Int. Journal of Math. Analysis, Vol. 5, 2011, no. 36, 1761 - 1766

On the Zeros of a Family of Self-Reciprocal Polynomials

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Abstract

The purpose of this paper is twofold. Firstly a sufficient condition is given to guarantee that all the zeros of self-reciprocal polynomials lie on the unit circle. Secondly it is shown that, for certain family of self-reciprocal polynomials with all its zeros on the unit circle, the convex combination also has all its zeros on the unit circle.

Mathematics Subject Classification: 30C10, 30C15

Keywords: Reciprocal polynomials, Zeros, Convex combination

1 Introduction

An nth-order real polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$ is called a self-reciprocal polynomial if $P(z) = z^n P(z^{-1})$ or equivalently $a_i = a_{n-i}$ for $0 \leq i \leq n$. As is well known all the zeros of a self-reciprocal polynomial lie on the unit circle or symmetric to the unit circle.

This paper concerns the following two interesting questions.

(Q1) Under what conditions does a self-reciprocal polynomial has all its zeros on the unit circle?
(Q2) Given two nth-order self-reciprocal real polynomials $P(z)$ and $Q(z)$, consider a convex combination

$$G(\lambda, z) = \lambda P(z) + (1 - \lambda(Q(z), \quad 0 \leq \lambda \leq 1.$$

Suppose all the zeros of $P(z)$ and $Q(z)$ lie on the unit circle. Then does $G(\lambda, z)$ inherits the same property for $0 \leq \lambda \leq 1$?

As an answer to (Q1), Lakatos [5] proved that if $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth-order self-reciprocal polynomial such that

$$|a_n| \geq \sum_{i=1}^{n-1} |a_n - a_i|,$$

then $P(z)$ has all its zeros on the unit circle. Lakatos and Losonczi [6] improved the result of [5] by showing that if $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth-order self-reciprocal polynomial of odd degree, the conclusion in [5] remains valid even if

$$|a_n| \geq \cos^2 \frac{1}{2(n+1)\pi} \sum_{i=1}^{n-1} |a_n - a_i|.$$

Lakatos and Losonczi [7] also proved that if $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth-order self-reciprocal polynomial such that

$$|a_n| \geq \frac{1}{2} \sum_{i=1}^{n-1} |a_i|,$$

then all the zeros of $P(z)$ lie on the unit circle.

A general answer to (Q2) was given by Fell [3]. Let $w_1, w_2, \cdots, w_n$ and $z_1, z_2, \cdots, z_n$ respectively denote the zeros of $P(z)$ and $Q(z)$ such that $0 < \arg(w_i) \leq \arg(w_{i+1}) < 2\pi$ and $0 < \arg(z_i) \leq \arg(z_{i+1}) < 2\pi$ for $1 \leq i \leq n-1$. Then Fell's lemma [3] states that all the zeros of $G(\lambda, z)$ also lie on the unit circle for $0 \leq \lambda \leq 1$ if and only if the open arcs bounded by w_i and z_i are disjoint. Recently Kim [4] studied the zero distribution of $G(\lambda, z)$ for the following families of self-reciprocal polynomials

$$P(z) = \frac{(z^\alpha - 1)(z^\beta - 1)}{(z-1)^2},$$

$$Q(z) = \frac{(z^\gamma - 1)(z^\eta - 1)}{(z-1)^2},$$

where $\alpha, \beta, \gamma, \eta \in \mathrm{N}$ and $\alpha + \beta = \gamma + \eta$. For the case where

$$(\alpha, \beta; \gamma, \eta) = (n, n+3; n+1, n+2),$$

it was verified through a straightforward but very lengthy procedure that the zeros of $P(z)$ and $Q(z)$ are distributed according to a particular pattern. Then Fell's lemma [3] was applied to conclude that all the zeros of $G(\lambda, z)$ lie on the unit circle. It was conjectured in [4] that the same conclusion may hold for general cases where $\alpha + \beta = \gamma + \eta$. But the proof was not given except for the above-mentioned case.

The purpose of this paper is twofold. Firstly we give a new sufficient condition to guarantee that a self-reciprocal polynomial $P(z)$ has all its zeros on the unit circle. Secondly it is shown that the conjecture in [4] is true. To this end more general convex combination is considered. The proof given here does not rely on Fell's lemma [3], and is much simpler than that of [4]. It is noted that Fell's lemma [3] (hence the proof of [4]) cannot be applied to general convex combinations of more than 2 polynomials.

2 Lemmas

We first state some preliminary results which will lead to the main results of this paper. Lemma 1 below is due to Cohn [2].

lemma 1. *A self-reciprocal polynomial has all its zeros on the unit circle if and only if all the zeros of its derivative lie in the disk* $|z| \leq 1$.

The following result is a consequence of [Lemma 2, 1].

Lemma 2. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be an nth-order polynomial such that* $|a_n| \geq \sum_{i=0}^{n-1} |a_i|$. *Then all the zeros of* $P(z)$ *lie in the disk* $|z| \leq 1$.

Lemma 3. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be an nth-order polynomial such that for some* k,

$$a_n \geq a_{n-1} \geq \cdots \geq a_k > 0,$$

$$a_k \geq |a_k - a_{k-1}| + |a_{k-1} - a_{k-2}| + \cdots + |a_1 - a_0| + |a_0|.$$

The all the zeros of $P(z)$ *lie in the disk* $|z| \leq 1$.

Proof: Consider a polynomial

$$\begin{aligned} \Phi(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0. \end{aligned}$$

Then it is easily seen that the hypothesis of Lemma 2 holds for $\Phi(z)$. Consequently all the zeros of $\Phi(z)$ lie in the disk $|z| \leq 1$ and the result follows.

