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On Gauge Laplace Transform

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Abstract

In this paper, the Laplace transform is considered as a gauge (Henstock-Kurzweil) integral. Different existential conditions are given. Elementary properties and analyticity are discussed. The inversion theorem is established using generalized differentiation. Finally, the existence of Gauge Laplace transform of some functions have been studied.

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1 Introduction

The Laplace transform of $f : [0, \infty) \to \mathbb{R}$, at $s \in \mathbb{C}$, is defined as the integral [11],

$$\int_0^\infty e^{-st} f(t)\, dt. \tag{1}$$

Many authors, in some previous decades, showed their interest in studying the Laplace transform in a classical sense; involving Lebesgue integral and/or Riemann integral on the real line.

In the present paper, we consider Laplace transform as a gauge integral. The gauge (Henstock-Kurzweil) integral is a generalization of Riemann, Lebesgue, Denjoy, and Perron's. But the definition, we take here, is in terms of Riemann sums. Note that in (1), the function e^{-st}, $s \in \mathbb{C}$, is not in $\mathcal{BV}[0, \infty)$, but it is in $\mathcal{BV}(I)$ for any compact interval $I \subset [0, \infty)$. Hence we can not consider (1) as a gauge (Henstock-Kurzweil) integral directly. So we could not get any specific condition for the existence of Laplace transform as a Gauge integral. Here we give some different existential conditions. All the elementary properties are retained by this method. The inverse Laplace transform is obtained by using Post's generalized differentiation method [4]. In this paper, sometimes we need to reverse the order of repeated integrals which is justified in [8].

2 Notations and Preliminary facts

Throughout this paper we shall mainly follow the notations and preliminary facts as given below:
Let $0 \le a < b \le \infty$, and denote the space of all gauge (Henstock-Kurzweil) integrable functions on $[a, b]$ as $\mathcal{HK}$([a,b]).
$\mathcal{HK}_{loc} = \{f : [0, \infty) \to \mathbb{R} \ : \ f \in \mathcal{HK}(I)$ for any compact interval $I \subset [0, \infty)\}$.
$\mathcal{BV}(I) = \{f : [0, \infty) \to \mathbb{R} \ : \ f$ is of bounded variation on I for any compact $I \subset [0, \infty)\}$.
$\mathcal{BV}(\infty) = \{f : [0, \infty) \to \mathbb{R} \ : \ f \in \mathcal{BV}(a, \infty]$ for some $a \in [0, \infty)\}$.

Definition 2.1 *[3] Let $E \subseteq I$. We say that a function $f : I \to \mathbb{R}$ belongs to the class $\mathcal{AC}_\delta(E)$ if for every $\epsilon > 0$, there exist $\eta > 0$ and a gauge δ on E such that $\sum_{i=1}^n |f(d_i) - f(c_i)| < \epsilon$ whenever $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^n$ is a δ-fine subpartition of E and $\sum_{i=1}^n (d_i - c_i) < \eta$.*
We say that f belongs to the class $\mathcal{ACG}_\delta(I)$ if I can be written as a countable union of sets on each of which the function f is $\mathcal{AC}_\delta$.

Definition 2.2 *[6] Let $\{g_n\}$ be a sequence of functions defined on [a,b]. We say that $\{g_n\}$ is of uniform bounded variation on [a,b], if there is a constant M such that $|g_n| \le M$ and $V(g_n) \le M$, $\forall n$.*

The lemma (2.3) and theorem (2.4) are holds in case of Riemann and Lebesgue integral and they are also true for gauge integral.

Lemma 2.3 *[10] If $a < b, \gamma > 0$, then*

$$\int_a^b e^{-k\gamma(x-a)^2}\, dx \longrightarrow \frac{1}{2}\sqrt{\frac{\pi}{k\gamma}} \tag{2}$$

as $k \to \infty$.

Theorem 2.4 *[10] Let η be such that $0 < \eta < b - a$, $\varphi(x) \in C^2$ $(a \le x \le a + \eta)$, $\varphi'(a) = 0$, $\varphi''(a) < 0$, $\varphi(x)$ is non-increasing on $(a, b]$. Then*

$$\int_a^b e^{k\varphi(x)}\, dx \longrightarrow e^{k\varphi(a)} \left(\frac{-\pi}{2\,k\,\varphi''(a)}\right)^{\frac{1}{2}} \tag{3}$$

as $k \to \infty$.

