Int. Journal of Math. Analysis, Vol. 5, 2011, no. 34, 1673 - 1681

Approximation of Fixed Points of Nonexpansive Semigroups Based on a General Iterative Process

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Abstract

Let C be a nonempty closed convex subset of a real Hilbert space H. Consider a nonexpansive semigroup $S = \{T(s) : 0 \leq s < \infty\}$ on C with a common fixed point, a contraction $f : C \to C$ with the coefficient $0 < \alpha < 1$ and a strongly positive linear bounded self-adjoint operator $A : C \to C$ with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. It is proved that the sequence $\{x_n\}$ generated in the iterative process:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = P_C\Big(\alpha_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \beta_n A\big)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)x_n ds\Big) \end{cases}$$

for each $n \geq 0$ converges strongly to a common fixed point $x^* \in F(S)$, where P_C denotes the metric projection from H onto C and $F(S)$ denotes the common fixed point of the nonexpansive semigroup. The point x^* solves the variational inequality $\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0$ for all $x \in F(S)$.

Mathematics Subject Classification: 47H09, 47J25

Keywords: Common fixed point, nonexpansive mapping, self-adjoint operator, variational inequality

1 Introduction and preliminaries

Throughout this paper, we assume that H is a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let C be a nonempty closed convex subset of H. Recall that the *metric projection* $P_C : H \to C$ assigns each point $x \in H$ with its unique nearest point in C which is denoted $P_C x$. Namely, $P_C x \in C$ is the unique point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : \forall y \in C\}.$$

Let $T : C \to C$ be a mapping. Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a mapping $f : C \to C$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

Recall that an operator A is said to be *strongly positive* on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Also, recall that a family $S = \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(a) $T(0)x = x$ for all $x \in C$;
(b) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
(c) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(d) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(S)$ the set of all common fixed points of S, that is, $F(S) = \bigcap_{0 \le s < \infty} F(T(s))$. It is known that $F(S)$ is closed and convex. We also know that $F(S)$ is nonempty if C is bounded; see [1].

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping. More precisely, take $t \in (0,1)$ and define a contraction T_t by

$$T_t x = t\gamma f(x) + (I - tA)Tx,$$

where f is a contraction and A is a strongly positive bounded linear self-adjoint operator. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t. That is,

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t.$$

In the case of T having a fixed point, Marino and Xu [4] proved $\{x_t\}$ converges strongly to a fixed point of T which solves some variational inequality and also is the optimality condition for some minimization problem; see [4] for more details.

Recently, Yao [8] introduced a general iterative process for an infinite family of nonexpansive mappings in Hilbert spaces as follows:

$$\begin{cases} u_0 \in H, \\ u_{n+1} = \alpha_n \gamma f(u_n) + \beta x_n + \big((1-\beta)I - \alpha_n A\big)W_n u_n \end{cases}$$

for each $n \ge 0$, where W_n is a W-mapping generated by an infinite family of nonexpansive mappings, A is a strongly positive bounded linear self-adjoint operator, $\beta, \gamma > 0$ are two constants and f is a contraction on H. He obtained a strong convergence theorem for the infinite family of nonexpansive mappings in Hilbert spaces under certain appropriate conditions on parameters.

Very recently, Plubtieng and Punpaeng [5] studied the following iterative process:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)x_n ds \end{cases} \tag{1.1}$$

for each $n \ge 0$, where $\{s_n\}$ is a positive real divergent sequence and f is a contraction on C. They showed the sequence $\{x_n\}$ generated in the above iterative process converges strongly to a common fixed point of the nonexpansive semigroup S, which solves the following variational inequality:

$$\langle (I - f)\bar{x}, x - \bar{x} \rangle \le 0, \quad \forall x \in F(S). \tag{1.2}$$

In this paper, motivated by Plubtieng and Punpaeng [5], Shimizu and Takahashi [6] and Yao [8], we introduce the following iterative process for a nonexpansive semigroup:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = P_C\Big(\alpha_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \alpha_n A\big)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)x_n ds\Big) \end{cases} \tag{1.3}$$

for each $n \geq 0$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{s_n\}$ is a positive real divergent sequence, $f : C \to C$ is a contraction with the coefficient $0 < \alpha < 1$ and $A : C \to C$ is a strongly positive bounded linear self-adjoint operator. We prove the sequence $\{x_n\}$ generated in (1.3) converges strongly to a common fixed point x^* of the nonexpansive semigroup S, which solves the following variational inequality:

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in F(S). \tag{1.4}$$

In order to prove our main result, we need the following lemmas.

