Int. Journal of Math. Analysis, Vol. 5, 2011, no. 34, 1683 - 1688

Positive Integral Operators with Analytic Kernels

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Abstract. In this work, as in [2] and [3], we construct examples of positive definite integral kernels which are also analytic using Fourier and Laplace Transforms

Mathematics Subject Classification: 45P05, 45H05

Keywords: Laplace Transform, Fourier Transform, Positive Integral Operators, Analytic Kernels

1 Introduction

Let $I, J \subset \mathbb{R}$ be intervals and suppose $k \in L^2(I \times J)$, i.e. $\int_I \int_J |k(s,u)|^2 \, duds < \infty$.

Then the formula

$$Sf(s) = \int_J k(s,u) f(u) du$$

where $s \in I, f \in L^2(J)$ defines a compact linear operator S mapping $L^2(J)$ into $L^2(I)$. The adjoint $S^* : L^2(I) \to L^2(J)$ is given by

$$S^* g(u) = \int_J g(t) \overline{k(t,u)} du.$$

Whenever $k \in L^2(I \times J), T = SS^*$ will be a positive integral operator on $L^2(I)$ with kernel

$$K(s,t) = \int_J k(s,u) \overline{k(t,u)} du.$$

This gives us a method of constructing examples of positive integral operators on $L^2(I)$.

We will use this theorem to give examples of positive definite kernels K using kernels k which arise in a natural way in mathematical analysis. (See [1])

2 Examples Suggested by the Fourier Transform

Here we will find positive definite kernels using Fourier transforms. We will give two examples of this method. Now we define the Fourier transform.

Definition 1 *Fourier transform of a function $f(u)$ is defined by*

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{isu} f(u) du. \tag{2.1}$$

Example 2 *Here (2.1) exists if $f \in L^1(\mathbb{R})$, because*

$$\left\| \hat{f} \right\|_\infty \leq \int_{\mathbb{R}} |f(u)| du = \|f\|_1 .$$

Suppose $\beta > 0$ and that ω is continuous function on $\mathbb{R}$ satisfying

$$\omega(u) = O(e^{-\beta |u|})$$

as $|u| \longrightarrow \infty$; then $\omega \in L^2(\mathbb{R})$. So if $f \in L^2(\mathbb{R}), f\omega \in L^1(\mathbb{R})$, so the Fourier transform of $f\omega$:

$$F(s) = \int_{\mathbb{R}} f(u)\omega(u)e^{isu} du$$

is continuous on $\mathbb{R}$; in fact F will be analytic on the strip $|\operatorname{Im} s| < \beta$.
Now let $I \subseteq \mathbb{R}$ be any bounded closed interval and define $S : L^2(\mathbb{R}) \longrightarrow L^2(I)$ by

$$Sf(s) = \int_{\mathbb{R}} f(u)\omega(u)e^{isu} du.$$

Here $k(s, u) = \omega(u)e^{isu}$. Now $k(s, u) \in L^2(I \times \mathbb{R})$, because

$$\begin{aligned} \int_I \int_{\mathbb{R}} |k(s,u)|^2 du &= \int_I \int_{\mathbb{R}} |\omega|^2 du ds \\ &= (b-a) \|\omega\|_2^2 < \infty. \end{aligned}$$

In this case

$$K(s,t) = \int_{\mathbb{R}} \omega(u)^2 e^{i(s-t)u} du \in L^2(I \times I).$$

For an explicit example we let $\omega = e^{-\alpha|u|/2}$. *Then*

$$\begin{aligned}
K(s,t) &= \int_{\mathbb{R}} e^{-\alpha|u|} e^{iu(s-t)} du \\
&= \int_0^\infty e^{-\alpha u} \left(e^{iu(s-t)} + e^{-iu(s-t)} \right) du \\
&= \int_0^\infty e^{-\alpha u} \left((\cos(s-t)u + i\sin(s-t)u) + (\cos(s-t)u - i\sin(s-t)u) \right) du \\
&= \int_0^\infty 2e^{-\alpha u} \cos(s-t)u du = 2\,\mathrm{Re} \left\{ \int_0^\infty e^{-(\alpha + i(s-t))u} du \right\} \qquad (\alpha > 0) \\
&= 2\,\mathrm{Re} \left\{ \frac{1}{-(\alpha+i(s-t))} e^{-(\alpha+i(s-t))u} \Big|_0^\infty \right\} = 2\,\mathrm{Re} \left\{ \frac{1}{\alpha + i(s-t)} \right\} \\
&= 2\,\mathrm{Re} \left\{ \frac{\alpha - i(s-t)}{\alpha^2 + (s-t)^2} \right\} = \frac{2\alpha}{\alpha^2 + (s-t)^2}.
\end{aligned}$$

Since $K(s,t)$ *is the kernel of* SS^* *it is positive definite on* $L^2(I)$.

For the next example we will let $\omega(u) = \dfrac{1}{(\cosh \lambda u)^{1/2}}$.

