.4). In the case of $F \equiv 0$, the problem (1.1.8) reduce to the classical variational inequality problem (1.1.5). The problem (1.1.8) is very general in the sense that it includes, as special case, variational inequalities, optimization problems and others; see, for example, [10].

Recently, Takahashi and Takahashi [20] introduced an iterative method to consider the problem (1.1.5), they proved the following results.

**Theorem 1.2.** Let $C$ be a closed convex subset of a real Hilbert spaces $H$ and let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $T$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$ and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap EP(F, T) \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by

$$
\begin{cases}
F(z_n, y) + \langle Tx_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \forall y \in C, \\
x_{n+1} = \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) z_n], \forall n \geq 1,
\end{cases}
$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, 2\alpha]$ satisfying

(i) $0 < c \leq \beta_n \leq d < 1$, $0 < a < \lambda_n \leq b < 2\alpha$,
(ii) $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1}) = 0$,
(iii) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then \( \{x_n\} \) converges strongly to $z = P_{F(S) \cap EP(F, T)}u$.

Very recently, Qin, Chang and Cho [15] proved the following results.
Theorem 1.3. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $F$ a bifunction from $C \times C$ to $\mathbb{R}$ which satisfying (A1)-(A4). Let $T$ be a $\theta$-inverse-strongly monotone mapping of $C$ into $H$, $A$ a $a$-inverse-strongly monotone mapping of $C$ into $H$ and $B$ a $b$-inverse-strongly monotone mapping of $C$ into $H$, respectively. Let $S : C \rightarrow C$ be a $k$-strict pseudo-contraction with a fixed point. Define a mapping $S_k : C \rightarrow C$ by

$$S_k x = kx + (1 - k)Sx, \text{ for all } x \in C.$$ 

Assume that $\Omega = EP(F, T) \cap F(S) \cap F(D) \neq \emptyset$, where the mapping $D$ is defined by Lemma 2.8. Let $u \in C, x_1 \in C$ and $\{x_n\}$ be a sequences generated by

$$F(u_n, y) + \langle Tx_n, y - u_n \rangle + \frac{1}{r}(y - u_n, u_n - x_n) \geq 0, \forall y \in C,$$

$$y_n = P_C(x_n - \eta Bx_n),$$

$$v_n = P_C(y_n - \lambda Ay_n),$$

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n [\mu_1 S_k x_n + \mu_2 u_n + \mu_3 v_n], \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1], \mu_1, \mu_2, \mu_3 \in [0, 1)$ such that $\mu_1 + \mu_2 + \mu_3 = 1, \lambda \in (0, 2a], \eta \in (0, 2b]$ and $r \in (0, 2\theta]$. If the above control sequences satisfy the following restrictions

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1$,

then the sequences $\{x_n\}$ defined by the iterative algorithm (1.1.9) converge strongly to $\bar{x} = P_\Omega u$ and $(\bar{x}, \bar{y})$, where $\bar{y} = P_C(\bar{x} - \eta B\bar{x})$ is the solution to the problem (1.1.6).

On the other hand, Moudafi [9] introduced the viscosity approximation method for nonexpansive mappings (see [31] for further developments in both Hilbert and Banach spaces). Let $f$ be a contraction on $C$. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \ n \geq 0,$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved [9, 31] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.1.9) strongly converges to the unique solution $q$ in $C$ of the variational inequality

$$\langle (I - f)q, p - q \rangle \geq 0, p \in C.$$
Recently, Marino and Xu [11] introduced the following general iterative method:

\[ x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \]  

(1.1.10)

where \( A \) is a strongly positive bounded linear operator on \( H \). They proved that if the sequence \( \{\alpha_n\} \) of parameters satisfies appropriate conditions, then the sequence \( \{x_n\} \) generated by (1.1.10) converges strongly to the unique solution of the variational inequality

\[ \langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in C \]  

(1.1.11)

which is the optimality condition for the minimization problem

\[ \min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x), \]

where \( h \) is a potential function for \( \gamma f \) (i.e., \( h'(x) = \gamma f(x) \) for \( x \in H \)).

In this paper, inspired and motivated by Zhou [36], Takahashi and Takahashi [20], and Marino and Xu [11], we introduce a general iterative method for finding a common element of the set of generalized equilibrium problems, of the set of a system of variational inequalities and of the set of fixed point of a strictly pseudocontractive mappings in Hilbert spaces. The results of this paper extend and improve the results of Zhou [36], Takahashi and Takahashi [20], and Marino and Xu [11] and many others.