Lemma 1.1. ([3]) *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $S = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on C. Then, for any $0 \leq h < \infty$,*

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x ds - T(h)\frac{1}{t}\int_0^t T(s)x ds \right\| = 0.$$

Lemma 1.2. ([2]) *Let H be a Hilbert space, C be a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C converges weakly to x and $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 1.3. ([4]) *Let H be a real Hilbert space and T be a nonexpansive mapping on H such that $F(T) \neq \emptyset$. Let f be a contraction with the coefficient α $(0 < \alpha < 1)$ and A be a strongly positive linear bounded self-adjoint operator of with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Then sequence $\{x_t\}$ defined by*

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t$$

converges strongly to $x^* \in F(T)$ as $t \to 0$, which uniquely solves the following variational inequality:

$$\langle (\gamma f - A)x^*, p - x^* \rangle \leq 0, \quad \forall p \in F(T).$$

Equivalently, we have $P_{F(T)}(I - A + \gamma f)x^* = x^*$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Lemma 1.4. ([7]) *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the following condition:*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n$$

for each $n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence of real numbers such that

(a) $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n = \infty$;

(b) *either* $\limsup_{n\to\infty}\sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}$ converges to zero.

2 Main results

Now, we are ready to give our main results.

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(S) \neq \emptyset$. Let $\{s_n\}$ be a positive real divergent sequence, $f : C \to C$ be a contraction with the coefficient α $(0 < \alpha < 1)$ and $A : C \to C$ be a strongly positive linear bounded operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$ be a sequence generated in* (1.3). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ such that*

(a) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$;

(b) $\lim_{n\to\infty}\beta_n = 0$ *and* $\alpha_n + \beta_n < 1$ *for each* $n \geq 0$.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(S)$, which solves the variational inequality (1.4).

Proof. Notice that $\{x_n\}$ is bounded. Indeed, from the conditions (a) and (b), we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$. Since A is a strongly positive bounded linear self-adjoint operator on C, we see that

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in C, \|x\| = 1\}.$$

Notice that

$$\begin{aligned}\big\langle\big((1-\beta_n)I - \alpha_n A\big)x, x\big\rangle &= 1 - \beta_n - \alpha_n\langle Ax, x\rangle \\ &\geq 1 - \beta_n - \alpha_n\|A\| \\ &\geq 0,\end{aligned}$$

that is, $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n A\| &= \sup\big\{\big\langle\big((1-\beta_n)I - \alpha_n A\big)x, x\big\rangle : x \in C, \|x\| = 1\big\} \\ &= \sup\big\{1 - \beta_n - \alpha_n\langle Ax, x\rangle : x \in C, \|x\| = 1\big\} \\ &\leq 1 - \beta_n - \alpha_n\bar{\gamma}.\end{aligned}$$

Fixing $p \in F(S)$, we see that

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\leq \Big\| \alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) \\
&\qquad + \big((1-\beta_n)I - \alpha_n A\big)\Big(\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - p\Big)\Big\| \\
&\leq \alpha_n\|\gamma f(x_n) - Ap\| + \beta_n\|x_n - p\| \\
&\quad + (1 - \beta_n - \alpha_n\bar{\gamma})\Big\|\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - p\Big\| \\
&\leq \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\
&\leq \alpha_n\|\gamma f(p) - Ap\| + \big(1 - \alpha_n(\bar{\gamma} - \alpha\gamma)\big)\|x_n - p\|.
\end{aligned}$$

