Int. Journal of Math. Analysis, Vol. 5, 2011, no. 35, 1719 - 1732

Strong Convergence Theorems for Two Relatively Asymptotically Nonexpansive Mappings in Banach Spaces[1]

Kriengsak Wattanawitoon[a,b]

[a]Department of Mathematics and Statistics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Tak, Tak 63000, Thailand
[b]Centre of Excellence in Mathematics, CHE, Si Ayuthaya Rd.
Bangkok 10400, Thailand
kriengsak.wat@rmutl.ac.th

Usa Humphries[c,b]

[c]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Bangkok 10140, Thailand
[b]Centre of Excellence in Mathematics, CHE, Si Ayuthaya Rd.
Bangkok 10400, Thailand
usa.wan@kmutt.ac.th

Poom Kumam[c,2]

[c]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Bangkok 10140, Thailand
poom.kum@kmutt.ac.th

Abstract

In this paper, we prove a strong convergence theorem of modified Ishikawa iteration processes by the new hybrid iterative method introduced by Takahashi et al. (2008) for two relatively asymptotically nonexpansive mappings in Banach spaces under some appropriate conditions. Furthermore, our result improves and generalizes the corresponding results announced by many others.

[1]The project was supported by the Centre of Excellence in Mathematics, under the Commission on Higher Education, Ministry of Education, Thailand.

[2]Corresponding author: email poom.kum@kmutt.ac.th

Mathematics Subject Classification: 46C05, 47D03, 47H09, 47H10, 47H20

Keywords: strong convergence; nonexpansive mapping; hybrid methods; relatively asymptotically nonexpansive

1 Introduction

Let E be a real Banach space, C be a nonempty closed convex subset of E, and $T : C \to C$ be a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \text{for all} \;\; x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in C : x = Tx\}$. A mapping T is said to be asymptotically nonexpansive [3] if there exists a sequence $\{k_n\}$ of positive real number with $\lim_{n\longrightarrow\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \qquad \text{for all} \;\; x, y \in C \quad \text{and} \;\; n \geq 1.$$

Some iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first iteration process is now known as Mann's iteration process [11] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0 \tag{1.1}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0, 1]$.

The second iteration process is referred to as Ishikawa's iteration process [4] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \quad n \geq 0 \end{cases} \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are in the interval $[0, 1]$.

Recall that the *normalized duality mapping* J from E to E^* is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \tag{1.3}$$

for $x \in E$.

Matsushita and Takahashi [12] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n) \tag{1.4}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They prove that the sequence $\{x_n\}$ converges weakly to a fixed point of T. Hence in order to have strong convergence, in recent years, the CQ iteration and the shrinking projection methods for approximating fixed points of nonlinear mappings has been introduced and studied by various authors [6, 7, 8, 9, 10, 14, 15, 16, 18, 19, 20, 21, 23, 25] and the references therein. Moreover, Matsushita and Takahashi [13] proposed the following modification of iteration (1.4):

$$\begin{cases} x_0 \in C \qquad \text{chosen arbitrary,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \qquad n = 0, 1, 2, \dots . \end{cases} \tag{1.5}$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}(x_0)$.

In 2007, Su and Qin [17] proved the following iteration for a relatively asymptotically nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n x_n), \\ C_n = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) \\ \qquad +(1-\alpha_n)(k_n^2\|z_n\|^2 - \|x_n\|^2 + (k_n^2 - 1)M - 2\langle v, k_n^2 Jz_n - Jx_n\rangle)\}, \\ Q_n = \{v \in C : \langle Jx_o - Jx_n, x_n - v\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x), \end{cases} \tag{1.6}$$

where M is an appropriate constant such that $M > \|v\|^2$ for each $v \in C$. They proved that if $\{\alpha_n\}$ and $\{\beta_n\} \subset [0, 1]$ and $\limsup_{n\longrightarrow\infty} \alpha_n < 1$ and $\beta_n \to 1$, then the sequence $\{x_n\}$ generate by (1.6) converges strongly to $P_{F(T)}x$.

The purpose of this paper is to employ Yongfu Su et al.'s idea [17] to modify iteration processes (1.6) for two closed relatively asymptotically nonexpansive mappings in Banach spaces, we introduce a new following hybrid iterative scheme:

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x), \qquad n = 0, 1, 2, \dots . \end{cases} \tag{1.7}$$

where J is the duality mapping on E and $\theta_n = [(1-\alpha_n)(t_n^2 - 1) - (1-\alpha_n)(1-\beta_n)t_n^2(s_n^2 - 1)]\phi(v, x_n) \to 0$ as $n \to \infty$. We shall prove the iteration process

(1.7) converg strongly to a $P_{F(T)}(x_0)$ under appropriate conditions by using the method recently introduced by Takahashi et al., [24]. Our results improve and extend the corresponding results of Su and Qin's result and connected with Su et al.'s result and many others.

