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On Nearly Quasi Einstein Manifold

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Abstract

The object of the present paper is to study a type of nearly quasi Einstein manifold.

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1 Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $(n > 2)$ is said to be an Einstein manifold if the condition

$$S(X,Y) = \frac{r}{n} g(X,Y) \tag{1}$$

holds on M^n, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to ([1], p. 432), condition (1) is called the

Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry, as well as in General Theory of Relativity. Also, Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([1], p.432-433) for instance every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{2}$$

where a, b are constants and A is a non-zero 1-from such that

$$g(X, U) = A(X) \tag{3}$$

for all vector fields X.

A non-flat Riemannian manifold (M^n, g), $(n > 2)$ is defined to be a quasi Einstein manifold [3] if its Ricci tensor S of type (0,2) satisfies the condition (2). We shall call A the associated 1-form and U as the generator of the manifold.

It is to be noted that M.C. Chaki and R.K. Maity[2] also initiated the study of quasi Einstein manifolds by considering a and b as scalars and the generator U of the manifold as a unit vector field.

In [5], U.C De and A.K. Gazi generalized the notion of quasi Einstein manifold by introducing the notion of nearly quasi Einstein manifold. According to them, nearly quasi Einstein manifold is a non-flat Riemannian manifold (M^n, g) $(n > 2)$, whose Ricci satisfies the condition

$$S(X, Y) = ag(X, Y) + bE(X, Y), \tag{4}$$

where a and b are non-zero scalars and E is a non-zero (0,2) tensor. Such a manifold is denoted by $N(QE)_n$.

It is known ([4], p-39) that the outer product of two covariant tensors of type (0,1) is a covariant tensor of type (0,2) but the converse is not true in general. Hence, the manifold which are quasi Einstein are also nearly quasi Einstein, but not conversely. This justifies the name nearly quasi Einstein manifolds given to this type of manifold.

2 A type of nearly quasi Einstein manifold

Throughout the paper, we consider a type of Nearly quasi Einstein manifold, whose associated tensor E of type (0,2) is given by

$$E(X, Y) = A(X)B(Y) + B(X)A(Y), \tag{5}$$

where A and B are non-zero 1-forms associated with orthogonal unit vector fields V and U, i.e.,

$$g(X,U) = A(X), \quad g(X,V) = B(X) \text{ and } g(U,V) = 0. \tag{6}$$

Thus, the equation (4), assumes the form

$$S(X,Y) = ag(X,Y) + b\big(A(X)B(Y) + B(X)A(Y)\big). \tag{7}$$

The above equation gives

$$S(X,U) = aA(X) + bB(X), \tag{8}$$

$$S(X,V) = aB(X) + bA(X), \tag{9}$$

$$S(U,V) = S(V,U) = a \tag{10}$$

and

$$r = na. \tag{11}$$

Let a Riemannian manifold (M^n, g) satisfies the following condition

$$\begin{aligned}[S(Y,Z)g(X,W) + S(X,W)g(Y,Z)] = \rho[&S(X,Z)g(Y,W) \\ &+ S(Y,W)g(X,Z)],\end{aligned} \tag{12}$$

where ρ is some scalar. Now putting $X = W = U$ in it, we get

$$S(Y,Z) = -\alpha g(Y,Z) + \rho[A(QZ)A(Y) + A(Z)A(QY)],$$

where, we have put $\alpha = S(U,U)$, Q is a Ricci tensor of type (1,1) given by

$$S(X,Y) = g(QX,Y) \tag{13}$$

and U is a unit vector associated with a 1-form, i.e., $A(X) = g(X,U)$. If, we put $B(X) = A(QX)$, where B is a 1-form, we obtain

$$S(Y,Z) = ag(Y,Z) + \rho[A(Y)B(Z) + B(Y)A(Z)],$$

which shows that the manifold is a nearly quasi Einstein manifold.

Hence, we have the following theorem:

Theorem 2.1. *If the Ricci tensor of the Riemannian manifold satisfies the relation*

$$\begin{aligned}[S(Y,Z)g(X,W) + S(X,W)g(Y,Z)] = \rho[&S(X,Z)g(Y,W) \\ &+ S(Y,W)g(X,Z)],\end{aligned} \tag{14}$$

where ρ is some scalar.Then the manifold is a nearly quasi Einstein manifold, whose Ricci tensor satisfies (7).

