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Common Fixed Points for Single-Valued and Multi-Valued Mappings in G-Metric Spaces

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Abstract

We prove the existence of common fixed point theorems for single-valued and multi-valued mappings satisfying the certain contractive conditions in G-metric spaces.

Keywords: Multi-valued mappings; Common fixed points; Weakly compatible mappings

1 Introduction

The metric fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for instant, variational inequalities, optimization, and approximation theory. There were many authors introduced the generalizations of metric spaces such as Gahler [5, 6] (called 2-metric spaces) and Dhage [3, 4] (called D-metric spaces). In 2003, Mustafa and Sims [12] found that most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect. Consequently, they introduced a generalization of metric spaces. Namely, G-metric spaces as the following:

Definition 1.1 Let X be a noempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying :

$(G1)$ $G(x,y,z)=0$ if $x=y=z$

$(G2)$ $0<G(x,x,y)$, for all $x,y\in X$ with $x\neq y$,

$(G3)$ $G(x,x,y)\leq G(x,y,z)$ for all $x,y,z\in X$ with $z\neq y$,

$(G4)$ $G(x,y,z)=G(x,z,y)=G(y,z,x)=\ldots$ (symmetry in all three variables), and

$(G5)$ $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X,G) is a G-metric space.

Since then the fixed point theory in G-metric space has been studied and developed by many authors (see [2, 13, 14, 16]). The common fixed point theorem for mappings satisfying certain contractive conditions in metric spaces has been continually studied for decade (see [1, 7, 8, 9, 10, 15] and references contained therein). Recently, Abbas and Rhoades [2], proved the common fixed point theorems for a pair of weakly compatible single-valued mappings on G-metric spaces. In this paper, we prove the existence of the common fixed points of a pair of weakly compatible single-valued and multi-valued mappings in G-metric spaces.

2 Preliminaries

We now recall some of the basic concepts and results in G-metric spaces that were introduced in [13].

Definition 2.1 A G-metric is said to be symmetric if $G(x,y,y)=G(y,x,x)$ for all $x,y,z\in X$.

Definition 2.2 Let (X,G) be a G-metric space. We say that a sequence $\{x_n\}$ in X is:

(i) a G-convergent sequence if, for any $\varepsilon>0$, there exist an $x\in X$ and $N\in\mathbb{N}$ such that $G(x,x_n,x_m)<\varepsilon$, for all $n,m\geq N$,

(ii) a G-Cauchy sequence if, for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $G(x_n,x_m,x_l)<\varepsilon$, for all $n,m,l\geq N$.

Theorem 2.3 *Let (X,G) be a G-metric space and $\{x_n\}$ be a sequence in X. Then the following are equivalent:*

(i) $\{x_n\}$ *is* G*-convergent to* x*,*

(ii) $G(x_n, x_n, x) \to 0$*, as* $n \to \infty$*,*

(iii) $G(x_n, x, x) \to 0$*, as* $n \to \infty$*,*

(iv) $G(x_m, x_n, x) \to 0$*, as* $m, n \to \infty$*.*

Proposition 2.4 *Every* G*-metric space* (X, G)*, defines a metric space* (X, d_G) *by*

$$d_G(x, y) = G(x, y, y) + G(x, x, y), \text{ for all } x, y \in X.$$

Theorem 2.5 *Let* (X, G) *be a* G*-metric space and* $\{x_n\}$ *be a sequence in* X*. Then the followings are equivalent:*

(i) $\{x_n\}$ *is* G*-Cauchy.*

(ii) *For every* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$*, for all* $n, m \geq N$*.*

(iii) $\{x_n\}$ *is a Cauchy sequence in the metric space* (X, d_G)*.*

A G-metric space X is said to be complete if every G-Cauchy sequence in X is a G-convergent sequence in X.

Proposition 2.6 *Let* (X, G) *be a* G*-metric space. Then the function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

Definition 2.7 Let f and g be single-valued self mappings on a set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Abbas and Rhoades [2] proved the existence of the common fixed points of a pair of weakly compatible single-valued mappings on G-metric spaces by using the following proposition as a main tool.

Proposition 2.8 *([2, Proposition 1.5]) Let* f *and* g *be weakly compatible self mappings on a set* X*. If* f *and* g *have a unique point of coincidence* $w = fx = gx$*, then* w *is the unique common fixed point of* f *and* g*.*

In this work, we assure the fixed point theorems for a pair of weakly compatible single-valued and multi-valued mappings.