By simple inductions, we have $\|x_n - p\| \leq \max\big\{\|x_0 - p\|, \frac{\|Ap - \gamma f(p)\|}{\bar{\gamma} - \gamma\alpha}\big\}$, which gives that the sequence $\{x_n\}$ is bounded. Setting $y_n = \frac{1}{s_n}\int_0^{s_n} T(s)x_n ds$, we have

$$\|y_n - p\| = \Big\|\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - p\Big\| \leq \|x_n - p\|.$$

Next, we show

$$\limsup_{n\to\infty}\langle(\gamma f - A)x^*, y_n - x^*\rangle \leq 0,$$

where $x^* \in F(S)$. Putting $z_0 = P_{F(S)}x_0$ and setting

$$M = \Big\{z \in C : \|z - z_0\| \leq \|x_0 - z_0\| + \frac{\|Az_0 - \gamma f(z_0)\|}{\bar{\gamma} - \gamma\alpha}\Big\},$$

we see that M is a nonempty closed convex bounded subset of C which is $T(s)$-invariant for each $s \in [0, \infty)$ and contains $\{y_n\}$. Therefore, we assume, without loss of generality, $S = \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on M. It follows from Lemma 1.1 that

$$\lim_{n\to\infty}\|y_n - T(s)y_n\| = 0 \tag{2.1}$$

for all $0 \leq s < \infty$. Take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)x^*, y_n - x^*\rangle = \lim_{i\to\infty}\langle(\gamma f - A)x^*, y_{n_i} - x^*\rangle.$$

We may also assume that $y_{n_i} \rightharpoonup \bar{x}$. From Lemma 1.2, we have $\bar{x} \in F(S)$. Therefore, we have from Lemma 1.3 that

$$\limsup_{n\to\infty}\langle(\gamma f - A)x^*, y_n - x^*\rangle = \langle(\gamma f - A)x^*, \bar{x} - x^*\rangle \leq 0, \tag{2.2}$$

where $\bar{x} \in F(S)$. Finally, we show $x_n \to x^*$ as $n \to \infty$. Observe that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \big\|\big((1-\beta_n)I - \alpha_n A\big)(y_n - x^*) + \beta_n(x_n - x^*) + \alpha_n(\gamma f(x_n) - Ax^*)\big\|^2 \\
&= \big\|\big((1-\beta_n)I - \alpha_n A\big)(y_n - x^*) + \beta_n(x_n - x^*)\big\|^2 \\
&\quad + \|\alpha_n(\gamma f(x_n) - Ax^*)\|^2 + 2\alpha_n(1-\beta_n)\langle y_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad - 2\alpha_n^2\langle A(y_n - x^*), \gamma f(x_n) - Ax^*\rangle + 2\alpha_n\beta_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&= \big\|\big((1-\beta_n)I - \alpha_n A\big)(y_n - x^*)\big\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\langle y_n - x^*, x_n - x^*\rangle - 2\beta_n\alpha_n\langle Ay_n - Ax^*, x_n - x^*\rangle \\
&\quad + \|\alpha_n(\gamma f(x_n) - Ax^*)\|^2 + 2\alpha_n(1-\beta_n)\langle y_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad - 2\alpha_n^2\langle Ay_n - Ax^*, \gamma f(x_n) - Ax^*\rangle + 2\alpha_n\beta_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\leq (1-\beta_n-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n\alpha_n\|Ay_n - Ax^*\|\|x_n - x^*\| \\
&\quad + \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 + 2\alpha_n\langle y_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\alpha_n^2\|Ay_n - Ax^*\|\|\gamma f(x_n) - Ax^*\| \\
&\quad + 2\alpha_n\beta_n\|x_n - x^*\|\|\gamma f(x_n) - Ax^*\| \\
&\quad + 2\alpha_n\beta_n\|y_n - x^*\|\|\gamma f(x_n) - Ax^*\| \\
&\leq (1-\beta_n-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n\alpha_n\|Ay_n - Ax^*\|\|x_n - x^*\| \\
&\quad + \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 + 2\alpha_n\langle y_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\alpha_n^2\|Ay_n - Ax^*\|\|\gamma f(x_n) - Ax^*\| \\
&\quad + 4\alpha_n\beta_n\|x_n - x^*\|\|\gamma f(x_n) - Ax^*\| \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 \\
&\quad + \alpha_n\big\{2\langle y_n - x^*, \gamma f(x_n) - Ax^*\rangle + \alpha_n\big(\|\gamma f(x_n) - Ax^*\|^2 \\
&\qquad + 2\|Ay_n - Ax^*\|\|\gamma f(x_n) - Ax^*\|\big) + 2\beta_n\big(\bar{\gamma}\|x_n - x^*\|^* \\
&\qquad + \|Ay_n - Ax^*\|\|x_n - x^*\| + 2\|x_n - x^*\|\|\gamma f(x_n) - Ax^*\|\big)\big\} \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 \\
&\quad + \alpha_n\big(2\langle y_n - x^*, \gamma f(x_n) - Ax^*\rangle + \alpha_n M_1 + 2\beta_n M_2\big),
\end{aligned} \tag{2.3}$$

