Optimal Convex Combination Bounds of Root-Square and Harmonic Root-Square Means for Seiffert Mean

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Abstract
In this paper, we find the greatest value $\alpha$ and least value $\beta$ such that the double inequality $\alpha S(a, b) + (1 - \alpha) H(a, b) < T(a, b) < \beta S(a, b) + (1 - \beta) H(a, b)$ holds for all $a, b > 0$ with $a \neq b$. Here, $S(a, b) = \sqrt{\frac{a^2 + b^2}{2}}$, $H(a, b) = \sqrt{\frac{2ab}{a^2 + b^2}}$, and $T(a, b) = \frac{a - b}{2 \arctan((a - b)/(a + b))}$ denote the root-square, harmonic root-square, and Seiffert means of two positive numbers $a$ and $b$ with $a \neq b$.

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1. Introduction
For $p \in \mathbb{R}$ the $p$-th power mean $M_p(a, b)$ and Seiffert mean $T(a, b)$ of two positive numbers $a$ and $b$ is defined by

$$M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2}\right)^\frac{1}{p}, & p \neq 0, \\ \sqrt{ab}, & p = 0 \end{cases}$$

(1.1)

and

$$T(a, b) = \begin{cases} \frac{a - b}{2 \arctan((\frac{a - b}{a + b})}\), & a \neq b, \\ \frac{a}{a}, & a = b. \end{cases}$$

(1.2)

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respectively.

Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a, b)$ and $T(a, b)$ can be found in the literature [1-13]. It is well known that $M_p(a, b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and many means are special cases of the power mean, for example,

$$M_{-2}(a, b) = \frac{\sqrt{2ab}}{\sqrt{a^2 + b^2}} = H(a, b),$$  \hspace{1cm} (1.3)

$$M_{-1}(a, b) = \frac{2ab}{a+b} = H(a, b), \quad M_0(a, b) = \sqrt{ab} = G(a, b), \quad M_1(a, b) = \frac{a+b}{2} = A(a, b), \text{ and}$$

$$M_2(a, b) = \sqrt{\frac{a^2 + b^2}{2}} = S(a, b)$$  \hspace{1cm} (1.4)

are the harmonic root-square, harmonic, geometric, arithmetic, and root-square means of $a$ and $b$, respectively.

In [1], Seiffert proved that

$$M_1(a, b) < T(a, b) < M_2(a, b)$$

for all $a, b > 0$ with $a \neq b$.

Chu, Wang and Qiu [3] find the greatest value $p = \frac{\log(3)}{\log(\pi/2)} \approx 2.4328$ and least value $q = 5/2$ such that

$$H_p(a, b) < T(a, b) < H_q(a, b)$$

for all $a, b > 0$ with $a \neq b$. Here, $H_p(a, b) = \left(\frac{a^{p+(ab)} + b^{p+(ab)}}{3}\right)^{\frac{1}{p}}, (p \neq 0)$ and $H_0(a, b) = \sqrt{ab}$ is the $p$-th power-type Heron mean of two positive numbers $a$ and $b$.

The following best possible Seiffert mean bounds in terms of Lehmer mean $L_p(a, b) = \frac{a^{p+1}+b^{p+1}}{a^p+b^p}$ are presented in [4]:

$$L_0(a, b) < T(a, b) < L_{1/3}(a, b)$$

for all $a, b > 0$ with $a \neq b$.

In [5], the authors prove that $\alpha = 3/5$ and $\beta = \pi/4$ are the best possible parameters such that the double inequality

$$\alpha T(a, b) + (1 - \alpha)G(a, b) < A(a, b) < \beta T(a, b) + (1 - \beta)G(a, b)$$

holds for all $a, b > 0$ with $a \neq b$. 

The main purpose of this paper is to answer the question: what are the greatest value $\alpha$ and least value $\beta$ such that the double inequality
\[
\alpha S(a, b) + (1 - \alpha) H(a, b) < T(a, b) < \beta S(a, b) + (1 - \beta) H(a, b)
\]
holds for all $a, b > 0$ with $a \neq b$.