Theorem 2.5 *[6] Suppose $f : [a, b] \to \mathbb{R}$ is gauge integrable and $\{g_n\}$ is a sequence of functions which are of uniform bounded variation on $[a, b]$ and such that $g_n \to g$. Then $\int_a^b f\, g_n \longrightarrow \int_a^b f\, g$ as $n \to \infty$.*

3 Existential Conditions

In this section we tackle the problem of existence of Laplace transform as a gauge integral. The classical sufficient condition for the existence of Laplace transform is that, the function f be locally integrable, i.e., $f \in \mathcal{L}_{loc}(0,\infty)$ and f is of exponential type, i.e., for some constants M, $t_0 > 0$ and real γ, f satisfies $|f(t)| \leq M\, e^{\gamma t}$, $\forall\ t \geq t_0$ [11]. However we give different existential conditions.

Proposition 3.1 *Let $f : [0,\infty) \to \mathbb{R}$ be any continuous function such that $F(x) = \int_0^x f$, $0 \leq x < \infty$, is bounded on $[0,\infty)$. Then the Laplace transform $\mathfrak{L}\{f(t)\}(s)$ of $f(t)$, i.e., $\mathfrak{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt$, $Re.\, s > 0$, exists.*

Proof: Since the function $e^{-st} : [0,\infty) \to \mathbb{R}$ is continuous on $[0,\infty)$ with $\lim_{t\to\infty} e^{-st} = 0$, $Re.\, s > 0$ and $(e^{-st})'$ is absolutely integrable over $[0,\infty)$. The result follows from Dedekind's test [5].

This result can also be drawn by using Du Bois-Reymond's test [1].

Proposition 3.2 *If the Laplace transform $\mathfrak{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt$ exists for $Re.\, s > 0$, then the function $f \in \mathcal{HK}_{loc}$.*

Proof: Note that the function $t \mapsto e^{-st}, s \in \mathbb{C}$, as a function of real variable t, is not of bounded variation on $[0,\infty)$ but is of bounded variation on any compact interval $[a,b]$.

Proposition 3.3 *Let $f \in \mathcal{HK}_{loc}$. Then the Laplace transform of $f(x)$ exists, if $f \in \mathcal{HK}([0,\infty)) \cap \mathcal{BV}(\infty)$.*

Proof: Proof follows from [8] and [9].

4 Some Properties

The usual elementary properties such as linearity, dilation, modulation, translation are remains same for the gauge Laplace transform, proofs of which are quite easy. Also there are some other results that are analogous to the classical one with some modification in the hypothesis which we discuss below.

Proposition 4.1 *(**gauge Laplace transform of derivative**) Let $f : [0,\infty) \to \mathbb{R}$ be a function which is in $\mathcal{ACG}_\delta(\mathbb{R})$ such that $\lim_{t\to\infty} \frac{f(t)}{e^{st}} = 0$. Then for $s \in \mathbb{C}$, both $\mathfrak{L}\{f(t)\}(s)$ and $\mathfrak{L}\left\{f'(t)\right\}(s)$ fail to exist or $\mathfrak{L}\left\{f'(t)\right\}(s) = s\,\mathfrak{L}\{f(t)\}(s) - f(0)$.*

Proof: Using the fact that the derivative of $\mathcal{ACG}_\delta$ function is always Gauge integrable and Hake's theorem [1].

The general case can be considered by imposing condition on f as: For $k = 0, 1, 2, \dots, (n-1)$, $f^{(k)}$ are $\mathcal{ACG}_\delta(\mathbb{R})$ and $\lim_{t\to\infty} \frac{f^{(k)}(t)}{e^{st}} = 0$.

Proposition 4.2 ***(Differentiation of gauge Laplace transform)*** *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $\mathfrak{L}\{f(t)\}(s) = \overline{f}(s)$ exists on a compact interval, say, $I = [\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$. Define $g(t) = tf(t)$ and assume that $g \in \mathcal{HK}(\mathbb{R}^+)$. Then $\mathfrak{L}\{g(t)\}(s)$ exists and $\mathfrak{L}\{g(t)\}(s) = -(\mathfrak{L}\{f(t)\}(s))'$ a.e. Here $s \in \mathbb{R}$.*