where M_1, M_2 are two appropriate constants such that

$$M_1 \geq \sup_{n\geq 1}\big(\|\gamma f(x_n) - Ax^*\|^2 + 2\|Ay_n - Ax^*\|\|\gamma f(x_n) - Ax^*\|\big)$$

and

$$M_2 \geq \sup_{n\geq 1}\big(\bar{\gamma}\|x_n - x^*\|^* + \|Ay_n - Ax^*\|\|x_n - x^*\| + 2\|x_n - x^*\|\|\gamma f(x_n) - Ax^*\|\big).$$

From the conditions (a), (b) and applying Lemma 1.4 to (2.3), we can obtain the desired conclusion easily. □

If $\beta_n = 0$ for each $n \geq 0$ in Theorem 2.1, then we have the following.

Corollary 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(S) \neq \emptyset$. Let $\{s_n\}$ be a positive real divergent sequence, $f : C \to C$ be a contraction with the coefficient α $(0 < \alpha < 1)$ and $A : C \to C$ be a strongly positive linear bounded operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = P_C\left(\alpha_n \gamma f(x_n) + (I - \alpha_n A)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)x_n ds\right) \end{cases}$$

for each $n \geq 0$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(S)$, which solves the variational inequality (1.4).

Taking $\gamma = 1$ and $A = I$, the identity mapping, in Theorem 2.1, we have the following.

Corollary 2.3. (Plubtieng and Punpaeng [5]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(S) \neq \emptyset$. Let $\{s_n\}$ be a positive real divergent sequence and $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$. Let $\{x_n\}$ be a sequence generated in* (1.1). *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ such that*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\lim_{n\to\infty} \beta_n = 0$ *and* $\alpha_n + \beta_n < 1$ *for each* $n \geq 0$.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(S)$, which solves the variational inequality (1.2).

If $\beta_n = 0$ for each $n \geq 0$ in Corollary 2.3, then we have the following.

Corollary 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(S) \neq \emptyset$. Let $\{s_n\}$ be a positive real divergent sequence and $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)x_n ds \end{cases}$$

for each $n \geq 0$*. Let* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$*. Then the sequence* $\{x_n\}$ *converges strongly to* $x^* \in F(S)$*, which solves the variational inequality* (1.2).

Remark 2.5. If $f(x) = u \in C$, a constant for all $x \in C$, then Corollary 2.4 is reduced to Theorem 2 of Shimizu and Takahashi [6].

References

[1] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Natl. Acad. Sci. USA* **54** (1965), 1041–1044.

[2] F. E. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.* **74** (1968), 660–665.

[3] R. E. Bruck, A simple proof of the mean egrodic theorem for nonlinear contractions in Banach spaces, *Israel J. Math.* **32** (1979), 107–116.

[4] G. Marino and H. K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **318** (2006), 43–52.

[5] S. Plubtiend and R. Punpaeng, Fixed point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces, *Math. Comput. Modelling* **48** (2008), 279–286.

[6] T. Shimizu and W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, *J. Math. Anal. Appl.* **211** (1997), 71–83.

[7] H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.* **66** (2002), 240–256.

[8] Y. Yao, A general iterative method for a finite family of nonexpansive mappings, *Nonlinear Anal.* **69** (2008), 1644–1654.

Received: March, 2011