Let X be a G-metric space. We shall denote $CB(X)$ the family of all nonempty closed bounded subsets of X and $K(X)$ the family of all nonempty compact subsets of X. Let $H(\cdot,\cdot,\cdot)$ be the Hausdorff G-distance on $CB(X)$, i.e.,

$$H_G(A,B,C) = \max\{\sup_{x\in A} G(x,B,C), \sup_{x\in B} G(x,C,A), \sup_{x\in C} G(x,A,B)\}$$

where

$$\begin{aligned} G(x,B,C) &= d_G(x,B) + d_G(B,C) + d_G(x,C) \\ d_G(x,B) &= \inf\{d_G(x,y) : y \in B\} \\ d_G(A,B) &= \inf\{d_G(a,b) : a \in A, b \in B\}. \end{aligned}$$

Recall that $G(x,y,C) = \inf\{G(x,y,z) : z \in C\}$. A mapping $T : X \to CB(X)$ is called a multi-valued mapping. A point $x \in X$ is called a fixed point of T if $x \in Tx$.

Remark 2.9 Let X be a G-metric space, $x \in X$ and $B \subseteq X$. For each $y \in B$, we have

$$\begin{aligned} G(x,B,B) &= d_G(x,B) + d_G(B,B) + d_G(x,B) \\ &\le 2\, d_G(x,y) \\ &= 2\,(G(x,x,y) + G(x,y,y)) \\ &\le 2\,(G(x,y,y) + G(x,y,y) + G(x,y,y)) \\ &\le 6\, G(x,y,y) \end{aligned}$$

3 Common fixed point theorems

Definition 3.1 Let X be a set. Assume that $g : X \to X$ and $T : X \to 2^X$. If $w = gx \in Tx$ for some $x \in X$, then x is called a coincidence point of g and T and w a point of coincidence of g and T.

Mappings g and T are called weakly compatible if $gx \in Tx$ for some $x \in X$ implies $gT(x) \subseteq Tg(x)$.

Proposition 3.2 *Let X be a set. Assume that $g : X \to X$ and $T : X \to 2^X$ are weakly compatible mappings. If g and T have a unique point of coincidence $w = gx \in Tx$, then w is the unique common fixed point of g and T.*

Proof. Assume that g and T have a unique point of coincidence $w = gx \in Tx$. Therefore $gw = g(gx) \in gT(x) \subseteq Tg(x) = Tw$. This implies that gw is a point of coincidence of g and T. Thus $w = gw \in Tw$ by uniqueness of point of coincidence of g and T. We obtain that w is a common fixed point of g and

T. Suppose that z is a common fixed point of g and T. Thus $z = gz \in Tz$. Therefore z is point of coincidence of g and T. Since w is the uniqueness of point of coincidence of g and T, we have $z = w$. Hence w is the unique common fixed point of g and T. □

Theorem 3.3 *Let (X, G) be a G-metric space. Assume that $g : X \to X$ and $T : X \to CB(X)$ satisfy the following condition*

$$H_G(Tx, Ty, Tz) \leq aG(gx, gy, gz) + bG(gx, Tx, Tx)$$

$$+cG(gy, Ty, Ty) + dG(gz, Tz, Tz), \qquad (1)$$

for all $x, y, z \in X$, where $a, b, c, d \in [0, 1), a$ or b is positive, and $a + 6b + 6c + 6d < 1$. If the range of g contains the range of T and $g(X)$ is a $G-$complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ implies $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then

(i) *g and T have a unique point of coincidence;*

(ii) *if g and T are weakly compatible, then g and T have a unique common fixed point*

Proof. Let $x_0 \in X$. Since $T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 \in Tx_0$. By definition of Hausdorff G-distance, we obtain that there exists $gx_2 \in Tx_1$ such that $G(gx_1, gx_2, gx_2) \leq H_G(Tx_0, Tx_1, Tx_1) + (a + 6b)$. Therefore, by (1), we have

$$\begin{aligned} G(gx_1, gx_2, gx_2) &\leq H_G(Tx_0, Tx_1, Tx_1) + (a + 6b) \\ &\leq aG(gx_0, gx_1, gx_1) + bG(gx_0, Tx_0, Tx_0) \\ &+ cG(gx_1, Tx_1, Tx_1) + dG(gx_1, Tx_1, Tx_1) + (a + 6b) \\ &\leq aG(gx_0, gx_1, gx_1) + 6bG(gx_0, gx_1, gx_1) \\ &+ 6(c + d)G(gx_1, gx_2, gx_2) + (a + 6b) \\ &= (a + 6b)G(gx_0, gx_1, gx_1) + 6(c + d)G(gx_1, gx_2, gx_2) + (a + 6b). \end{aligned}$$