Proof: Suppose that $\mathfrak{L}\{f(t)\}(s) = \overline{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$ exists on $[\alpha,\beta]$, $\alpha, \beta \in \mathbb{R}$. Define $g(t) = tf(t)$. We know that the Necessary and Sufficient condition for differentiating under the gauge integral is that

$$\int_{t=0}^{\infty}\int_{s=a}^{b} e^{-st}\, t\, f(t)\, ds\, dt = \int_{s=a}^{b}\int_{t=0}^{\infty} e^{-st}\, t\, f(t)\, dt\, ds \tag{4}$$

for all $[a,b] \in [\alpha,\beta]$ [7]. Observe that the L.H.S. of equation (4) exists. Therefore, by lemma 25(a) [8], equation (4) holds. Hence we can differentiate under the gauge integral. In doing so we get that $\int_{s=a}^{b}\int_{t=0}^{\infty} e^{-st}\, t\, f(t)\, dt\, ds = -\int_a^b (\overline{f}(s))'\, ds$ for all $[a,b] \in [\alpha,\beta]$. Therefore $\mathfrak{L}\{tf(t)\}(s) = -(\mathfrak{L}\{f(t)\}(s))'$ a.e. on $[\alpha,\beta]$. Hence the derivative of the Laplace transform of $f(t)$ exists a.e. on $[\alpha,\beta]$.

We can generalize this result as follows: If for $n \in \mathbb{N}$, $t^n f(t)$ is gauge integrable, then $\mathfrak{L}\{t^n f(t)\}$ exists and $\mathfrak{L}\{t^n f(t)\} = (-1)^n (\mathfrak{L}\{f(t)\})^{(n)}$. Hence derivative of any order of $\mathfrak{L}\{f(t)\}$ exists a.e. so $\mathfrak{L}\{f(t)\}$ is analytic a.e.

Proposition 4.3 ***(Multiplication by $\frac{1}{t}$)*** *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $\mathfrak{L}\{f(t)\}(s) = \overline{f}(s)$ exists and assume that $\frac{f(t)}{t} \in \mathcal{HK}(\mathbb{R}^+)$. Then $\mathfrak{L}\left\{\frac{f(t)}{t}\right\}(s)$ exists and $\int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \int_s^\infty \overline{f}(u)\,du$. Here $s \in \mathbb{R}$.*

Proof: Since the function $\frac{f(t)}{t}$ is gauge integrable and e^{-st} is of bounded variation on a compact interval, it is seen that $\mathfrak{L}\{f(t)\}(s)$ exists. And the equation $\int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \int_s^\infty \overline{f}(u)\,du$ can be justified by lemma 25(a) in [8].

Proposition 4.4 ***(gauge Laplace transform of integral)*** *Let $f : [0,\infty) \to \mathbb{R}$ be a gauge integrable function such that $\mathfrak{L}\{f(t)\}(s)$ exists. Then $\mathfrak{L}\left\{\int_0^t f(u)\,du\right\}(s)$ exists and $\mathfrak{L}\left\{\int_0^t f(u)\,du\right\}(s) = \frac{1}{s}\overline{f}(s)$.*

Proof: Let $F(t) = \int_0^t f(u)\,du$, $t \in \mathbb{R}$. Since the function f is gauge integrable, we have $\lim_{t\to\infty} F(t)$ is bounded a.e. on $\mathbb{R}$ and so integration by parts gives the required result.

5 Inversion

Since, in case of gauge (Henstock-Kurzweil) integral, the integration process and differentiation process are now inverse of each other. Here we shall establish the inverse gauge Laplace transform by using Post's generalized differentiation method [4].

Proposition 5.1 *Suppose* $0 < \eta < b - a$ *and* $\varphi(x) \in C^2$, $(a \leq x \leq a + \eta)$, $\varphi'(a) = 0$, $\varphi''(a) < 0$, $\varphi(x)$ *is non-increasing on* $(a, b]$. *Suppose* $f(x) \in \mathcal{HK}([a, b])$. *Then* $\int_a^b f(x)\, e^{k\varphi(x)}\, dx \longrightarrow f(a)\, e^{k\varphi(a)} \left(\frac{-\pi}{2\, k\, \varphi''(a)}\right)^{\frac{1}{2}}$ *as* $k \to \infty$.

