On the Zeros of a Family of
Self-Reciprocal Polynomials

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Abstract

The purpose of this paper is twofold. Firstly a sufficient condition is given to guarantee that all the zeros of self-reciprocal polynomials lie on the unit circle. Secondly it is shown that, for certain family of self-reciprocal polynomials with all its zeros on the unit circle, the convex combination also has all its zeros on the unit circle.

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1 Introduction

An \( n \)th-order real polynomial \( P(z) = \sum_{i=0}^{n} a_i z^i \) is called a self-reciprocal polynomial if \( P(z) = z^n P(z^{-1}) \) or equivalently \( a_i = a_{n-i} \) for \( 0 \leq i \leq n \). As is well known all the zeros of a self-reciprocal polynomial lie on the unit circle or symmetric to the unit circle.

This paper concerns the following two interesting questions.

(Q1) Under what conditions does a self-reciprocal polynomial has all its zeros on the unit circle?

(Q2) Given two \( n \)th-order self-reciprocal real polynomials \( P(z) \) and \( Q(z) \), consider a convex combination

\[
G(\lambda, z) = \lambda P(z) + (1 - \lambda) Q(z), \quad 0 \leq \lambda \leq 1.
\]

Suppose all the zeros of \( P(z) \) and \( Q(z) \) lie on the unit circle. Then does \( G(\lambda, z) \) inherits the same property for \( 0 \leq \lambda \leq 1 \)?
As an answer to (Q1), Lakatos [5] proved that if $P(z) = \sum_{i=0}^{n} a_i z^i$ is an \textit{n}th-order self-reciprocal polynomial such that
\[ |a_n| \geq \sum_{i=1}^{n-1} |a_n - a_i|, \]
then $P(z)$ has all its zeros on the unit circle. Lakatos and Losonczi [6] improved the result of [5] by showing that if $P(z) = \sum_{i=0}^{n} a_i z^i$ is an \textit{n}th-order self-reciprocal polynomial of odd degree, the conclusion in [5] remains valid even if
\[ |a_n| \geq \cos^2 \frac{1}{2(n+1)\pi} \sum_{i=1}^{n-1} |a_n - a_i|. \]
Lakatos and Losonczi [7] also proved that if $P(z) = \sum_{i=0}^{n} a_i z^i$ is an \textit{n}th-order self-reciprocal polynomial such that
\[ |a_n| \geq \frac{1}{2} \sum_{i=1}^{n-1} |a_i|, \]
then all the zeros of $P(z)$ lie on the unit circle.

A general answer to (Q2) was given by Fell [3]. Let $w_1, w_2, \ldots, w_n$ and $z_1, z_2, \ldots, z_n$ respectively denote the zeros of $P(z)$ and $Q(z)$ such that $0 < \arg(w_i) \leq \arg(w_{i+1}) < 2\pi$ and $0 < \arg(z_i) \leq \arg(z_{i+1}) < 2\pi$ for $1 \leq i \leq n - 1$. Then Fell’s lemma [3] states that all the zeros of $G(\lambda, z)$ also lie on the unit circle for $0 \leq \lambda \leq 1$ if and only if the open arcs bounded by $w_i$ and $z_i$ are disjoint. Recently Kim [4] studied the zero distribution of $G(\lambda, z)$ for the following families of self-reciprocal polynomials
\[ P(z) = \frac{(z^\alpha - 1)(z^\beta - 1)}{(z - 1)^2}, \]
\[ Q(z) = \frac{(z^\gamma - 1)(z^\eta - 1)}{(z - 1)^2}, \]
where $\alpha, \beta, \gamma, \eta \in \mathbb{N}$ and $\alpha + \beta = \gamma + \eta$. For the case where
\[ (\alpha, \beta; \gamma, \eta) = (n, n + 3; n + 1, n + 2), \]
it was verified through a straightforward but very lengthy procedure that the zeros of $P(z)$ and $Q(z)$ are distributed according to a particular pattern. Then Fell’s lemma [3] was applied to conclude that all the zeros of $G(\lambda, z)$ lie on the unit circle. It was conjectured in [4] that the same conclusion may hold for general cases where $\alpha + \beta = \gamma + \eta$. But the proof was not given except for the above-mentioned case.

The purpose of this paper is twofold. Firstly we give a new sufficient condition to guarantee that a self-reciprocal polynomial $P(z)$ has all its zeros
on the unit circle. Secondly it is shown that the conjecture in [4] is true. To this end more general convex combination is considered. The proof given here does not rely on Fell’s lemma [3], and is much simpler than that of [4]. It is noted that Fell’s lemma [3] (hence the proof of [4]) cannot be applied to general convex combinations of more than 2 polynomials.

2 Lemmas

We first state some preliminary results which will lead to the main results of this paper. Lemma 1 below is due to Cohn [2].

Lemma 1. A self-reciprocal polynomial has all its zeros on the unit circle if and only if all the zeros of its derivative lie in the disk $|z| \leq 1$.

The following result is a consequence of [Lemma 2, 1].

Lemma 2. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be an $n$th-order polynomial such that $|a_n| \geq \sum_{i=0}^{n-1} |a_i|$. Then all the zeros of $P(z)$ lie in the disk $|z| \leq 1$.