2. Preliminaries

Let \( H \) be a real Hilbert space with the norm \( \| \cdot \| \) and the inner product \( \langle \cdot, \cdot \rangle \) and let \( C \) be a closed convex subset of \( H \). We denote weak convergence and strong convergence by notations \( \rightharpoonup \) and \( \rightarrow \), respectively.

A space \( X \) is said to satisfy Opial’s condition [13] if for each sequence \( \{x_n\} \) in \( X \) which converges weakly to a point \( x \in X \), we have

\[ \liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x. \]

For every point \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_Cx \), such that

\[ \|x - P_Cx\| \leq \|x - y\| \quad \text{for all} \ y \in C. \]

\( P_C \) is called the metric projection of \( H \) onto \( C \). It is well known that \( P_C \) is a firmly nonexpansive mapping of \( H \) onto \( C \) and satisfies

\[ \|P_Cx - P_Cy\|^2 \leq \langle x - y, P_Cx - P_Cy \rangle \]  

(2.2.1)

for every \( x, y \in H \). In addition, \( P_Cx \) is characterized by the following properties: \( P_Cx \in C \) and

\[ \langle x - P_Cx, y - P_Cx \rangle \leq 0, \]  

(2.2.2)

\[ \|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2 \]  

(2.2.3)
for all $x \in H, y \in C$. It is obvious to see that the following is true:

$$\tag{2.2.4} u \in VI(C,T) \Leftrightarrow u = P_C(u - \lambda Tu), \lambda > 0.$$ 

If $A$ an $\alpha$–inverse-strongly monotone mapping of $C$ into $H$, then it is obvious that $A$ is $\frac{1}{\alpha}$–Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\| (I - \lambda A)x - (I - \lambda A)y \|^2 = \| x - y - \lambda(Ax - Ay) \|^2$$

$$\leq \| x - y \|^2 - 2\lambda \langle Ax - Ay, x - y \rangle + \lambda^2 \| Ax - Ay \|^2$$

$$\tag{2.2.5} \leq \| x - y \|^2 + \lambda(\lambda - 2\alpha)\| Ax - Ay \|^2$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of $C$ into $H$.

The following lemmas will be useful for proving the convergence result of this paper.

**Lemma 2.1.** Let $H$ be a real Hilbert space. Then for all $x, y \in H$,

1. $\| x + y \|^2 \leq \| x \|^2 + 2\langle y, x + y \rangle$
2. $\| x + y \|^2 \geq \| x \|^2 + 2\langle y, x \rangle$.

**Lemma 2.2.** ([17]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 \leq \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) < 0$. Then $\lim n \to \infty \|y_n - x_n\| = 0$.

**Lemma 2.3.** ([5]) Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \{ z \in C : F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \forall y \in C \}$$

for all $z \in H$. Then, the following hold:

1. $T_r$ is single-valued;
2. $T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\| T_r x - T_r y \|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$
3. $F(T_r) = EP(F)$;
4. $EP(F)$ is closed and convex.

**Lemma 2.4.** ([13]) Let $H$ be a Hilbert space, $C$ a closed convex subset of $H$, and $S : C \to C$ a nonexpansive mapping with $F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in $C$ weakly converging to $x \in C$ and if $\{(I - S)x_n\}$ converges strongly to $y$, then $(I - S)x = y$.

**Lemma 2.5.** ([31]). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n, \quad n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that
Lemma 2.6. ([11]) Assume $A$ is a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$. 

Lemma 2.7. ([11]) Let $C$ be a closed convex subset of a real Hilbert space $H$ and $T : C \rightarrow C$ a $k$-strict pseudo-contraction. Define $S : C \rightarrow H$ by

$$Sx = \alpha x + (1 - \alpha)Tx, \text{ for all } x \in C.$$ 

Then, as $\alpha \in [k, 1)$, $S$ is nonexpansive such that $F(S) = F(T)$. 

Lemma 2.8. ([4]) For given $x^*, y^* \in C$, where $y^* = P_C(x^* - \mu Bx^*)$, is a solution of problem (1.1.6) if and only if $x^*$ is a fixed point of the mapping $D : C \rightarrow C$ defined by

$$D(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)].$$

Lemma 2.9. ([1]) Let $E$ be a uniformly convex Banach space, $C$ be a nonempty closed convex subset of $E$ and $S : C \rightarrow C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero. 