2. Main Result

**Theorem 2.1.** The double inequality
\[
\alpha S(a, b) + (1 - \alpha) H(a, b) < T(a, b) < \beta S(a, b) + (1 - \beta) H(a, b)
\]
holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{2\sqrt{2}}{\pi} = 0.900316 \cdots$ and $\beta \geq \frac{11}{12} = 0.916666 \cdots$.

**Proof.** Firstly, we prove that
\[
T(a, b) < \frac{11}{12} S(a, b) + \frac{1}{12} H(a, b)
\]
and
\[
T(a, b) > \frac{2\sqrt{2}}{\pi} S(a, b) + \frac{\pi - 2\sqrt{2}}{\pi} H(a, b)
\]
for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $p \in \{11/12, 2\sqrt{2}/\pi\}$, then from (1.1)-(1.4) one has
\[
\begin{align*}
T(a, b) &- [pS(a, b) + (1 - p) H(a, b)] \\
&= b[p(1 + t^2) + 2(1 - p)t] \\
&= \frac{2 \arctan \left( \frac{1}{\sqrt{t^2 + 1}} \right)}{\sqrt{t^2 + 1}} \times \frac{(t - 1)\sqrt{t^2 + 1}}{p(1 + t^2) + 2(1 - p)t - \sqrt{2} \arctan \left( \frac{t - 1}{t + 1} \right)}.
\end{align*}
\]

Let
\[
f(t) = \frac{(t - 1)\sqrt{t^2 + 1}}{p(1 + t^2) + 2(1 - p)t - \sqrt{2} \arctan \left( \frac{t - 1}{t + 1} \right)},
\]
then simple computations lead to
\[
f(1) = 0,
\]
\[
\lim_{t \to +\infty} f(t) = \frac{1}{p} - \frac{\sqrt{2\pi}}{4}.
\]
\[ f'(t) = \frac{f_1(t)}{(t^2 + 1)^{3/2}[p(t^2 + 1) + 2(1 - p)t]^2}, \quad (2.7) \]

where
\[
f_1(t) = (t + 1)(t^2 + 1)[(2 - p)t^2 + 2(p - 1)t + 2 - p] \]
\[
- \sqrt{2}[p(1 + t^2) + 2(1 - p)t]^2 \sqrt{t^2 + 1}. \quad (2.8)\]

We divide the proof into two cases.

**Case 1.** If \( p = 11/12 \), then we clearly see that
\[
(2 - p)t^2 + 2(p - 1)t + 2 - p = \frac{1}{12}(13t^2 - 2t + 13) > 0 \quad (2.9)\]

and
\[
\left\{ (t + 1)(t^2 + 1)[(2 - p)t^2 + 2(p - 1)t + 2 - p] \right\}^2 \\
- \left\{ \sqrt{2}[p(1 + t^2) + 2(1 - p)t]^2 \sqrt{t^2 + 1} \right\}^2 \\
= \frac{1}{10368}(t^2 + 1)(t - 1)^4[2421t^4 + 52(t - 1)t^3 + 4898t^2 \\
+ 52(t - 1)t + 2473] < 0 \quad (2.10)\]

for \( t > 1 \).

Equation (2.8) together with inequalities (2.9) and (2.10) imply that
\[
f_1(t) < 0 \quad (2.11)\]

for \( t > 1 \).

Therefore, inequality (2.1) follows from (2.3)-(2.5) and (2.7) together with (2.11).

**Case 2.** If \( p = 2\sqrt{2}/\pi = 0.900316 \cdots \), then simple computations lead to
\[
(2 - p)t^2 + 2(p - 1)t + 2 - p = 2(1 - \sqrt{2}/\pi)(t - 1)^2 + 2t > 0 \quad (2.12)\]

and
\[
\left\{ (t + 1)(t^2 + 1)[(2 - p)t^2 + 2(p - 1)t + 2 - p] \right\}^2 \\
- \left\{ \sqrt{2}[p(1 + t^2) + 2(1 - p)t]^2 \sqrt{t^2 + 1} \right\}^2 \\
= (t^2 + 1)(t - 1)^2 g(t). \quad (2.13)\]