Lemma 3. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be an $n$th-order polynomial such that for some $k$,

$$a_n \geq a_{n-1} \geq \cdots \geq a_k > 0,$$

$$a_k \geq |a_k - a_{k-1}| + |a_{k-1} - a_{k-2}| + \cdots + |a_1 - a_0| + |a_0|.$$  

The all the zeros of $P(z)$ lie in the disk $|z| \leq 1$.

Proof: Consider a polynomial

$$\Phi(z) = (1 - z)P(z)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0.$$  

Then it is easily seen that the hypothesis of Lemma 2 holds for $\Phi(z)$. Consequently all the zeros of $\Phi(z)$ lie in the disk $|z| \leq 1$ and the result follows.

3 Main results

Theorem 1. For an $n$th-order self-reciprocal polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$, assume that for some $0 \leq k \leq n - 1$,

$$na_n \geq (n - 1)a_{n-1} \geq \cdots \geq (k + 1)a_{k+1} > 0,$$

$$(k + 1)a_{k+1} \geq \sum_{i=0}^{k} |(i + 1)a_{i+1} - ia_i|.$$
Then \( P(z) \) has all its zeros on the unit circle.

**Proof:** Consider the derivative of \( P(z) \)

\[
P'(z) = \sum_{i=0}^{n-1} (i+1)a_{i+1}z^i.
\]

Then the hypotheses of Lemma 3 are satisfied for the derivative for \( P'(z) \) and so \( P'(z) \) has all its zeros in the disk \(|z| \leq 1\). Then the proof is completed by Lemma 1.

If we take \( k = 0 \) in Theorem 1, then we have Corollary 1 below.

**Corollary 1.** If \( P(z) = \sum_{i=0}^{n} a_i z^i \) is an \( n \)-th-order self-reciprocal polynomial such that

\[
na_n \geq (n-1)a_{n-1} \geq \cdots \geq 2a_2 \geq a_1 > 0,
\]

then \( P(z) \) has all its zeros on the unit circle.

If we take \( k = n - 1 \) in Theorem 1, then we have the following result.

**Corollary 2.** If \( P(z) = \sum_{i=0}^{n} a_i z^i \) is an \( n \)-th-order self-reciprocal polynomial such that

\[
na_n \geq \sum_{i=0}^{n-1} \left| (i+1)a_{i+1} - ia_i \right|,
\]

then \( P(z) \) has all its zeros on the unit circle.

**Example 1.** Consider a seventh-order self-reciprocal polynomial given by

\[
P(z) = 5z^7 + 3z^6 + 3z^5 + 2z^4 + 2z^3 + 3z^2 + 3z + 5.
\]

The results of [5], [6], [7] do not apply to this example. However the hypothesis of Corollary 1 is satisfied and all the zero of \( P(z) \) lie on the unit circle.

**Example 2.** Consider a seventh-order self-reciprocal polynomial given by

\[
P(z) = 10z^7 + 5.9z^6 + 2.9z^5 + 2z^4 + 2z^3 + 2z^2 + 5.9 + 10.
\]

It is easily seen that the hypotheses of [5], [6], [7] and Corollary 1 are not satisfied. On the other hand the hypothesis of Corollary 2 holds and all the zero of \( P(z) \) lie on the unit circle.

**Theorem 2.** Let

\[
P_i(z) = \frac{(z^{\alpha_i} - 1)(z^{\beta_i} - 1)}{(z - 1)^2}, \quad \alpha_i, \beta_i \in \mathbb{N}, \ 1 \leq i \leq m,
\]
and consider a convex combination

\[ G(\lambda_1, \lambda_2, \cdots, \lambda_m, z) = \sum_{i=1}^{m} \lambda_i P_i(z), \]

where \( 0 \leq \lambda_i \leq 1, \ 1 \leq i \leq m \) and \( \sum_{i=1}^{m} \lambda_i = 1 \). If \( \alpha_i + \beta_i = n \) for \( 1 \leq i \leq m \),
then \( G(\lambda_1, \lambda_2, \cdots, \lambda_m, z) \) has all its zeros on the unit circle.

Proof: It suffices to consider the convex combination

\[ \hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z) = \sum_{i=1}^{m} \lambda_i \hat{P}_i(z), \]

where

\[ \hat{P}_i(z) = (z^{\alpha_i} - 1)(z^{\beta_i} - 1). \]

Since each \( \hat{P}_i(z) \) is self-reciprocal, \( \hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z) \) is also self-reciprocal.

The derivative of \( \hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z) \) is computed by

\[
\frac{d}{dz} \hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z) = \sum_{i=1}^{m} \lambda_i \frac{d}{dz} \hat{P}_i(z) \\
= \sum_{i=1}^{m} \lambda_i (nz^{n-1} - \alpha_i z^{\alpha_i-1} - \beta_i z^{\beta_i-1}) \\
= nz^{n-1} - \sum_{i=1}^{m} \lambda_i (\alpha_i z^{\alpha_i-1} + \beta_i z^{\beta_i-1}).
\]

Since \( \sum_{i=1}^{m} \lambda_i (\alpha_i + \beta_i) = n \), \( \hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z) \) has all its zeros on or inside
the unit circle by Lemma 2. Then the proof is completed by Lemma 1.

References

[1] A. Aziz and Q.G. Mohammad, Zero-free regions for polynomials and

[2] A. Cohn, Über die Anzahl der Wurzeln einer algebraischen Gleichung


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