This implies that $G(gx_1, gx_2, gx_2) \leq \frac{a+6b}{1-6c-6d}G(gx_0, gx_1, gx_1) + \frac{a+6b}{1-6c-6d}$. Let $k = \frac{a+6b}{1-6c-6d}$. Since $a + 6b + 6c + 6d < 1$, we obtain that $k < 1$. Thus

$$G(gx_1, gx_2, gx_2) \leq kG(gx_0, gx_1, gx_1) + k.$$

Since $gx_2 \in Tx_1$, there exists $gx_3 \in Tx_2$ such that

$$G(gx_2, gx_3, gx_3) \leq H_G(Tx_1, Tx_2, Tx_2) + \frac{(a + 6b)^2}{1 - 6c - 6d}.$$

Therefore, by (1), we obtain that

$$\begin{aligned} G(gx_2, gx_3, gx_3) &\leq H_G(Tx_1, Tx_2, Tx_2) + \frac{(a+6b)^2}{1-6c-6d} \\ &\leq aG(gx_1, gx_2, gx_2) + bG(gx_1, Tx_1, Tx_1) + cG(gx_2, Tx_2, Tx_2) \\ &+ dG(gx_2, Tx_2, Tx_2) + \frac{(a+6b)^2}{1-6c-6d} \\ &\leq (a+6b)G(gx_1, gx_2, gx_2) + 6(c+d)G(gx_2, gx_3, gx_3) + \frac{(a+6b)^2}{1-6c-6d}. \end{aligned}$$

Thus

$$G(gx_2, gx_3, gx_3) \leq \frac{a+6b}{1-6c-6d}G(gx_1, gx_2, gx_2) + \frac{(a+6b)^2}{(1-6c-6d)^2}.$$

So

$$G(gx_2, gx_3, gx_3) \leq kG(gx_1, gx_2, gx_2) + k^2.$$

Therefore, for each $n \in \mathbb{N}$, there exists $gx_{n+1} \in Tx_n$ such that

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq kG(gx_{n-1}, gx_n, gx_n) + k^n.$$

Next, we will show that $\{gx_n\}$ is a Cauchy sequence. Since

$$\begin{aligned} d_G(gx_n, gx_{n+1}) &= G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_n, gx_n, gx_{n+1}) \\ &\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_n, gx_{n+1}) \\ &= 3G(gx_n, gx_{n+1}, gx_{n+1}) \\ &\leq 3(kG(gx_{n-1}, gx_n, gx_n) + k^n) \\ &\leq 3(k(kG(gx_{n-2}, gx_{n-1}, gx_{n-1}) + k^{n-1}) + k^n) \\ &= 3(k^2G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + k^n + k^n) \\ &= 3(k^2G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + 2k^n) \\ &= 3(k^2(kG(gx_{n-3}, gx_{n-2}, gx_{n-2}) + k^{n-2}) + 2k^n) \\ &= 3(k^3G(gx_{n-3}, gx_{n-2}, gx_{n-2}) + 3k^n) \\ &\ \ \vdots \\ &= 3(k^nG(gx_0, gx_1, gx_1) + nk^n), \end{aligned}$$

we have

$$\Sigma_{n=0}^{\infty} d_G(gx_n, gx_{n+1}) \leq 3\{\Sigma_{n=0}^{\infty} k^n G(gx_0, gx_1, gx_1) + \Sigma_{n=0}^{\infty} nk^n\} < \infty.$$

Therefore $\{gx_n\}$ is Cauchy. By Theorem 2.6, we have $\{gx_n\}$ is G-Cauchy. Since $g(X)$ is G-complete, there exists $q \in g(X)$ such that $gx_n \to q$ as $n \to \infty$.