3 $N(QE)n$, whose generators are parallel vector fields

In this section, we consider a nearly quasi Einstein manifold, whose Ricci tensor is given by (7). Also, let the generators U and V of the $N(QE)_n$ be parallel vector fields, i.e.,

$$\nabla_X U = 0 \quad \text{and} \quad \nabla_X V = 0. \tag{15}$$

Then, we have

$$(\nabla_X A)(Y) = (\nabla_X B)(Y) = 0 \tag{16}$$

and

$$R(X, Y, U) = R(X, Y, V) = 0. \tag{17}$$

The relation (17) yields

$$S(X, U) = S(X, V) = 0. \tag{18}$$

Comparing (18) with (8) and (9), we get

$$aA(X) + bB(X) = 0$$

and

$$aB(X) + bA(X) = 0.$$

The above two relations gives

$$a = b.$$

Thus, we have the following theorem:

Theorem 3.1. *If generators of a $N(QE)_n$ manifold satisfying (7) are parallel vector fields, then the associated scalars a and b are equal.*

In view of theorem (3.1), differentiating the equation (7) covariantly, we get

$$\begin{aligned}(\nabla_X S)(Y, Z) = (\nabla_X a)[g(Y, Z) + A(X)B(Y) + B(X)A(Y)] \\ + a[(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z) \\ + (\nabla_X A)(Z)B(Y) + (\nabla_X B)(Y)A(Z)].\end{aligned} \tag{19}$$

If the generators U and V are parallel vector fields, then using (16) in above, we get

$$(\nabla_X S)(Y, Z) = D(X)S(Y, Z), \tag{20}$$

where, we have put $D(X) = \nabla_X a$.

Thus, we have the following theorem:

Theorem 3.2. *If generators of a $N(QE)_n$ manifold satisfying (7) are parallel vector fields then manifold is Ricci recurrent.*

Also, in view of the equation (20), we may write the following corollary:

Corollary 3.3. *If generators of a $N(QE)_n$ manifold satisfying (7) are parallel vector fields then the manifold is Ricci symmetric if and only if associated scalars are constant.*

4 $N(QE)n$ Manifold with constant associated scalars

In this section, we consider a $N(QE)_n$ manifold , whose Ricci tensor is given by (7) and associated scalars are constants. Further, if we take generators to be Killing vector fields, then

$$(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0 \tag{21}$$

and

$$(\nabla_X B)(Y) + (\nabla_Y B)(X) = 0. \tag{22}$$

Also, covariant differentation of (7), gives

$$\begin{aligned}(\nabla_X S)(Y,Z) = b[&(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z)\\ &+ (\nabla_X A)(Z)B(Y) + (\nabla_X B)(Y)A(Z)].\end{aligned} \tag{23}$$

From above equation, we have

$$\begin{aligned}\sigma_{(X,Y,Z)}(\nabla_X S)(Y,Z) = b\big[&\{(\nabla_X A)(Y) + (\nabla_Y A)(X)\}B(Z)\\ &+ \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}B(Y)\\ &+ \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}B(X)\\ &+ \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}A(Y)\\ &+ \{(\nabla_X B)(Y) + (\nabla_Y B)(X)\}A(Z)\\ &+ \{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}A(X)],\end{aligned}$$

where $\sigma_{(X,Y,Z)}$ denotes the cyclic sum with respect to X, Y and Z. Using (21) and (22) in above equation, we get

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_X S)(X,Y) = 0.$$

Thus, we may write the following theorem

Theorem 4.1. *If generators of an $N(QE)_n$ manifold satisfying (7) and with constant associated scalars are Killing vector fields, then it satisfies cyclic Ricci tensor.*

Next, if we suppose that $N(QE)_n$ manifold with constant associated scalars is Ricci symmetric, i.e.,

$$(\nabla_X S)(Y, Z) = 0, \tag{24}$$

then (23) yields

$$\begin{aligned} 0 =& b[(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z) \\ &+ (\nabla_X A)(Z)B(Y) + (\nabla_X B)(Y)A(Z)]. \end{aligned}$$

Since $b \neq 0$, therefore putting $Z = U$, in above, we get

$$A(Y)(\nabla_X B)(U) + (\nabla_X A)(U)B(Y) + (\nabla_X B)(U) = 0.$$

Again putting $Y = V$, in the above equation, we get.

$$(\nabla_X B)(U) = 0,$$

which proves that $\nabla_X U$ and V are perpendicular vector fields.
Again, since

$$g(U, V) = 0.$$

Therefore, we have

$$g(U, \nabla_X V) = 0.$$

Hence $\nabla_X V$ and U are mutually perpendicular vector fields.

Thus, we can state the following theorem:

Theorem 4.2. *In a Ricci symmetric $N(QE)_n$ manifold satisfying (7) and with constant associated scalars, $\nabla_X U$ and V as well as $\nabla_X V$ and U are mutually perpendicular vector fields.*

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