Lemma 2.10. ([3]) Let $C$ be a closed convex subset of a real Banach space $E$. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on $C$. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive number with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping $S$ on $C$ defined by

$$Sx = \sum_{n=1}^{\infty} \lambda_n T_n x$$

for $x \in C$ is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds. 

3. Main Results

In this section, we prove the strong convergence theorem for solving the generalized equilibrium problems for strictly pseudocontractive mappings in a real Hilbert spaces.

Theorem 3.1. Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $T$ be a $\theta$-inverse-strongly monotone mapping of $C$ into $H$ Let $f : C \rightarrow C$ be a contraction with coefficient $\beta \in (0, 1)$, $A$ an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$ and $B$ an $\theta$-inverse-strongly monotone mapping of $C$ into $H$ and let $S$ be a $k$-strict pseudocontraction of $C$ into itself such that $\Omega = EP(F,T) \cap F(S) \cap F(D) \neq \emptyset$, where the mapping $D$ is defined by Lemma 2.8. Define $S_k : C \rightarrow C$ by

$$S_k x = kx + (1 - k)Sx, \text{ for all } x \in C.$$
Let $G : C \rightarrow C$ be a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma} > 0$ with $0 < \gamma < \frac{2}{\alpha}$. Suppose the sequences $\{x_n\}, \{y_n\}$ are given by

\[
(3.3.1) \begin{cases}
    x_1 \in C \text{ chosen arbitrary}, \\
    F(u_n, y) + \langle Tx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\
    y_n = P_C(x_n - \eta Bx_n), \\
    v_n = P_C(y_n - \lambda Ay_n), \\
    x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n G)\left[\mu_1 S_k x_n + \mu_2 u_n + \mu_3 v_n\right], \forall n \geq 1,
\end{cases}
\]

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0, 1]$, $\mu_1, \mu_2, \mu_3 \in [0, 1)$ such that $\mu_1 + \mu_2 + \mu_3 = 1$ and $\{r_n\}$ is a sequence in $(0, 2\alpha)$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ are chosen so that $r_n \in [a, b]$ for some $a, b$ with $0 < a < b < 2\alpha$ satisfying

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $0 < \lim \inf_{n \rightarrow \infty} \beta_n \leq \lim \sup_{n \rightarrow \infty} \beta_n < 1$,

(iii) $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ converges strongly to a point $\bar{x} \in \Omega$ which is the unique solution of the variational inequality

\[
(3.3.2) \langle (G - \gamma f)\bar{x}, \bar{x} - z \rangle \leq 0, \quad z \in \Omega.
\]

Equivalently, one has $\bar{x} = P_\Omega(I - G + \gamma f)(\bar{x})$.

Proof. Since $\alpha_n \rightarrow 0$ by the condition (i), we may assume that, without loss of generality, that $\alpha_n \leq \|G\|^{-1}$ for all $N$. From Lemma 2.7, we know that if $0 < \rho < \|G\|^{-1}$, then $\|I - \rho G\| \leq 1 - \rho$. We will assume that $\|I - G\| \leq 1 - \bar{\gamma}$. We devise the proof into six steps as follows.

**Step 1.** Show that $\{x_n\}$ is well defined.

First, we show that the mappings $I - \lambda A$, $I - \eta B$ and $I - rT$ are nonexpansive, respectively.

For any $x, y \in C$, we have

\[
\|(I - \lambda A)x - (I - \lambda A)y\|^2 = \|x - y - \lambda(Ax - Ay)\|^2 \\
= \|x - y\|^2 - 2\lambda\langle Ax - Ay, x - y \rangle + \lambda^2 \|Ax - Ay\|^2 \\
\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \\
\leq \|x - y\|^2.
\]

This shows that $I - \lambda A$ is nonexpansive. Similarly we can show that the mappings $I - \eta B$ and $I - rT$ are also nonexpansive. From Lemma 2.7, we see that $S_k$ is nonexpansive with $F(S) = F(S_k)$. It follows that $F(S)$ is closed and convex. On the other hand, we have
\(EP(F,T) = F(T_r(I - rT))\). Since \(T_r(I - rT)\) is nonexpansive, we see that \(EP(F,T)\) is closed and convex. From Lemma 2.8, we see that
\[
D = P_C [P_C(I - \eta B) - \lambda AP_C(I - \eta B)] = P_C(I - \lambda A)P_C(I - \eta B).
\]
It follows that the mapping \(D\) is nonexpansive. This shows that \(F(D)\) is closed and convex. It follows that \(P_D\) is well defined.