Consequently, we can find a p in X such that $gp = q$. We will show that $gp \in Tp$. For each $n \in \mathbb{N}$, by (1), it follows that

$$\begin{aligned} G(gx_{n+1}, Tp, Tp) &\leq H_G(Tx_n, Tp, Tp) \\ &\leq aG(gx_n, gp, gx_n) + bG(gx_n, Tx_n, Tx_n) \\ &+ cG(gp, Tp, Tp) + dG(gp, Tp, Tp) \\ &\leq aG(gx_n, gp, gx_n) + 6bG(gx_n, gx_{n+1}, gx_{n+1}) \\ &+ cG(gp, Tp, Tp) + dG(gp, Tp, Tp). \end{aligned}$$

Taking the limit as $n \to \infty$ implies that $G(gp, Tp, Tp) \leq (c+d)G(gp, Tp, Tp)$. Since $c + d < 1$, we obtain that $gp \in Tp$. That is T and g have a point of coincidence. Now, assume that if $gp \in Tp$ and $gq \in Tq$, then $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$. We will prove the uniqueness of a point of coincidence of g and T. Suppose that $gp \in Tp$ and $gq \in Tq$. By (1) and the assumption, we have

$$\begin{aligned} G(gq, gp, gp) &\leq H_G(Tq, Tp, Tp) \\ &\leq aG(gq, gp, gp) + bG(gq, Tq, Tq) + (c+d)G(gp, Tp, Tp). \end{aligned}$$

Therefore $G(gq, gp, gp) \leq aG(gq, gp, gp)$, and so $gp = gq$. Since

$$\begin{aligned} H_G(Tq, Tp, Tp) &\leq aG(gq, gp, gp) + bG(gq, Tq, Tq) + (c+d)G(gp, Tp, Tp) \\ &= 0, \end{aligned}$$

we can conclude that $Tq = Tp$. Suppose that g and T are weakly compatible. By applying Proposition 3.2, we obtain that g and T have a unique common fixed point. □

Theorem 3.4 *Let (X, G) be a G-metric space. Assume that $g : X \to X$ and $T : X \to CB(X)$ satisfy the following condition*

$$\begin{aligned} H_G(Tx, Ty, Tz) &\leq aG(gx, gy, gz) + bG(gx, gx, Tx) \\ &+cG(gy, gy, Ty) + dG(gz, gz, Tz) \end{aligned} \tag{2}$$

for all $x, y, z \in X$, where $a, b, c, d \in [0, 1)$, a, b or c is positive, and $a+b+c+d < 1$. If the range of g contains the range of T and $g(X)$ is a $G-$complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ implies $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then

(i) *g and T have a unique point of coincidence;*

(ii) *if g and T are weakly compatible, then g and T have a unique common fixed point*

Proof. Let $x_0 \in X$. Since $T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 \in Tx_0$. By definition of Hausdorff G-distance, we obtain that there exists $gx_2 \in Tx_1$ such that $G(gx_1, gx_1, gx_2) \leq H_G(Tx_0, Tx_0, Tx_1) + (a + b + c)$. Therefore, by (1), we have

$$\begin{aligned} G(gx_1, gx_1, gx_2) &\leq H_G(Tx_0, Tx_0, Tx_1) + (a+b+c) \\ &\leq aG(gx_0, gx_0, gx_1) + bG(gx_0, gx_0, Tx_0) \\ &+ cG(gx_0, gx_0, Tx_0) + dG(gx_1, gx_1, Tx_1) + (a+b+c) \\ &\leq aG(gx_0, gx_0, gx_1) + bG(gx_0, gx_0, gx_1) \\ &+ cG(gx_0, gx_0, gx_1) + dG(gx_1, gx_1, gx_2) + (a+b+c) \\ &= (a+b+c)G(gx_0, gx_0, gx_1) + dG(gx_1, gx_2, gx_2) + (a+b+c). \end{aligned}$$

This implies that

$$G(gx_1, gx_1, gx_2) \leq \frac{a+b+c}{1-d} G(gx_0, gx_0, gx_1) + \frac{a+b+c}{1-d}.$$

Let $k = \frac{a+b+c}{1-d}$. Since $a + b + c + d < 1$, we obtain that $k < 1$. Thus

$$G(gx_1, gx_1, gx_2) \leq kG(gx_0, gx_0, gx_1) + k.$$

Therefore, for each $n \in \mathbb{N}$, there exists $gx_{n+1} \in Tx_n$ such that

$$G(gx_n, gx_n, gx_{n+1}) \leq kG(gx_{n-1}, gx_{n-1}, gx_n) + k^n.$$

The rest of proof is similar to the proof of Theorem 3.3. □

The following corollary is the consequence of Theorem 3.4.