where
\[
g(t) = (-2p^4 + p^2 - 4p + 4)t^6 + 4(3p^4 - 4p^3 - p + 2)t^5 + (-30p^4 + 64p^3 \\
- 49p^2 - 4p + 16)t^4 + 8(5p^4 - 12p^3 + 12p^2 - 9p + 4)t^3 \\
+ (-30p^4 + 64p^3 - 49p^2 - 4p + 16)t^2 + 4(3p^4 - 4p^3 - p + 2)t \\
- 2p^4 + p^2 - 4p + 4. \quad (2.14)\]
$$\begin{align*}
-2p^4 + p^2 - 4p + 4 &= -0.104741 \cdots < 0. \\
(2.15) \\
\text{It follows from (2.14) and (2.15) that} \\
g(1) &= 8(11 - 12p) = 1.569633 \cdots > 0, \\
(2.16) \\
\lim_{t \to +\infty} g(t) &= -\infty, \\
(2.17) \\
g'(t) &= 6(-2p^4 + p^2 - 4p + 4)t^5 + 20(3p^4 - 4p^3 - p + 2)t^4 + 4(-30p^4 \\
&+ 64p^3 - 9p^2 - 4p + 16)t^3 + 24(5p^4 - 12p^3 + 12p^2 - 9p + 4)t^2 \\
&+ 2(-30p^4 + 64p^3 - 49p^2 - 4p + 16)t + 12p^4 - 16p^3 - 4p + 8, \\
g'(1) &= 24(11 - 12p) > 0, \\
(2.18) \\
\lim_{t \to +\infty} g'(t) &= -\infty, \\
(2.19) \\
g''(t) &= 30(-2p^4 + p^2 - 4p + 4)t^4 + 80(3p^4 - 4p^3 - p + 2)t^3 + 12(-30p^4 \\
&+ 64p^3 - 9p^2 - 4p + 16)t^2 + 48(5p^4 - 12p^3 + 12p^2 - 9p + 4)t \\
&+ 2(-30p^4 + 64p^3 - 49p^2 - 4p + 16), \\
g''(1) &= 8(-10p^2 - 86p + 87) = 11.736817 \cdots > 0, \\
(2.20) \\
\lim_{t \to +\infty} g''(t) &= -\infty, \\
(2.21) \\
g'''(t) &= 120(-2p^4 + p^2 - 4p + 4)t^3 + 240(3p^4 - 4p^3 - p + 2)t^2 \\
&+ 24(-30p^4 + 64p^3 - 49p^2 - 4p + 16)t \\
&+ 48(5p^4 - 12p^3 + 12p^2 - 9p + 4), \\
g'''(1) &= 96(-5p^2 - 13p + 16) = 23.331892 \cdots > 0, \\
(2.22) \\
\lim_{t \to +\infty} g'''(t) &= -\infty, \\
(2.23) \\
g^{(4)}(t) &= 360(-2p^4 + p^2 - 4p + 4)t^2 + 480(3p^4 - 4p^3 - p + 2)t \\
&+ 24(-30p^4 + 64p^3 - 49p^2 - 4p + 16), \\
(2.24)
\end{align*}$$
\[
g^{(4)}(1) = 48(-17p^2 - 42p + 58 - 8p^3) = 27.306355\cdots > 0, \quad (2.24)
\]

\[
\lim_{t \to +\infty} g^{(4)}(t) = -\infty, \quad (2.25)
\]

\[
g^{(5)}(t) = 720(-2p^4 + p^2 - 4p + 4)t + 480(3p^4 - 4p^3 - p + 2), \quad (2.26)
\]

\[
g^{(5)}(1) = 240(3p^2 - 14p + 16 - 8p^3) = -2.609127\cdots < 0. \quad (2.27)
\]

From (2.15) and (2.26) we clearly see that \(g^{(5)}(t) < 0\) is strictly decreasing in \([1, +\infty)\), then (2.27) leads to the conclusion that \(g^{(4)}(t)\) is strictly decreasing in \([1, +\infty)\).

It follows from (2.24) and (2.25) together with the monotonicity of \(g^{(4)}(t)\) we clearly see that there exists \(\lambda_1 > 1\) such that \(g^{(4)}(t) > 0\) for \(t \in [1, \lambda_1)\) and \(g^{(4)}(t) < 0\) for \(t \in (\lambda_1, +\infty)\). Hence, \(g''(t)\) is strictly increasing in \([1, \lambda_1]\) and strictly decreasing in \([\lambda_1, +\infty)\).