**Step 2.** Show that \(\{x_n\}\) is bounded.
We note that by hypothesis \(EP(F,T) \cap F(S) \cap F(D) \neq \emptyset\). Put \(u_n = T_r(I - rT)x_n, n \geq 1\). Let \(x^* \in \Omega\), we obtain that \(x^* = T_r(I - rT)x^* = Sx^* = Dx^*\). From Lemma 2.7, we see that \(x^* = S_kx^*\). It follows from Lemma 2.8 that
\[
x^* = P_C [P_C(x^* - \eta Bx^*) - \lambda AP_C(x^* - \eta Bx^*)].
\]
Putting \(y^* = P_C(x^* - \eta Bx^*)\), we obtain that
\[
x^* = P_C(y^* - \lambda Ay^*).
\]
Put \(e_n = \mu_1S_kx_n + \mu_2u_n + \mu_3v_n\) for each \(n \geq 1\). It follows that
\[
\|e_n - x^*\| \leq \|\mu_1S_kx_n + \mu_2u_n + \mu_3v_n - x^*\|
\]
\[
\leq \mu_1\|S_kx_n - x^*\| + \mu_2\|u_n - x^*\| + \mu_3\|v_n - x^*\|
\]
\[
\leq \mu_1\|x_n - x^*\| + \mu_2\|T_r(I - rT)x_n - T_r(I - rT)x^*\| + \mu_3\|P_C(I - \lambda A)y_n - P_C(I - \lambda A)y^*\|
\]
\[
\leq \mu_1\|x_n - x^*\| + \mu_2\|(I - \lambda B)x_n - (I - \lambda B)x^*\| + \mu_3\|y_n - y^*\|
\]
\[
= \mu_1\|x_n - x^*\| + \mu_2\|x_n - x^*\| + \mu_3\|P_C(I - \eta B)x_n - P_C(I - \eta B)x^*\|
\]
\[
\leq \mu_1\|x_n - x^*\| + \mu_2\|x_n - x^*\| + \mu_3\|x_n - x^*\|
\]
\[
= (3\|x^*\|) - x^*.
\]
It follows from the last inequality that
\[
\|x_{n+1} - x^*\| = \|\alpha_n\gamma f(x_n) + \beta_nx_n + (1 - \beta_n)(I - \alpha_nA)e_n - x^*\|
\]
\[
= \|\alpha_n\gamma f(x_n) - Ax^* + \beta_n(x_n - x^*) + (1 - \beta_n)(I - \alpha_nA)(e_n - x^*)\|
\]
\[
\leq (1 - \beta_n - \alpha_n\tilde{\gamma})\|x_n - x^*\| + \beta_n\|x_n - x^*\| + \alpha_n\|\gamma f(x_n) - Ax^*\|
\]
\[
\leq (1 - \alpha_n\tilde{\gamma})\|x_n - x^*\| + \alpha_n\gamma\alpha\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Ax^*\|
\]
\[
= (1 - \alpha_n(\tilde{\gamma} - \gamma\alpha))\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Ax^*\|.
\]
By induction, we have
\[
\|x_n - x^*\| \leq \max \left\{ \|x_1 - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\tilde{\gamma} - \gamma\alpha} \right\}, \quad n \geq 1.
\]
This shows that the sequence \(\{x_n\}\) is bounded, so are \(\{u_n\}\), \(\{v_n\}\) and \(\{y_n\}\).
Step 3. Show that \( x_{n+1} - x_n \rightarrow 0 \) as \( n \rightarrow \infty \).

From the algorithm (3.3.1), we have
\[
\|v_{n+1} - v_n\| = \|P_C(y_{n+1} - \lambda Ay_{n+1}) - P_C(y_n - \lambda Ay_n)\|
\leq \|(y_{n+1} - \lambda Ay_{n+1}) - (y_n - \lambda Ay_n)\|
\leq \|y_{n+1} - y_n\|
= \|P_C(x_{n+1} - \eta Bx_{n+1}) - P_C(x_n - \eta Bx_n)\|
\leq \|(x_{n+1} - \eta Bx_{n+1}) - (x_n - \eta Bx_n)\|
\leq \|x_{n+1} - x_n\|.
\]