3 Main results

Theorem 1. *For an nth-order self-reciprocal polynomial* $P(z) = \sum_{i=0}^{n} a_i z^i$, *assume that for some* $0 \leq k \leq n-1$,

$$na_n \geq (n-1)a_{n-1} \geq \cdots \geq (k+1)a_{k+1} > 0,$$

$$(k+1)a_{k+1} \geq \sum_{i=0}^{k} |(i+1)a_{i+1} - ia_i|.$$

Then $P(z)$ has all its zeros on the unit circle.
Proof: Consider the derivative of $P(z)$

$$P'(z) = \sum_{i=0}^{n-1}(i+1)a_{i+1}z^i.$$

Then the hypotheses of Lemma 3 are satisfied for the derivative for $P'(z)$ and so $P'(z)$ has all its zeros in the disk $|z| \le 1$. Then the proof is completed by Lemma 1.

If we take $k = 0$ in Theorem 1, then we have Corollary 1 below.

Corollary 1. *If $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth-order self-reciprocal polynomial such that*

$$na_n \ge (n-1)a_{n-1} \ge \cdots \ge 2a_2 \ge a_1 > 0,$$

then $P(z)$ has all its zeros on the unit circle.

If we take $k = n - 1$ in Theorem 1, then we have the following result.

Corollary 2. *If $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth-order self-reciprocal polynomial such that*

$$na_n \ge \sum_{i=0}^{n-1} |(i+1)a_{i+1} - ia_i|,$$

then $P(z)$ has all its zeros on the unit circle.

Example 1. Consider a seventh-order self-reciprocal polynomial given by

$$P(z) = 5z^7 + 3z^6 + 3z^5 + 2z^4 + 2z^3 + 3z^2 + 3z + 5.$$

The results of [5], [6], [7] do not apply to this example. However the hypothesis of Corollary 1 is satisfied and all the zero of $P(z)$ lie on the unit circle.

Example 2. Consider a seventh-order self-reciprocal polynomial given by

$$P(z) = 10z^7 + 5.9z^6 + 2.9z^5 + 2z^4 + 2z^3 + 2.9z^2 + 5.9z + 10.$$

It is easily seen that the hypotheses of [5], [6], [7] and Corollary 1 are not satisfied. On the other hand the hypothesis of Corollary 2 holds and all the zero of $P(z)$ lie on the unit circle.

Theorem 2. *Let*

$$P_i(z) = \frac{(z^{\alpha_i} - 1)(z^{\beta_i} - 1)}{(z-1)^2}, \ \ \alpha_i, \beta_i \in \mathrm{N}, \ 1 \le i \le m,$$

and consider a convex combination

$$G(\lambda_1, \lambda_2, \cdots, \lambda_m, z) = \sum_{i=1}^{m} \lambda_i P_i(z),$$

where $0 \leq \lambda_i \leq 1$*,* $1 \leq i \leq m$ *and* $\sum_{i=1}^{m} \lambda_i = 1$*. If* $\alpha_i + \beta_i = n$ *for* $1 \leq i \leq m$ *, then* $G(\lambda_1, \lambda_2, \cdots, \lambda_m, z)$ *has all its zeros on the unit circle.*

Proof: It suffices to consider the convex combination

$$\hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z) = \sum_{i=1}^{m} \lambda_i \hat{P}_i(z),$$

where

$$\hat{P}_i(z) = (z^{\alpha_i} - 1)(z^{\beta_i} - 1).$$

Since each $\hat{P}_i(z)$ is self-reciprocal, $\hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z)$ is also self-reciprocal. The derivative of $\hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z)$ is computed by

$$\begin{aligned} \frac{d}{dz}\hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z) &= \sum_{i=1}^{m} \lambda_i \frac{d}{dz}\hat{P}_i(z) \\ &= \sum_{i=1}^{m} \lambda_i (nz^{n-1} - \alpha_i z^{\alpha_i - 1} - \beta_i z^{\beta_i - 1}) \\ &= nz^{n-1} - \sum_{i=1}^{m} \lambda_i (\alpha_i z^{\alpha_i - 1} + \beta_i z^{\beta_i - 1}). \end{aligned}$$

Since $\sum_{i=1}^{m} \lambda_i(\alpha_i + \beta_i) = n$, $\hat{G}(\lambda_1, \lambda_2, \cdots, \lambda_m, z)$ has all its zeros on or inside the unit circle by Lemma 2. Then the proof is completed by Lemma 1.

References

[1] A. Aziz and Q.G. Mohammad, *Zero-free regions for polynomials and some generalizations of Eneström-Kakeya theorem*, Canad. Math. Bull., **27** (1984), 265–272.

[2] A. Cohn, *Über die Anzahl der Wurzeln diner algebraischen Gleichung in einem Kreise*, Math. Z., **14** (1922), 110–148.

[3] H.J. Fell, *On the zeros of convex combinations of polynomials*, Pacific J. Math., **89** (1980), 43–50.

[4] S.H. Kim, *The zeros of certain family of self-reciprocal polynomials*, Bull. Korean Math. Soc., **44** (2007), 461–473.

[5] P. Lakatos, *On zeros of reciprocal polynomials,* Publ. Math. Debrecen, **61** (2002), 645–661.

[6] P. lakatos and L. Losonczi, *On zeros of reciprocal polynomials of odd degree,* J. Inequal. Pure Appl. Math., **4** (3) Art. 60 (2003), 8 pp.

[7] P. lakatos and L. Losonczi, *Self-inversive polynomials whose zeros are on the unit circle,* Publ. Math. Debrecen, **65** (2004), 409–420.

[8] M. Marden, *Geometry of polynomials*, Math. Surveys, No. 3, Amer. Math. Society, Providence, R.I., 1966.

Received: February, 2011