Corollary 3.5 *([2, Theorem 2.3]) Let (X, G) be a G-metric. Assume that $f : X \to X$ and $g : X \to X$ satisfy*

$$G(fx, fy, fz) \leq aG(gx, gy, gz) + bG(gx, gx, fx) + cG(gy, gy, fy) + dG(gz, gz, fz), \quad (3)$$

for all $x, y, z \in X$, where $a + b + c + d < 1$. If the range of g contains the range of f and $g(X)$ is a $G-$complete subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Theorem 3.6 *Let (X, G) be a G-metric space. Assume that $g : X \to X$ and $T : X \to K(X)$ satisfy*

$$H_G(Tx, Ty, Tz) \leq k \max\{G(gx, Tx, Tx), G(gy, Ty, Ty), G(gz, Tz, Tz)\} \quad (4)$$

for all $x, y, z \in X$, where $0 \leq k < \frac{1}{6}$. If the range of g contains the range of T and $g(X)$ is a $G-$complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ implies $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then

(i) *g and T have a unique point of coincidence;*

(ii) *if g and T are weakly compatible, then g and T have a unique common fixed point*

Proof. Let $x_0 \in X$. Since $T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 \in Tx_0$. By the definition of Hausdorff G-distance, there exists $gx_2 \in Tx_1$ such that

$$G(gx_1, gx_2, gx_2) \leq H_G(Tx_0, Tx_1, Tx_1).$$

Thus, for each $n \in \Gamma$, we can find $gx_{n+1} \in Tx_n$ such that

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq H_G(Tx_{n-1}, Tx_n, Tx_n).$$

For any $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &\leq H_G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq k \max\{G(gx_{n-1}, Tx_{n-1}, Tx_{n-1}), G(gx_n, Tx_n, Tx_n), \\ &\quad G(gx_n, Tx_n, Tx_n)\} \\ &\leq k \max\{6G(gx_{n-1}, gx_n, gx_n), 6G(gx_n, gx_{n+1}, gx_{n+1})\}. \end{aligned}$$

Suppose that there exists $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$. Since $gx_{n+1} \in Tx_n$, we have T and g have a point of coincidence. Now, assume that $n \in \mathbb{N}$ for which $gx_n \neq gx_{n+1}$. This implies that

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq 6kG(gx_{n-1}, gx_n, gx_n).$$

Let $h = 6k$. Thus $h < 1$. By continuing the above process, we obtain that

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq h^n G(gx_0, gx_1, gx_1).$$

For each $m, n \in \mathbb{N}$ and $m > n$,

$$\begin{aligned} G(gx_n, gx_m, gx_m) &\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\ &\quad + G(gx_{n+2}, gx_{n+3}, gx_{n+3}) + ... + G(gx_{m-1}, gx_m, gx_m) \\ &\leq (h^n + h^{n+1} + ... + h^{m-1})G(gx_0, gx_1, gx_1) \\ &= \frac{h^n}{1-h} G(gx_0, gx_1, gx_1). \end{aligned}$$

We have $G(gx_n, gx_m, gx_m) \to 0$ as $m, n \to \infty$. Consequently, we obtain that $\{gx_n\}$ is G-Cauchy. Because of the G-completeness of $g(X)$, there exists $q \in g(X)$ for which $gx_n \to q$ as $n \to \infty$. Since $q \in g(X)$, we can find $p \in X$ such that $gp = q$. We claim that $gp \in Tp$. By (4), it follows that

$$\begin{aligned} G(gx_{n+1}, Tp, Tp) &\leq H_G(Tx_n, Tp, Tp) \\ &\leq k \max\{G(gx_n, Tx_n, Tx_n), G(gp, Tp, Tp), \\ &\quad G(gp, Tp, Tp)\} \\ &\leq k \max\{6G(gx_n, gx_{n+1}, gx_{n+1}), G(gp, Tp, Tp).\} \end{aligned}$$