It follows that
\[
\|e_{n+1} - e_n\| = \|\mu_1 S_k x_{n+1} + \mu_2 u_{n+1} + \mu_3 v_{n+1} - (\mu_1 S_k x_n + \mu_2 u_n + \mu_3 v_n)\|
= \|\mu_1(S_k x_{n+1} - S_k x_n) + \mu_2(u_{n+1} - u_n) + \mu_3(v_{n+1} - v_n)\|
\leq \mu_1\|S_k x_{n+1} - S_k x_n\| + \mu_2\|u_{n+1} - u_n\| + \mu_3\|v_{n+1} - v_n\|
= \mu_1\|S_k x_{n+1} - S_k x_n\| + \mu_2\|T_r(I - rT)x_{n+1} - T_r(I - rT)x_n\| + \mu_3\|v_{n+1} - v_n\|
\leq \mu_1\|x_{n+1} - x_n\| + \mu_2\|x_{n+1} - x_n\| + \mu_3\|x_{n+1} - x_n\|
= \|x_{n+1} - x_n\|.
\]

Put \( l_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \), for all \( n \geq 1 \). that is,
(3.3.4) \( x_{n+1} = (1 - \beta_n) l_n + \beta_n x_n \), \( \forall n \geq 1 \),
we see that
\[
l_{n+1} - l_n = \frac{\alpha_{n+1} \gamma f(x_{n+1}) + [(1 - \beta_{n+1})I - \alpha_{n+1} G] e_{n+1} - \alpha_n \gamma f(x_n) + [(1 - \beta_n)I - \alpha_n G] e_n}{1 - \beta_{n+1}}.
\]
(3.3.5) \[
\left(1 - \beta_{n+1}\right) \left[\|\gamma f(x_{n+1}) - Ge_{n+1}\| + e_{n+1} - \frac{\alpha_n}{1 - \beta_n} \|\gamma f(x_n) - Ge_n\| - e_n\right].
\]
and so,
(3.3.6) \[
\|l_{n+1} - l_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - Ge_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|\gamma f(x_n) - Ge_n\| + \|e_{n+1} - e_n\|,
\]
which combines with (3.3.6) yields that
\[
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - Ge_{n+1}\| + \frac{\alpha_n}{1 - \beta_n} \|\gamma f(x_n) - Ge_n\|.
\]
It follows from the conditions (i) that \( \lim \sup_{n \rightarrow \infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0 \).
Hence, from Lemma 2.2, one obtains
(3.3.7) \[
\lim_{n \rightarrow \infty} \|l_n - x_n\| = 0.
\]
From (3.3.7), we obtain that
(3.3.8) \[
\|x_{n+1} - x_n\| = (1 - \beta_n) \|l_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.
\]
Step 4. Show that \(\|x_n - e_n\| \to 0\) as \(n \to \infty\).

From the algorithm (3.3.1), we have
\[
x_{n+1} - x_n = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n G]e_n - x_n
\]
(3.3.9)
\[
= \alpha_n (\gamma f(x_n) - Ge_n) + (1 - \beta_n)(t_n - x_n).
\]

It follows that
(3.3.10) \( (1 - \beta_n)\|e_n - x_n\| \leq \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(x_n) - Ge_n\| \).

From the conditions (i) and (3.3.8), we see that
(3.3.11) \( \lim_{n \to \infty} \|e_n - x_n\| = 0. \)

We observe that \(P_\Omega(\gamma f + (I - G))\) is a contraction. Indeed, for all \(x, y \in H\), we have
\[
\|P_\Omega(\gamma f + (I - G))(x) - P_\Omega(\gamma f + (I - G))(y)\| \leq \|\gamma f(x) - f(y)\| + \|I - G\|\|x - y\|
\]
\[
\leq \gamma \alpha \|x - y\| + (1 - \gamma \alpha)\|x - y\|
\]
\[
< \gamma \|x - y\|.
\]

Banach’s Contraction Mapping Principle guarantees that \(P_\Omega(\gamma f + (I - G))\) has a unique fixed point, say \(\bar{x} \in H\). That is, \(\bar{x} = P_\Omega(\gamma f + (I - G))(\bar{x})\).