2 Preliminaries

Let E be a real Banach space with dual E^*. Denote by $\langle \cdot, \cdot \rangle$ the inner product. If C is a nonempty closed convex subset of a real Hilbert space H and $P_C : H \to C$ is the metric projection, then P_C is nonexpansive. Alber [1] has recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue representation of the metric projection in Hilbert spaces. Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \qquad \text{for } x, y \in E. \tag{2.1}$$

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem

$$\phi(x^*, x) = \min_{y \in C} \phi(y, x),$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\| y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \qquad \text{for } x, y \in E. \tag{2.2}$$

Remark 2.1. ([22]). If E is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(y, x) = 0$ then $x = y$. From (2.2), we have $\|x\| = \|y\|$. This implies $\langle y, Jx\rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$

Let C be a closed convex subset of E, a point p in C is said to be an asymptotic fixed point of T [2] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. We say that the mapping T is relatively nonexpansive if the following conditions are satisfied:
(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p, Tx) \leq \phi(p, x)$ for each $x \in C, p \in F(T)$;
(R3) $F(T) = \hat{F}(T)$.
A mapping T from C into itself is called relatively asymptotically nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, T^n x) \leq k_n^2 \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Lemma 2.2. (*Kamimura and Takahashi* [5]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.3. (*Alber* [1]). *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \qquad \forall y \in C. \tag{2.3}$$

Lemma 2.4. (*Alber* [1]). *Let E be a reflexive, strict convex, and smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \qquad \forall y \in C. \tag{2.4}$$

Lemma 2.5. (*Matsushita and Takahashi* [13]). *Let E be a strictly convex and uniformly smooth real Banach space, let C be a closed convex subset of E, and let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.6. (*Plutieng and Ungchittrakool* [16]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a closed convex subset of E. Then, for points $w, x, y, z \in E$ and a real number $a \in \mathbb{R}$, the set $K := \{v \in C : \phi(v, y) \leq \phi(v, x) + \langle v, Jz - Jw \rangle + a\}$ is closed and convex.*

3 Main Results

In this section, we prove a strong convergence theorem which is main result.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty bounded closed convex subset of E. Let S and T be two relatively asymptotically nonexpansive mappings from C into itself with sequence $\{s_n\}$ and $\{t_n\}$ such that $s_n, t_n \to 1$ as $n \to \infty$, respectively and $F(\mathcal{T}) = F(S) \cap F(T)$ is $F(\mathcal{T}) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequence in [0,1] such that $\limsup_{n \longrightarrow \infty} \alpha_n < 1$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x), \qquad n = 0, 1, 2, \ldots . \end{cases} \tag{3.1}$$

where J is the duality mapping on E and $\theta_n = [(1-\alpha_n)(t_n^2 - 1) - (1-\alpha_n)(1-\beta_n)t_n^2(s_n^2 - 1)]\phi(v, x_n) \to 0$ as $n \to \infty$. Then $\{x_n\}$ converges strongly to $\Pi_{F(\mathcal{T})}x$, where $\Pi_{F(\mathcal{T})}$ is the generalized projection from C onto $F(\mathcal{T})$.

Proof. We first show that C_{n+1} is closed and convex for each $n \geq 0$. From the definition of C_{n+1} it is obvious that C_{n+1} is closed for each $n \geq 0$. By Lemma 2.6, C_{n+1} is convex for any $n \geq 0$.

Next, we show that $F(\mathcal{T}) \subset C_n$ for all $n \geq 0$. Indeed, let $p \in F(\mathcal{T})$ and where S and T are relatively asymptotically nonexpansive, we have