This implies that $G(gp, Tp, Tp) \leq kG(gp, Tp, Tp)$. That is $gp \in Tp$ and then g and T have a point of coincidence. Suppose that if $gp \in Tp$ and $gq \in Tq$, then $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$. We next prove that g and T have a unique point of coincidence. Let $gp \in Tp$ and $gq \in Tq$. By (4), it follows that

$$\begin{aligned} G(gq, gp, gp) &\leq H_G(Tq, Tp, Tp) \\ &\leq k \max\{G(gq, Tq, Tq), G(gp, Tp, Tp), G(gp, Tp, Tp)\} \\ &= k \max\{G(gq, Tq, Tq), G(gp, Tp, Tp)\} \\ &= 0, \end{aligned}$$

and so $gp = gq$. Since

$$\begin{aligned} H_G(Tq, Tp, Tp) &\leq k \max\{G(gq, Tq, Tq), G(gp, Tp, Tp), G(gp, Tp, Tp)\} \\ &= k \max\{G(gq, Tq, Tq), G(gp, Tp, Tp)\} \\ &= 0, \end{aligned}$$

we obtain that $Tq = Tp$. Finally, assume that g are T weakly compatible. By applying Proposition 3.2, we can conclude that g and T have a unique common fixed point. □

Theorem 3.7 *Let (X, G) be a G-metric space. Assume that $g : X \to X$ and $T : X \to K(X)$ satisfy*

$$H_G(Tx, Ty, Tz) \leq k \max\{G(gx, gx, Tx), G(gy, gy, Ty), G(gz, gz, Tz)\} \quad (5)$$

for all $x, y, z \in X$, where $0 \leq k < 1$. If the range of g contains the range of T and $g(X)$ is a $G-$complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ implies $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then

(i) *g and T have a unique point of coincidence;*

(ii) *if g and T are weakly compatible, then g and T have a unique common fixed point*

Proof. Let $x_0 \in X$. Since $T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 \in Tx_0$. By the definition of Hausdorff G-metric, there exists $gx_2 \in Tx_1$ such that

$$G(gx_1, gx_1, gx_2) \leq H_G(Tx_0, Tx_0, Tx_1).$$

Thus for each $n \in \Gamma$, we can find $gx_{n+1} \in Tx_n$ such that

$$G(gx_n, gx_n, gx_{n+1}) \leq H_G(Tx_{n-1}, Tx_{n-1}, Tx_n).$$

For any $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} G(gx_n, gx_n, gx_{n+1}) &\leq H_G(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq k \max\{G(gx_{n-1}, gx_{n-1}, Tx_{n-1}), G(gx_{n-1}, gx_{n-1}, Tx_{n-1}), \\ &\quad G(gx_n, gx_n, Tx_n)\} \\ &\leq k \max\{G(gx_{n-1}, gx_{n-1}, gx_n), G(gx_n, gx_n, gx_{n+1})\}. \end{aligned}$$

Suppose that there exists $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$. Since $gx_{n+1} \in Tx_n$, we have T and g have a coincidence point. Now, assume that $n \in \mathbb{N}$ for which $gx_n \neq gx_{n+1}$. This implies that

$$G(gx_n, gx_n, gx_{n+1}) \leq kG(gx_{n-1}, gx_{n-1}, gx_n),$$

and continuing the above process, we obtain that

$$G(gx_n, gx_n, gx_{n+1}) \leq k^n G(gx_0, gx_0, gx_1),$$

for all $n \in \mathbb{N}$. The rest of the proof is similar to Theorem 3.6. □

The following corollary is the particular case of Theorem 3.7.

Corollary 3.8 *([2, Theorem 2.4]) Let (X, G) be a G-metric. Assume that $f : X \to X$ and $g : X \to X$ satisfy*

$$G(fx, fy, fz) \leq k \max\{G(gx, gx, fx), G(gy, gy, fy), G(gz, gz, fz)\} \tag{6}$$

for all $x, y, z \in X$, where $0 \leq k < 1$. If the range of g contains the range of f and $g(X)$ is a $G-$complete subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Theorem 3.9 *Let (X, G) be a G-metric space. Assume that $g : X \to X$ and $T : X \to CB(X)$ satisfy one of the following condition*

$$H(Tx, Ty, Ty) \leq a\{G(gx, Ty, Ty) + G(gy, Tx, Tx)\} \tag{7}$$

for all $x, y \in X$, where $0 < a < \frac{1}{12}$. If the range of g contains the range of T and $g(X)$ is a $G-$complete subspace of X, then g and T have a point of coincidence in X.
Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ implies $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then