Proof: Consider the integral

$$I_k = \int_a^b [f(x) - f(a)]\; e^{k[\varphi(x) - \varphi(a)]}\, dx. \tag{5}$$

We show that $I_k \to 0$ as $k \to \infty$.
Let

$$g_k(x) = \begin{cases} e^{k[\varphi(x) - \varphi(a)]}, & \text{if } x \in (a, b] \\ 0, & \text{if } x = a. \end{cases}$$

Then $g_k(x) = e^{k[\varphi(x) - \varphi(a)]} \to 0$, $\forall\, x \in [a, b]$ as $k \to \infty$.
Set $g(x) = 0$, $\forall\, x \in [a, b]$.
Observe that the sequence $\{g_k(x)\} = \left\{e^{k[\varphi(x) - \varphi(a)]}\right\}$ is of uniform bounded variation on $\mathbb{R}$ with $g_k(x) \to 0$ as $k \to \infty$. Now by theorem 2.5 we can say that $I_k \to 0$ as $k \to \infty$ and the result follows by theorem 2.4.

Also, there is a result of the similar kind as

Proposition 5.2 *Suppose* $0 < \eta < b - a$, *and* $\varphi(x) \in C^2$, $(b - \eta \leq x \leq b)$, $\varphi'(b) = 0$, $\varphi''(b) < 0$, $\varphi(x)$ *is non-decreasing on* $[a, b)$ *and suppose* $f(x) \in \mathcal{HK}([a, b])$. *Then* $\int_a^b f(x)\, e^{k\varphi(x)}\, dx \longrightarrow f(b)\, e^{k\varphi(b)} \left(\frac{-\pi}{2\, k\, \varphi''(b)}\right)^{\frac{1}{2}}$ *as* $k \to \infty$.

Proposition 5.3 ***(Inversion Theorem)*** *If* $f(x) \in \mathcal{HK}\left((o, \infty)\right)$ *and if the Laplace transform of* $f(t)$, $\mathfrak{L}\left\{f(t)\right\}(s) = \int_0^\infty e^{-st} f(t)\, dt$ *exists, then*

$$\lim_{k \to \infty} \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \int_0^\infty e^{-\frac{ku}{t}}\, u^k\, f(u)\, du = f(t). \tag{6}$$

Proof: The inversion theorem follows immediately, if we can prove the following:

I. If $f(t) \in \mathcal{HK}\left([t, \infty)\right)$, $0 < t \leq x < \infty$, then we have

$$\lim_{k \to \infty} \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \int_t^\infty e^{-\frac{ku}{t}}\, u^k\, f(u)\, du = \frac{f(t)}{2}.$$

II. If $f(t) \in \mathcal{HK}\left((0,t]\right),\ (0 < x \leq t)$ then we have

$$\lim_{k\to\infty} \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^t e^{-\frac{ku}{t}}\, u^k\, f(u)\, du = \frac{f(t)}{2}.$$

I. We have the relation $\int_a^b f(t)\, e^{k\varphi(x)}\, dx = f(a)\, e^{k\varphi(a)} \left(\frac{-\pi}{2\, k\, \varphi''(a)}\right)^{\frac{1}{2}}$ as $k \to \infty$. Take $a = t$ so that

$$\int_t^b f(t)\, e^{k\varphi(x)}\, dx = f(t)\, e^{k\varphi(t)} \left(\frac{-\pi}{2\, k\, \varphi''(t)}\right)^{\frac{1}{2}}$$

as $k \to \infty$. Now take $\varphi(x) = \ln x - \frac{x}{t}$, $\forall x \in [t, \infty)$ and use the Stirling's formula we get that

$$\lim_{k\to\infty} \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_t^b f(x)\, x^k\, e^{\frac{-kx}{t}}\, dx = \frac{f(t)}{2}.$$

And by Hake's theorem we have

$$\lim_{k\to\infty} \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_t^\infty f(x)\, e^{-\frac{kx}{t}}\, x^k\, dx = \frac{f(t)}{2}. \tag{7}$$

II. Here again we have the relation $\int_a^b f(t)\, e^{k\varphi(x)}\, dx = f(b)\, e^{k\varphi(b)} \left(\frac{-\pi}{2\, k\, \varphi''(b)}\right)^{\frac{1}{2}}$ as $k \to \infty$. Take $b = t$ and $\varphi(x) = \ln x - \frac{x}{t}$, $\forall\, x \in (0, t]$ and use Hake's theorem, we get

$$\lim_{k\to\infty} \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^t f(t)\, e^{-\frac{kx}{t}}\, x^k\, dx = \frac{f(t)}{2}. \tag{8}$$

and so in this way we get equation (6).

6 Examples

Here we give some examples of gauge Laplace transform of functions whose classical Laplace transform do not exist. However we could not able to find the exact form of the gauge Laplace transforms of these functions.