From (2.22) and (2.23) together with the piecewise monotonicity of \(g''(t)\) we know that there exists \(\lambda_2 > 1\) such that \(g''(t)\) is strictly increasing in \([1, \lambda_2]\) and strictly decreasing in \([\lambda_2, +\infty)\).

Inequality (2.20) and equation (2.21) together with the piecewise monotonicity of \(g''(t)\) lead to the conclusion that there exists \(\lambda_3 > 1\) such that \(g'(t)\) is strictly increasing in \([1, \lambda_3]\) and strictly decreasing in \([\lambda_3, +\infty)\).

It follows from (2.18) and (2.19) together with the piecewise monotonicity of \(g'(t)\) that there exists \(\lambda_4 > 1\) such that \(g(t)\) is strictly increasing in \([1, \lambda_4]\) and strictly decreasing in \([\lambda_4, +\infty)\).

From (2.16) and (2.17) together with the piecewise monotonicity of \(g(t)\) we clearly see that there exists \(\lambda_5 > 1\) such that \(g(t) > 0\) for \(t \in [1, \lambda_5)\) and \(g(t) < 0\) for \(t \in (\lambda_5, +\infty)\). Then (2.7) and (2.8) together with (2.12)-(2.14) lead to the conclusion that \(f(t)\) is strictly increasing in \([1, \lambda_5]\) and strictly decreasing in \([\lambda_5, +\infty)\).

Note that (2.6) becomes

\[
\lim_{t \to +\infty} f(t) = 0. \quad (2.28)
\]

Equations (2.5) and (2.28) together with the piecewise monotonicity of \(f(t)\) imply that

\[
f(t) > 0 \quad (2.29)
\]

for \(t > 1\).

Therefore, inequality (2.2) follows from (2.3) and (2.4) together with (2.29).
Next, we prove that \( \alpha = 2\sqrt{2}/\pi = 0.900316 \cdots \) and \( \beta = 11/12 = 0.916666 \cdots \) are the best possible parameters such that the double inequality
\[
\alpha S(a, b) + (1 - \alpha)\overline{H}(a, b) < T(a, b) < \beta S(a, b) + (1 - \beta)\overline{H}(a, b)
\]
holds for all \( a, b > 0 \) with \( a \neq b \).

For any \( x > 0 \), \( \alpha > 2\sqrt{2}/\pi \) and \( \beta < 11/12 \), from (1.1)-(1.4) one has
\[
\lim_{x \to +\infty} \frac{\alpha S(1, x) + (1 - \alpha)\overline{H}(1, x)}{T(1, x)} = \frac{\sqrt{2\pi}}{4} - \alpha > 1 \tag{2.30}
\]
and
\[
\beta S(1, 1 + x) + (1 - \beta)\overline{H}(1, 1 + x) - T(1, 1 + x) = \beta [1 + \frac{1}{2}x + \frac{1}{8}x^2 + o(x^2)] + (1 - \beta) [1 + \frac{x}{2} - \frac{3}{8}x^2 + o(x^2)]
\]
\[= [1 + \frac{1}{2}x + \frac{1}{12}x^2 + o(x^2)] \tag{2.31}
\]
\[\leq -\frac{1}{2} \left( \frac{11}{12} - \beta \right)x^2 + o(x^2) \quad (x \to 0). \tag{2.32}
\]

Inequality (2.30) implies that for any \( \alpha > 2\sqrt{2}/\pi \) there exists \( X = X(\alpha) > 1 \) such that \( \alpha S(1, x) + (1 - \alpha)\overline{H}(1, x) > T(1, x) \) for \( x \in (X, +\infty) \), and equation (2.31) implies that for any \( \beta < 11/12 \) there exists \( \delta = \delta(\beta) > 0 \) such that \( \beta S(1, 1 + x) + (1 - \beta)\overline{H}(1, 1 + x) < T(1, 1 + x) \) for \( x \in (0, \delta) \).

References


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