(i) *g and T have a unique point of coincidence;*

(ii) *if g and T are weakly compatible, then g and T have a unique common fixed point*

Proof. Let $x_0 \in X$. Since $T(X) \subseteq g(X)$, there exists $x_1 \in X$ for which $gx_1 \in Tx_0$. By the definition of Hausdorff G-distance, we can find $gx_2 \in Tx_1$ such that $G(gx_1, gx_2, gx_2) \leq H_G(Tx_0, Tx_1, Tx_1) + 6a$. Therefore

$$\begin{aligned} G(gx_1, gx_2, gx_2) &\leq H_G(Tx_0, Tx_1, Tx_1) + 6a \\ &\leq a\{G(gx_0, Tx_1, Tx_1) + G(gx_1, Tx_0, Tx_0)\} + 6a \\ &\leq 6a\{G(gx_0, gx_2, gx_2) + G(gx_1, gx_1, gx_1)\} + 6a \\ &\leq 6aG(gx_0, gx_1, gx_1) + 6aG(gx_1, gx_2, gx_2)\} + 6a. \end{aligned}$$

This implies that $G(gx_1, gx_2, gx_2) \leq \frac{6a}{1-6a}G(gx_0, gx_1, gx_1) + \frac{6a}{1-6a}$. So $0 \leq k = \frac{6a}{1-6a} < 1$ and then

$$G(gx_1, gx_2, gx_2) \leq kG(gx_0, gx_1, gx_1) + k.$$

Since $gx_2 \in Tx_1$, there exists $gx_3 \in Tx_2$ such that

$$\begin{aligned} G(gx_2, gx_3, gx_3) &\leq H_G(Tx_1, Tx_2, Tx_2) + \frac{6a^2}{1-6a} \\ &\leq a\{G(gx_1, Tx_2, Tx_2) + G(gx_2, Tx_1, Tx_1)\} + \frac{6a^2}{1-6a} \\ &\leq 6a\{G(gx_1, gx_3, gx_3) + G(gx_2, gx_2, gx_2)\} + \frac{6a^2}{1-6a} \\ &\leq 6aG(gx_1, gx_2, gx_2) + aG(gx_2, gx_3, gx_3)\} + \frac{6a^2}{1-6a}. \end{aligned}$$

Therefore $G(gx_2, gx_3, gx_3) \leq \frac{6a}{1-6a}G(gx_1, gx_2, gx_2) + \frac{6a^2}{(1-6a)^2}$.
Hence $G(gx_2, gx_3, gx_3) \leq kG(gx_1, gx_2, gx_2) + k^2$. So, for each $n \in \mathbb{N}$, there exists $gx_{n+1} \in Tx_n$ such that

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq kG(gx_{n-1}, gx_n, gx_n) + k^n.$$

As the proof of Theorem 3.3, we obtain that $\{gx_n\}$ is G-Cauchy. Because of the G-completeness of $g(X)$, there exists $q \in g(X)$ such that $gx_n \to q$ as $n \to \infty$. Thus, we can find $p \in X$ for which $gp = q$. We will show that $gp \in Tp$. By (7), it follows that

$$\begin{aligned} G(gx_{n+1}, Tp, Tp) &\leq H_G(Tx_n, Tp, Tp) \\ &\leq a\{G(gx_n, Tp, Tp) + G(gp, Tx_n, Tx_n)\} \\ &\leq aG(gx_n, Tp, Tp) + 6aG(gp, gx_{n-1}, gx_{n-1}), \end{aligned}$$

we can conclude that $G(gp, Tp, Tp) \leq aG(gp, Tp, Tp)$ which is a contradiction and then $gp \in Tp$. That is g and T have a point of coincidence. Now, suppose that if $gp \in Tp$ and $gq \in Tq$, then $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$. We will prove that g and T have a unique point of coincidence. Assume that $gp \in Tp$ and $gq \in Tq$. Since

$$\begin{aligned} G(gq, gp, gp) &\leq H_G(Tq, Tp, Tp) \\ &\leq a\{G(gq, Tp, Tp) + G(gp, Tq, Tq)\} \\ &\leq 6aG(gq, gp, gp) + 6aG(gp, gq, gq), \end{aligned}$$

we obtain that $G(gq, gp, gp) \leq \frac{6a}{1-6a} G(gp, gq, gq)$, and

$$\begin{aligned} G(gp, gq, gq) &\leq H_G(Tp, Tq, Tq) \\ &\leq a\{G(gp, Tq, Tq) + G(gq, Tp, Tp)\} \\ &\leq 6aG(gp, gq, gq) + 6aG(gq, gp, gp). \end{aligned}$$