Step 5. Show that \(\limsup_{n \to \infty} \langle \gamma f(\bar{x}) - G\bar{x}, x_n - \bar{x} \rangle \leq 0\). We can choose a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) such that
\[
\limsup_{n \to \infty} \langle \gamma f(\bar{x}) - G\bar{x}, x_n - \bar{x} \rangle = \limsup_{i \to \infty} \langle \gamma f(\bar{x}) - G\bar{x}, x_{n_i} - \bar{x} \rangle.
\]

Since \(\{x_{n_i}\}\) is bounded, there exists a subsequence \(\{x_{n_{i_j}}\}\) of \(\{x_{n_i}\}\) which converge weakly to \(w \in C\). Without loss of generality, we can assume that \(x_{n_i} \rightharpoonup w \in C\). Next, we show that
\[
w \in \Omega = F(S) \cap F(D) \cap EP(F, T).
\]

In fact, define a mapping \(Q : C \to C\) by
\[
Qx = \mu_1 S_k x + \mu_2 T_r x + \mu_3 P_C(I - \lambda A)P_C(I - \mu B)x, \forall x \in C.
\]

From Lemma 2.10, we see that \(Q\) is a nonexpansive mapping such that
\[
F(Q) = F(S_k) \cap F(T_r(I - r t)) \cap F(P_C(I - \lambda A)P_C(I - \mu B))
\]
\[
= F(S) \cap EP(F, T) \cap F(D).
\]

From (3.3.11), we obtain
(3.3.12) \( \lim_{i \to \infty} \|Qx_{n_i} - x_{n_i}\| = 0. \)
From (3.3.12) and $x_n \rightarrow w$, we have by Lemma 2.9, $w \in F(Q) = F(S) \cap F(D) \cap EP(F, T)$. Hence
\[
\limsup_{n \rightarrow \infty} \langle \gamma f(\tilde{x}) - G\tilde{x}, x_n - \tilde{x} \rangle = \limsup_{i \rightarrow \infty} \langle \gamma f(\tilde{x}) - G\tilde{x}, x_i - \tilde{x} \rangle \\
= \langle \gamma f(\tilde{x}) - G\tilde{x}, w - \tilde{x} \rangle \\
\leq 0.
\]

**Step 6.** Finally, we show that $x_n \rightarrow \tilde{x}$ as $n \rightarrow \infty$. Indeed, we note that
\[
\|x_{n+1} - \tilde{x}\|^2 = \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n G)e_n - \tilde{x}\|^2 \\
= \|((1 - \beta_n)I - \alpha_n G)(e_n - \tilde{x}) + \beta_n (x_n - \tilde{x}) + \alpha_n \gamma f(x_n) - G\tilde{x}\|^2 \\
= \|((1 - \beta_n)I - \alpha_n G)(e_n - \tilde{x}) + \beta_n (x_n - \tilde{x})\|^2 + \alpha_n^2 \|\gamma f(x_n) - G\tilde{x}\|^2 \\
+ 2\beta_n \alpha_n \langle x_n - \tilde{x}, \gamma f(x_n) - G\tilde{x} \rangle \\
+ 2\alpha_n (((1 - \beta_n)I - \alpha_n G)(e_n - \tilde{x}), \gamma f(x_n) - G\tilde{x}) \\
\leq (1 - \beta_n - \alpha_n)^2 \|x_n - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \alpha_n^2 \|\gamma f(x_n) - G\tilde{x}\|^2 \\
+ 2\beta_n \alpha_n \gamma \alpha \|x_n - \tilde{x}\|^2 + 2\beta_n \alpha_n \langle x_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x} \rangle \\
+ 2(1 - \beta_n) \gamma \alpha \alpha \|e_n - \tilde{x} - x_n - \tilde{x} \|^2 + 2(1 - \beta_n) \alpha_n \langle e_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x} \rangle \\
- 2\alpha_n^2 \langle G(e_n - \tilde{x}), \gamma f(\tilde{x}) - G\tilde{x} \rangle \\
= [(1 - \beta_n - \alpha_n)^2 + 2\beta_n \alpha_n \gamma \alpha + 2(1 - \beta_n) \gamma \alpha \alpha \|x_n - \tilde{x}\|^2 + \alpha_n^2 \|\gamma f(x_n) - G\tilde{x}\|^2 \\
+ 2\beta_n \alpha_n \langle x_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x} \rangle + 2(1 - \beta_n) \alpha_n \langle e_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x} \rangle \\
- 2\alpha_n^2 \langle G(e_n - \tilde{x}), \gamma f(\tilde{x}) - G\tilde{x} \rangle \\
\leq [1 - 2(\gamma - \alpha_n)^2 \|x_n - \tilde{x}\|^2 + \alpha_n^2 \|\gamma f(x_n) - G\tilde{x}\|^2 \\
+ 2\beta_n \alpha_n \langle x_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x} \rangle + 2(1 - \beta_n) \alpha_n \langle e_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x} \rangle \\
+ 2\alpha_n^2 \langle G(e_n - \tilde{x}), \gamma f(\tilde{x}) - G\tilde{x} \rangle \\
= [1 - 2(\gamma - \alpha_n)^2 \|x_n - \tilde{x}\|^2 + \alpha_n \{\alpha_n (\gamma^2 \|x_n - \tilde{x}\|^2 + \|\gamma f(x_n) - G\tilde{x}\|^2 \\
+ 2\|G(e_n - \tilde{x})\| \|\gamma f(\tilde{x}) - G\tilde{x}\| + 2\beta_n (x_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x}) \\
\}
+ 2(1 - \beta_n) \langle e_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x} \rangle \}
\}
\]