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)JT^n z_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JT^n z_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, JT^n z_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|T^n z_n\|^2 \\
&= \alpha_n(\|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2) + (1-\alpha_n)(\|p\|^2 - 2\langle p, JT^n z_n\rangle + \|T^n z_n\|^2) \\
&\leq \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, T^n z_n) \\
&\leq \alpha_n\phi(p, x_n) + t_n^2(1-\alpha_n)\phi(p, z_n) \\
&\leq \phi(p, x_n) + (1-\alpha_n)(t_n^2\phi(p, z_n) - \phi(p, x_n))
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n x_n)) \\
&= \|p\|^2 - 2\langle p, \beta_n Jx_n + (1-\beta_n)JS^n x_n\rangle + \|\beta_n Jx_n + (1-\beta_n)JS^n x_n\|^2 \\
&\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1-\beta_n)\langle p, JS^n x_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|S^n x_n\|^2 \\
&= \beta_n(\|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2) + (1-\beta_n)(\|p\|^2 - 2\langle p, JS^n x_n\rangle + \|S^n x_n\|^2) \\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, S^n x_n) \\
&\leq \beta_n\phi(p, x_n) + s_n^2(1-\beta_n)\phi(p, x_n) \\
&= \phi(p, x_n) + (1-\beta_n)(s_n^2\phi(p, x_n) - \phi(p, x_n)) \\
&\leq \phi(p, x_n) + (1-\beta_n)(s_n^2 - 1)\phi(p, x_n).
\end{aligned} \tag{3.3}$$

Substituting (3.3) in (3.2), we have

$$\begin{aligned}
\phi(p, y_n) &\leq \phi(p, x_n) + (1-\alpha_n)(t_n^2(\phi(p, x_n) + (1-\beta_n)(s_n^2 - 1)\phi(p, x_n)) - \phi(p, x_n)) \\
&= \phi(p, x_n) + (1-\alpha_n)(t_n^2(1 + (1-\beta_n)(s_n^2 - 1)) - 1)\phi(p, x_n) \\
&= \phi(p, x_n) + (1-\alpha_n)(t_n^2 + (1-\beta_n)t_n^2(s_n^2 - 1) - 1)\phi(p, x_n) \\
&= \phi(p, x_n) + [(1-\alpha_n)(t_n^2 - 1) - (1-\alpha_n)(1-\beta_n)t_n^2(s_n^2 - 1)]\phi(p, x_n) \\
&= \phi(p, x_n) + \theta_n,
\end{aligned} \tag{3.4}$$

where $\theta_n = [(1-\alpha_n)(t_n^2 - 1) - (1-\alpha_n)(1-\beta_n)t_n^2(s_n^2 - 1)]\phi(p, x_n)$. This mean that, $p \in C_{n+1}$ for all $n \geq 0$. Thus, $\{x_n\}$ is well defined. Since $x_{n+1} = \Pi_{C_{n+1}}x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, we get

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0),$$

for all $n \geq 0$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing.
By definition of x_n and Lemma 2.4, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(p, x_0) - \phi(p, \Pi_{C_n}x_0) \leq \phi(p, x_0),$$

for all $p \in F(T) \subset C_n$. Thus, $\phi(x_n, x_0)$ is bounded. So, $\{x_n\}$ and $\{Tx_n\}$ are bounded.
Thus $\lim_{n\longrightarrow\infty}\phi(x_n, x_0)$ exists. By Lemma 2.4, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n}x_0) &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n}, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0),\end{aligned}$$

for all $n \geq 0$. Thus, $\phi(x_{n+1}, x_n) \to \infty$ as $n \to \infty$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Assuming not, hence there exists $\varepsilon_0 > 0$ and subsequence $\{n_k\}, \{m_k\} \subset \{n\}$ such that

$$\|x_{n_k+m_k} - x_{n_k}\| \geq \varepsilon_0,$$

for all $k \geq 1$. Applying Lemma 2.4 that

$$\phi(x_{n_k+m_k}, x_{n_k}) \leq \phi(x_{n_k+m_k}, x) - \phi(x_{n_k}, x) \to 0, \text{as } k \to \infty. \tag{3.5}$$

Since $\phi(x_n, x_0)$ is bounded and the limit of $\phi(x_n, x_0)$ exists, we obtain

$$\lim_{n\to\infty} \phi(x_{n_k+m_k} - x_{n_k}) = 0.$$

Hence, by Lemma 2.2, we have

$$\lim_{k\to\infty} \|x_{n_k+m_k} - x_{n_k}\| = 0.$$

This is a contradiction, so that $\{x_n\}$ is a Cauchy sequence, such that $\{x_n\}$ converges strongly to p for some a point p in C.
However, since $\lim_{n\longrightarrow\infty}\beta_n = 1$ and $\{x_n\}$ is bounded, we obtain

$$\begin{aligned}\phi(x_{n+1}, z_n) &= \phi(x_{n+1}, J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n x_n))\\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n Jx_n + (1-\beta_n)JS^n x_n\rangle + \|\beta_n Jx_n + (1-\beta_n)JS^n x_n\|^2\\ &\leq \|x_{n+1}\|^2 - 2\beta_n\langle x_{n+1}, Jx_n\rangle - 2(1-\beta_n)\langle x_{n+1}, JS^n x_n\rangle + \beta_n\|x_n\|^2\\ &\quad +(1-\beta_n)\|S^n x_n\|^2\\ &= \beta_n\phi(x_{n+1}, x_n) + (1-\beta_n)\phi(x_{n+1}, S^n x_n)\end{aligned} \tag{3.6}$$