This implies that $G(gp, gq, gq) \leq \frac{6a}{1-6a} G(gq, gp, gp)$. Therefore

$$\begin{aligned} G(gq, gp, gp) &\leq \frac{6a}{1-6a} G(gp, gq, gq) \\ &\leq (\frac{6a}{1-6a})^2 G(gq, gp, gp). \end{aligned}$$

Since $(\frac{6a}{1-6a})^2 < 1$, $G(gq, gp, gp) = 0$ and so $gq = gp$. By (7), we have

$$\begin{aligned} H_G(Tq, Tp, Tp) &\leq a\{G(gp, Tq, Tq) + G(gq, Tp, Tp)\} \\ &\leq 6aG(gp, gq, gq) + 6aG(gq, gp, gp) \\ &= 0. \end{aligned}$$

It follows that $Tq = Tp$. Finally, assume that g are T weakly compatible. By applying Proposition 3.2, we can conclude that g and T have a unique common fixed point. □

Theorem 3.10 *Let (X, G) be a G-metric space. Assume that $g : X \to X$ and $T : X \to CB(X)$ satisfy*

$$H(Tx, Ty, Ty) \leq a\{G(gx, gx, Ty) + G(gy, gy, Tx)\} \tag{8}$$

for all $x, y \in X$, where $0 < a < \frac{1}{2}$. If the range of g contains the range of T and $g(X)$ is a $G-$complete subspace of X, then g and T have a point of coincidence in X.
Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ implies $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then

(i) *g and T have a unique point of coincidence;*

(ii) *if g and T are weakly compatible, then g and T have a unique common fixed point*

Proof. Let $x_0 \in X$. Since $T(X) \subseteq g(X)$, there exists $x_1 \in X$ for which $gx_1 \in Tx_0$. By the definition of Hausdorff G-distance, we can find $gx_2 \in Tx_1$ such that $G(gx_1, gx_1, gx_2) \leq H_G(Tx_0, Tx_0, Tx_1) + a$. Therefore

$$\begin{aligned} G(gx_1, gx_1, gx_2) &\leq H_G(Tx_0, Tx_0, Tx_1) + a \\ &\leq a\{G(gx_0, gx_0, Tx_1) + G(gx_1, gx_1, Tx_0)\} + a \\ &\leq a\{G(gx_0, gx_0, gx_2) + G(gx_1, gx_1, gx_1)\} + a \\ &\leq a\{G(gx_0, gx_0, gx_1) + G(gx_1, gx_1, gx_2)\} + a. \end{aligned}$$

This implies that $G(gx_1, gx_1, gx_2) \leq \frac{a}{1-a} G(gx_0, gx_1, gx_1) + \frac{a}{1-a}$. So $0 \leq k = \frac{a}{1-a} < 1$ and then

$$G(gx_1, gx_2, gx_2) \leq kG(gx_0, gx_1, gx_1) + k.$$

By continuing this process, we have for each $n \in \mathbb{N}$, there exists $gx_{n+1} \in Tx_n$ such that

$$G(gx_n, gx_n, gx_{n+1}) \leq kG(gx_{n-1}, gx_{n-1}, gx_n) + k^n.$$

The rest of the proof is similar to Theorem 3.9. ∎

The following corollary is immediately followed by Theorem 3.10.

Corollary 3.11 *([2, Theorem 2.6]) Let (X, G) be a G-metric. Assume that $f : X \to X$ and $g : X \to X$ satisfy either*

$$G(fx, fy, fy) \leq a\{G(gx, gx, fy) + G(gy, gy, fx)\} \tag{9}$$

for all $x, y \in X$, where $0 \leq a < \frac{1}{2}$. If the range of g contains the range of f and $g(X)$ is a $G-$complete subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

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