Since $\{x_n\}, \{f(x_n)\}$ and $\{e_n\}$ are bounded, we can take a constant $M > 0$ such that
\[
\gamma^2 \|x_n - \tilde{x}\|^2 + \|\gamma f(x_n) - G\tilde{x}\|^2 + 2\|G(e_n - \tilde{x})\| \|\gamma f(\tilde{x}) - G\tilde{x}\| \leq M
\]
Using the condition (i), and (3.3.13), we get
\[
\limsup_{n \to \infty} \| x_n - \tilde{x} \|^2 \leq \left[ 1 - 2(\bar{\gamma} - \alpha_n \gamma) \right] \| x_n - \tilde{x} \|^2 + \alpha_n \sigma_n,
\]
where
\[
\sigma_n = 2\beta_n \langle x_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x} \rangle + 2(1 - \beta_n) \langle e_n - \tilde{x}, \gamma f(\tilde{x}) - G\tilde{x} \rangle + \alpha_n M.
\]
Using the condition (i), and (3.3.13), we get \( \limsup_{n \to \infty} \beta_n \leq 0 \). Now applying Lemma 2.1 to (3.3.15), we conclude that \( x_n \to \tilde{x} \) as \( n \to \infty \). The proof is now complete. \( \square \)

By Theorem 3.1, we can obtain some new and interesting strong convergence theorems. Now we give some examples as follows:

Setting \( f(x) := u \) for all \( x \in C \), \( G = I \), the identity mapping, and \( \gamma = 1 \), we obtain the following result.

**Corollary 3.2.** [15] Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and \( F \) a bifunction mapping from \( C \times C \) to \( \mathbb{R} \) which satisfying (A1)-(A4). Let \( T \) be a \( \theta \)-inverse-strongly monotone mapping of \( C \) into \( H \), \( A \) a \( \alpha \)-inverse-strongly monotone mapping of \( C \) into \( H \) and \( B \) be a \( \gamma \)-inverse-strongly monotone mapping of \( C \) into \( H \), respectively. Let \( S : C \to C \) be a \( k \)-strict pseudo-contraction with a fixed point. Define a mapping \( S_k : C \to C \) by
\[
S_kx = kx + (1 - k)Sx, \text{ for all } x \in C.
\]
Assume that \( \Omega = EP(F,T) \cap F(S) \cap F(D) \neq \emptyset \), where the mapping \( D \) is defined by Lemma 2.8. Let \( u \in C, x_1 \in C \) and \( \{ x_n \} \) be a sequences generated by

\[
F(u_n, y) + \langle Tx_n, y - u_n \rangle + \frac{1}{n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,
\]
\[
y_n = P_C(x_n - \eta Bx_n),
\]
\[
v_n = P_C(y_n - \lambda Ay_n),
\]
\[
x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n [\mu_1 S_k x_n + \mu_2 u_n + \mu_3 v_n], \quad \forall n \geq 1,
\]

where \( \{ \alpha_n \}, \{ \beta_n \} \) and \( \{ \gamma_n \} \) are sequences in \( [0, 1] \), \( \mu_1, \mu_2, \mu_3 \in [0, 1] \) such that \( \mu_1 + \mu_2 + \mu_3 = 1, \lambda \in (0, 2a], \eta \in (0, 2b] \) and \( r \in (0, 2\theta] \). If the above control sequences satisfy the following restrictions

(i) \( \alpha_n + \beta_n + \gamma_n = 1 \);

(ii) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);

(iii) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \),

then the sequences \( \{ x_n \} \) defined by the iterative algorithm (??) converge strongly to \( \bar{x} = P_{\Omega} u \) and \( (\bar{x}, \bar{y}) \), where \( \bar{y} = P_C(\bar{x} - \eta B\bar{x}) \) is the solution to the problem (1.1.6).
Setting $T = 0$, the zero mapping, in Theorem 3.1, we have the following result.