Therefore $\phi(x_{n+1}, z_n) \to 0$.
Since $x_{n+1} = \Pi_{C_{n+1}} \in C_{n+1}$, from the definition of C_n, we have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \theta_n,$$

for all $n \geq 0$. Thus

$$\phi(x_{n+1}, y_n) \to 0, \text{as } n \to \infty.$$

By using Lemma 2.2, we also have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jz_n\| = 0. \quad (3.7)$$

We observe that

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)JT^n z_n)\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1}) - JT^n z_n\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JT^n z_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JT^n z_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|. \end{aligned}$$

It follows that

$$\|Jx_{n+1} - JT^n z_n\| \leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|).$$

By (3.7) and $\limsup_{n\longrightarrow\infty}\alpha_n < 1$, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JT^n z_n\| = 0.$$

Again since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - T^n z_n\| = 0. \quad (3.8)$$

By triangle inequality, we get

$$\|x_n - T^n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^n z_n\| + k_n\|z_n - x_n\|.$$

Since

$$\|z_n - x_n\| = \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0.$$

From (3.8), we have $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$. We now show that $\|x_{n+1} - S^n x_n\| \to 0$. From (3.7), we have

$$\begin{aligned} \|Jx_{n+1} - Jz_n\| &= \|Jx_{n+1} - (\beta_n Jx_n + (1-\beta_n)JS^n x_n)\| \\ &= \|\beta_n(Jx_{n+1} - Jx_n) + (1-\beta_n)(Jx_{n+1}) - JS^n x_n\| \\ &= \|(1-\beta_n)(Jx_{n+1} - JS^n x_n) - \beta_n(Jx_n - Jx_{n+1})\| \\ &\geq (1-\beta_n)\|Jx_{n+1} - JS^n x_n\| - \beta_n\|Jx_n - Jx_{n+1}\|. \end{aligned}$$

It follows that

$$\|Jx_{n+1} - JS^n x_n\| \leq \frac{1}{1-\beta_n}(\|Jx_{n+1} - Jz_n\| + \beta_n\|Jx_n - Jx_{n+1}\|).$$

From (3.7), we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JS^n x_n\| = 0$$

and we also have $\lim_{n\to\infty} \|x_{n+1} - S^n x_n\| = 0$.
Putting $t_\infty = \sup\{t_n : n > 1\} < \infty$, we deduce that

$$\begin{aligned} \|Tx_n - x_n\| &\leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \\ &\leq t_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1+t_\infty)\|x_n - x_{n+1}\|. \end{aligned}$$

Since T is uniformly continuous, we have

$$\|Tx_n - x_n\| \to 0, \qquad \text{as} \quad n \to \infty. \tag{3.9}$$

Finally, we prove that $x_n \to p = \Pi_{F(\mathcal{T})}x_0$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_i}\} \rightharpoonup q \in C$. Putting $q' = \Pi_{F(\mathcal{T})}x_0$ from $x_{n+1} = \Pi_{C_{n+1}}x_0$ and $q' \in F \subset C_{n+1}$, we have $\phi(x_{n+1}, x) \leq \phi(q', x)$.
On the other hand, from weakly lower semicontinuity of the norm, we have

$$\begin{aligned} \phi(q, x_0) &= \|q\|^2 - 2\langle q, Jx_0\rangle + \|x_0\|^2 \\ &\leq \liminf_{i\longrightarrow\infty}(\|x_{n_i}\|^2 - \langle x_{n_i}, Jx_0\rangle + \|x_0\|^2) \\ &\leq \liminf_{i\longrightarrow\infty}\phi(x_{n_i}, x_0) \leq \limsup_{i\longrightarrow\infty}\phi(x_{n_i}, x_0) \\ &\leq \phi(q', x_0). \end{aligned}$$

From the definition of $\Pi_{F(T)}x_0$, that $q = \Pi_{F(T)}x_0$ and hence

$$\lim_{n\longrightarrow\infty}\phi(x_{n_i}, x_0) = \phi(q, x_0)$$

so, we have $\lim_{n\longrightarrow\infty}\|x_{n_i}\| = \|q\|$. Using the Kadec-Klee property of E, we obtain that $\{x_{n_i}\}$ converges strongly to $\Pi_{F(\mathcal{T})}x_0$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent sequence of $\{x_n\}$, we can conclude that $\{x_n\}$ convergences strongly to $\Pi_{F(\mathcal{T})}x$. □

Every class of relatively nonexpansive mapping is relatively asymptotically nonexpansive mapping when $k_n = 1$ and setting $\beta_n = 1$ in (3.1). By using Theorem we obtain the following corollary.