Example 3 *We let* $\omega(u) = \dfrac{1}{(\cosh \lambda u)^{1/2}}$ $(\lambda > 0)$ *so that* $K(s,t)$ *becomes*

$$K(s,t) = \int_{-\infty}^{\infty} \frac{e^{iu(s-t)}}{\cosh \lambda u} du.$$

To solve this we consider the following integral:

$$I(x) = \int_{-\infty}^{\infty} \frac{e^{ixu}}{\cosh u} du = I(-x) \qquad (x > 0)$$

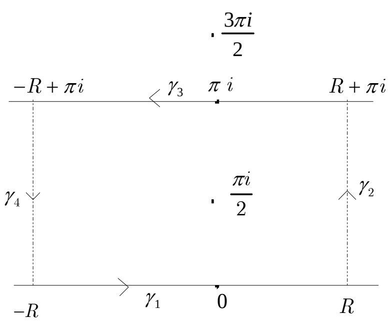

Figure 1: Integration Contour

$$\int_{\gamma_R} f(u)du = 2\pi i.Res(f(u), \pi i/2)$$

$$f(u) = \frac{e^{isu}}{\cosh u} = -e^{-\pi s} f(u+\pi i)$$

$$\left|\int_{\gamma_2/\gamma_4} f(u)du\right| \leq \frac{2\pi}{\cosh R} \longrightarrow 0 \quad as \quad R \to \infty.$$

$$(1+e^{-\pi s})I\,(s) = \frac{2\pi i e^{-\pi s/2}}{\sinh \pi i/2} = 2\pi e^{-\pi s/2}$$

$$\int_{\mathbb{R}} \frac{e^{isu}}{\cosh u} du = \frac{\pi}{\cosh \pi s/2}.$$

Then we get the following integral kernel

$$K(s,t) = \int_{\mathbb{R}} \frac{e^{iu(s-t)}}{\cosh u} du = \frac{\pi}{\cosh \pi(s-t)/2}.$$

More generally, for $\lambda > 0$ we have

$$\begin{aligned} \int_{\mathbb{R}} \frac{e^{iu(s-t)}}{\cosh \lambda u} du &= 1/\lambda \int_{\mathbb{R}} \frac{e^{iu(s-t)/\lambda}}{\cosh u} du \\ &= \frac{\pi}{\lambda \cosh \pi(s-t)/2\lambda}. \end{aligned}$$

Since K is the kernel of SS^*, K is positive definite on $L^2(I)$.

3 Examples Suggested by the Laplace Transform

Here we shall use Laplace transform to find some examples of positive definite kernels. As in the previous section we will give two examples of such positive definite kernels.
First, we define the Laplace transform which is similar to Fourier transform.

Definition 4 *For g belonging to $L^2((0,\infty))$ we write $G(s) = (Lg)(s)$ for the Laplace transform of g*

$$G(s) = (Lg)(s) = \int_0^\infty g(t)e^{-st}dt.$$

Example 5 *In this example we have* $0 < a < b$ *and* $I = [a, b], J = [0, \infty)$.
We define the operator $S : L^2([0, \infty)) \longrightarrow L^2(I)$ *by*

$$Sf(s) = \int_0^\infty f(u)e^{-su}du$$

so that

$$k(s, u) = e^{-su} \in L^2(J \times I).$$

This is because

$$\begin{aligned} \int_a^b \int_0^\infty e^{-2su} duds &\leq \int_a^b \left(\int_0^\infty e^{-2au} du \right) ds \\ &= \int_a^b \frac{1}{2a} ds = \frac{b-a}{2a} < \infty. \end{aligned}$$

Then

$$K(s, t) = \int_0^\infty e^{-u(s+t)} du = \frac{1}{s+t}.$$

since it is the kernel of SS^* *we know that it is positive definite on* $L^2(I)$. *Now we give our last example.*

Example 6 *Let* $I = [a, b]$ *where* $a > 0$ *and* $J = [0, \infty)$. *Now we consider the following integral*

$$\begin{aligned} \int_o^\infty u^\alpha e^{-su} du &= \frac{1}{s^{\alpha+1}} \int_0^\infty u^\alpha e^{-u} du \qquad \alpha > -1 \\ &= \frac{1}{s^{\alpha+1}} \Gamma(\alpha + 1) \qquad (\Gamma(\alpha + 1) = \alpha!). \end{aligned}$$

Similarly,

$$\begin{aligned} \int_o^\infty u^{\alpha-1} e^{-su} du &= \frac{1}{s^\alpha} \int_0^\infty u^{\alpha-1} e^{-u} du \qquad \alpha > 0 \\ &= \frac{1}{s^\alpha} \Gamma(\alpha). \end{aligned}$$

In this example we take

$$k(s, u) = u^{\frac{\alpha-1}{2}} e^{-us},$$

which is in $L^2(J \times I)$, because

$$\begin{aligned} \int_a^b \int_0^\infty u^{\alpha-1} e^{-2us} duds &\leq \int_a^b \int_0^\infty u^{\alpha-1} e^{-2au} duds \\ &= (b-a)C < \infty \end{aligned}$$

where C is a constant. *Then we have,*

$$\begin{aligned} K(s,t) &= \int_a^b \int_0^\infty u^{\alpha-1} e^{-u(s+t)} duds \\ &= \frac{\Gamma(\alpha)}{(s+t)^\alpha} \qquad \alpha > 0. \end{aligned}$$

Since $K(s,t)$ is the kernel of SS^*, it is positive definite on $L^2(I)$.

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Received: March, 2011