Example 6.1 *Consider the function* $g(t) = \frac{e^{-st}\sin t}{\ln(t+2)}$, $s \in \mathbb{C}$.
Let $f(t) = e^{-st}\, \sin t$ *and* $\varphi(t) = \frac{1}{\ln(t+2)}$.
Observe that the function $\varphi(t) = \frac{1}{\ln(t+2)}$ *is differentiable on* $(0, \infty)$ *and* $\varphi' \in \mathcal{L}(\mathbb{R}^+)$. *And if* $F(u) = \int_0^u e^{-st} \sin t\, dt$, *then* $|F(t)| \leq M,\ \forall\, t \in (0, \infty)$, $Re.\, s > 0$ *for some constant* $M > 0$. *Also* $\lim_{t\to\infty} F(t)\varphi(t)$ *exists. Therefore by Du Bois-Reymond's test [1],* $\mathfrak{L}\left\{\frac{\sin t}{\ln(t+2)}\right\}(s)$ *exists,* $Re.\, s > 0$.

Example 6.2 *Let* $f(t) = \frac{\cos t}{\sqrt{t}}$ *and* $\varphi(t) = e^{-st}$, $s \in \mathbb{C}$. *Then by Du Bois-Reymond's test [1],* $\mathfrak{L}\left\{\frac{\cos t}{\sqrt{t}}\right\}(s)$ *exists for Re.* $s > 0$.

Example 6.3 *Let* $\varphi : (0, \infty) \to \mathbb{R}$ *be a function defined as* $\varphi(t) = \frac{e^{-\sigma t}}{t}$, $t \in (0, \infty)$, $\sigma > 0$. *This function is not in* $\mathcal{HK}((0, \infty))$.
Now let $n_0 \in \mathbb{N}$ *be such that* $0 < (1 + 4n_0)\frac{\pi}{4\omega}$ *for* $t \in (0, \infty)$, *we have* $|\sin \omega t| \geq \frac{1}{\sqrt{2}}$ *if and only if* $\omega t \in \bigcup_{n=n_0}^{\infty} \left[(1 + 4n)\frac{\pi}{4}, (3 + 4n)\frac{\pi}{4}\right]$.
We show that $\frac{e^{-\sigma t}}{t} \sin \omega t \notin \mathcal{L}((0, \infty))$, $\sigma > 0$ *and* $\frac{e^{-\sigma t}}{t} \cos \omega t \notin \mathcal{L}((0, \infty))$, $\sigma > 0$ *follows similarly.*
If $n \in \mathbb{N}$, *then* $(3 + 4n)\frac{\pi}{4} < (1 + n)\pi$ *and we get*

$$\int_0^{(n+1)\pi} \frac{e^{-\sigma t}}{t} |\sin \omega t| \, dt \geq \frac{\pi}{2\sqrt{2}} \sum_{i=n_0}^{n} \frac{1}{(1 + i)\pi e^{\sigma(1+i)\pi}} \tag{9}$$

On the other hand, we have

$$\int_0^{(n+1)\pi} \frac{e^{-\sigma t}}{t} dt = \int_0^{n_0\pi} \frac{e^{-\sigma t}}{t} dt + \int_{n_0\pi}^{(1+n)\pi} \frac{e^{-\sigma t}}{t} dt.$$

But, Since $\frac{e^{-\sigma t}}{t} \notin \mathcal{HK}((0, \infty))$, *we have* $\int_0^{\infty} \frac{e^{-\sigma t}}{t} dt = \infty$.
Therefore $\sum_{i=n_0}^{\infty} \frac{e^{-\sigma i\pi}}{i\pi} = \infty$, *consequently, we have* $\frac{e^{-\sigma t}}{t} \sin \omega t \notin \mathcal{L}((0, \infty))$, $\sigma > 0$. *Similarly* $\frac{e^{-\sigma t}}{t} \cos \omega t \notin \mathcal{L}((0, \infty))$, $\sigma > 0$. *But by Chartier-Dirichlet's test [1], we have both* $\frac{e^{-\sigma t}}{t} \sin \omega t \in \mathcal{HK}((0, \infty))$, $\sigma > 0$ *and* $\frac{e^{-\sigma t}}{t} \cos \omega t \in \mathcal{HK}(0, \infty)$, $\sigma > 0$. *Hence the integral* $\int_0^{\infty} \frac{e^{-st}}{t} dt$, $s = \sigma + i\omega$, $\sigma > 0$ *exists as gauge integral.*

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