**Corollary 3.3.** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $F$ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $f : C \rightarrow C$ be a contraction with coefficient $\beta \in (0, 1)$, $A$ an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$ and $B$ an $b$-inverse-strongly monotone mapping of $C$ into $H$ and let $S$ be a $k$-strict pseudocontraction of $C$ into itself such that $\Omega = EP(F) \cap F(S) \cap F(D) \neq \emptyset$, where the mapping $D$ is defined by Lemma 2.8. Define $S_k : C \rightarrow C$ by

$$S_k x = kx + (1 - k)Sx, \text{ for all } x \in C.$$ 

Let $G : C \rightarrow C$ be a strongly positive linear bounded self adjoint operator with coefficient $\gamma > 0$ with $0 < \gamma < \frac{2}{\alpha}$. Suppose the sequences $\{x_n\}, \{y_n\}$ are given by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \forall y \in C, \\ y_n = P_C(x_n - \eta Bx_n), \\ v_n = P_C(y_n - \lambda Ay_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n G)[\mu_1 S_k x_n + \mu_2 u_n + \mu_3 v_n], \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0, 1]$, $\mu_1, \mu_2, \mu_3 \in [0, 1)$ such that $\mu_1 + \mu_2 + \mu_3 = 1$ and $\{r_n\}$ is a sequence in $(0, 2\alpha]$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ are chosen so that $r_n \in [a, b]$ for some $a, b$ with $0 < a < b < 2\alpha$ satisfying

(i) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$,

(iii) $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ converges strongly to a point $\bar{x} \in \Omega$ which is the unique solution of the variational inequality

$$\langle (G - \gamma f)\bar{x}, \bar{x} - z \rangle \leq 0, \ z \in \Omega.$$ 

Equivalently, one has $\bar{x} = P_\Omega(I - G + \gamma f)(\bar{x}).$

Setting $F = 0$, $T = 0$, the zero mapping, and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1, we have the following result.

**Corollary 3.4.** Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $f : C \rightarrow C$ be a contraction with coefficient $\beta \in (0, 1)$, $A$ an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$ and $B$ an $b$-inverse-strongly monotone mapping of $C$ into $H$ and let $S$ be
a $k$-strict pseudocontraction of $C$ into itself such that $\Omega = F(S) \cap F(D) \neq \emptyset$, where the mapping $D$ is defined by Lemma 2.8. Define $S_k : C \to C$ by

$$S_kx = kx + (1 - k)Sx, \text{ for all } x \in C.$$ 

Let $G : C \to C$ be a strongly positive linear bounded self adjoint operator with coefficient $\bar{\gamma} > 0$ with $0 < \gamma < \frac{\bar{\gamma}}{a}$. Suppose the sequences $\{x_n\}, \{y_n\}$ are given by

$$\begin{cases} 
\begin{align*}
x_1 &\in C \text{ chosen arbitrary}, \\
y_n &= P_C(x_n - \eta Bx_n), \\
v_n &= P_C(y_n - \lambda Ay_n), \\
x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n G)[\mu_1 S_k x_n + \mu_2 x_n + \mu_3 v_n], \forall n \geq 1,
\end{align*}
\end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0,1]$, $\mu_1, \mu_2, \mu_3 \in [0,1)$ such that $\mu_1 + \mu_2 + \mu_3 = 1$ and $\{r_n\}$ is a sequence in $(0,2\alpha)$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ are chosen so that $r_n \in [a,b]$ for some $a, b$ with $0 < a < b < 2\alpha$ satisfying

(i) $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1,$

Then $\{x_n\}$ converges strongly to a point $\tilde{x} \in \Omega$ which is the unique solution of the variational inequality

$$\langle (G - \gamma f)\tilde{x}, \tilde{x} - z \rangle \leq 0, \ z \in \Omega.$$ 

Equivalently, one has $\tilde{x} = P_{\Omega}(I - G + \gamma f)(\tilde{x}).$

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References


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