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty bounded closed convex subset of E. Let T be* *a relatively nonexpansive mapping from C into itself with sequence $\{t_n\}$ such that $t_n \to 1$ as $n \to \infty$ and $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is sequence in* [0,1] *such that* $\limsup_{n\longrightarrow\infty}\alpha_n < 1$. *Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x), \qquad n = 0, 1, 2, \dots . \end{cases} \tag{3.10}$$

where J is the duality mapping on E and $\theta_n = [(1-\alpha_n)t_n^2]\phi(v, x_n) \to 0$ as $n \to \infty$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Remark 3.3. Our theorem extends and generalizes result of Su and Qin in [17, Theorem 2.1] from one relatively asymptotically nonexpansive mapping to two relatively asymptotically nonexpansive mappings in Banach spaces and from the CQ hybrid method to the new hybrid method.

4 Applications in Hilbert spaces

In this section, we obtain strong convergence theorems by the hybrid method of modified Ishikawa iterations for relatively nonexpansive mappings in Hilbert spaces.

Theorem 4.1. *Let C be a nonempty bounded closed convex subset of Hilbert space H. Let S and T be two asymptotically nonexpansive mapping from C into itself with sequence $\{s_n\}$ and $\{t_n\}$ such that $s_n, t_n \to 1$ as $n \to \infty$, respectively and $F(\mathcal{T}) = F(S) \cap F(T)$ is $F(\mathcal{T}) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequence in* [0,1] *such that* $\limsup_{n\longrightarrow\infty}\alpha_n < 1$ *and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}(x), \qquad n = 0, 1, 2, \dots . \end{cases} \tag{4.1}$$

where $\theta_n = [(1-\alpha_n)(t_n^2 - 1) - (1-\alpha_n)(1-\beta_n)t_n^2(s_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$. Then $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})}x$, where $P_{F(\mathcal{T})}$ is the metric projection from C onto $F(\mathcal{T})$.

Proof. Since J is an identity operator, we have

$$\phi(x, y) = \|x - y\|^2$$

for every $x, y \in H$. therefore

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \Leftrightarrow \phi(T^n x, T^n y) \leq k_n^2 \phi(x, y)$$

for every $x \in C$ and $p \in F(\mathcal{T})$. Hence, T and S are asymptotically nonexpansive if and only if T and S are relatively asymptotically nonexpansive. Then, by Theorem 3.1, we obtain the result. □

Theorem 4.2. *Let C be a nonempty bounded closed convex subset of Hilbert space H. Let T be an asymptotically nonexpansive mapping from C into itself with sequence $\{t_n\}$ such that $t_n \to 1$ as $n \to \infty$ and $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequence in [0,1] such that $\limsup_{n\longrightarrow\infty}\alpha_n < 1$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T^n z_n, \\ z_n = \beta_n x_n + (1 - \beta_n) T^n x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}(x), \qquad n = 0, 1, 2, \dots . \end{cases} \tag{4.2}$$

where $\theta_n = [(1 - \alpha_n)(t_n^2 - 1) - (1 - \alpha_n)(1 - \beta_n) t_n^2 (t_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$.

Proof. In Theorem 4.1 if $S \equiv T$, then (4.1) reduced to (4.2). □

Corollary 4.3. *Let C be a nonempty bounded closed convex subset of Hilbert space H. Let T be an asymptotically nonexpansive mapping from C into itself with sequence $\{t_n\}$ such that $t_n \to 1$ as $n \to \infty$ and $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is sequence in [0,1] such that $\limsup_{n\longrightarrow\infty}\alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \leq \|x_n - v\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x), \qquad n = 0, 1, 2, \dots . \end{cases} \tag{4.3}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. By taking $\beta_n = 1$ in Theorem 4.1, we can obtain the desired conclusion. □

Remark 4.4. Our theorem extends and improves result of Su and Qin in [17, Theorem 3.1-3-3] from the CQ hybrid method to the new hybrid method.

ACKNOWLEDGEMENTS. The authors would like to thank the Centre of Excellence in Mathematics, under the Commission on Higher Education, Ministry of Education, Thailand. Furthermore, the third author would like to thank the National Research University Project of Thailand's Office of the Higher Education Commission (under NRU-CSEC project no.54000267) for their financial support during the preparation of this paper.

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Received: March, 2011