Int. Journal of Math. Analysis, Vol. 5, 2011, no. 35, 1741 - 1760

Some Results on the Zeros of Polynomials and Related Analytic Functions

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Abstract

Many results on the location of zeros of polynomials and analytic functions are available in the literature. In this paper we extend some existing results on the zeros of polynomials by considering more general coefficient conditions. As special cases the extended results yield much simpler expressions for the upper bounds of zeros than those of the existing results. The zero-free regions of analytic functions subject to similar coefficient conditions are also investigated.

Mathematics Subject Classification: 30C10, 30C15

Keywords: Polynomials, Zeros, Eneström-Kakeya theorem, Analytic functions

1 Introduction

Many results on the location of zeros of polynomials are available in the literature. Among them the Eneström-Kakeya theorem [16] given below is well known in the theory of zero distribution of polynomials.

Theorem A. *For an nth-order polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$, assume*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0. \tag{1}$$

Then $P(z)$ has all its zeros in the disk $|z| \leq 1$.

In the literature [1–15], [17], [18], diverse attempts have been made for generalizing the Eneström-Kakeya theorem to polynomials and analytic functions. Among others, Gardner and Govil [11], [12] generalized the Eneström-Kakeya theorem and proved the following two theorems.

Theorem B. *Consider an nth-order complex polynomial* $P(z) = \sum_{i=0}^{n} a_i z^i$ *with* $\mathrm{Re}\{a_i\} = \alpha_i$ *and* $\mathrm{Im}\{a_i\} = \beta_i$, $i = 0, 1, 2, \cdots, n$, *and assume that for some* k *and* r, *and for some* $t > 0$,

$$t^n\alpha_n \le t^{n-1}\alpha_{n-1} \le \cdots \le t^{k+1}\alpha_{k+1} \le t^k\alpha_k \ge \cdots \ge t\alpha_1 \ge \alpha_0,$$

$$t^n\beta_n \le t^{n-1}\beta_{n-1} \le \cdots \le t^{r+1}\beta_{r+1} \le t^r\beta_r \ge \cdots \ge t\beta_1 \ge \beta_0.$$

Then $P(z)$ *has all its zeros in* $R_1 \le |z| \le R_2$ *where*

$$R_1 = \min\left\{\frac{t|a_0|}{t^n|a_n|^n + 2(t^k\alpha_k + t^r\beta_r) - t^n(\alpha_n + \beta_n) - (\alpha_0 + \beta_0)}, t\right\},$$

and

$$\begin{aligned} R_2 \;=\; & \max\left\{\frac{1}{|a_n|}\left[t^{n+1}|a_0| - t^{n-1}(\alpha_0 + \beta_0) - t(\alpha_n + \beta_n)\right.\right. \\ & +(t^2+1)(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r) + (t^2-1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j\right. \\ & \left.\left.\left. + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right) + (1-t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right)\right], \frac{1}{t}\right\}. \end{aligned}$$

Theorem C. *Consider an nth-order complex polynomial* $P(z) = \sum_{i=0}^{n} a_i z^i$ *with* $\mathrm{Re}\{a_i\} = \alpha_i$ *and* $\mathrm{Im}\{a_i\} = \beta_i$, $i = 0, 1, 2, \cdots, n$. *If for some* k *and for some* $t > 0$,

$$t^n\alpha_n \le t^{n-1}\alpha_{n-1} \le \cdots \le t^{k+1}\alpha_{k+1} \le t^k\alpha_k \ge \cdots \ge t\alpha_1 \ge \alpha_0,$$

then $P(z)$ *has all its zeros in* $R_1 \le |z| \le R_2$

$$R_1 = \frac{t|a_0|}{2t^k\alpha_k - t^n\alpha_n - \alpha_0 + |\beta_0| + t^n|\beta_n| + 2\sum_{j=1}^{n-1} t^j|\beta_j|},$$

and

$$\begin{aligned} R_2 \;=\; & \max\left\{\frac{1}{|a_n|}\left[t^{n+1}|a_0| - t^{n-1}\alpha_0 - t\alpha_n + (t^2+1)t^{n-k-1}\alpha_k\right.\right. \\ & +(t^2-1)\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + (1-t^2)\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j \\ & \left.\left. + \sum_{j=1}^{n} t^{n-j}(|\beta_{j-1}| + t|\beta_j|)\right], \frac{1}{t}\right\}. \end{aligned}$$

Recently Cao and Gardner [6] investigated the location of zeros of polynomials subject to some coefficient conditions on the even and odd power of the variable, and proved the following two theorems.

Theorem D. *Consider an nth-order polynomial* $P(z)=\sum_{i=0}^{n}a_iz^i$ *with* $\mathrm{Re}\{a_i\}=\alpha_i$ *and* $\mathrm{Im}\{a_i\}=\beta_i$, $i=0,1,2,\cdots,n$, *such that for some* $t>0$ *and some nonnegative integers* k *and* s, *and positive integers* l *and* q,

$$t^{2[n/2]}\alpha_{2[n/2]}\le\cdots\le t^{2k+2}\alpha_{2k+2}\le t^{2k}\alpha_{2k}\ge\cdots\ge t^2\alpha_2\ge\alpha_0,$$
$$t^{2[n/2]}\alpha_{2[n/2]-1}\le\cdots\le t^{2l}\alpha_{2l+1}\le t^{2l-2}\alpha_{2l-1}\ge\cdots\ge t^2\alpha_3\ge\alpha_1,$$
$$t^{2[n/2]}\beta_{2[n/2]}\le\cdots\le t^{2s+2}\beta_{2s+2}\le t^{2s}\beta_{2s}\ge\cdots\ge t^2\beta_2\ge\beta_0,$$
$$t^{2[n/2]}\beta_{2[n/2]-1}\le\cdots\le t^{2q}\beta_{2q+1}\le t^{2q-2}\beta_{2q-1}\ge\cdots\ge t^2\beta_3\ge\beta_1.$$

Then $P(z)$ *has all its zeros in the disk* $R_1\le|z|\le R_2$, *where*

$$R_1=\min\left\{\frac{t|a_0|}{M_1},t\right\}\quad\text{and}\quad R_2=\max\left\{\frac{M_2}{|a_n|},\frac{1}{t}\right\},$$

and

$$\begin{aligned}M_1 &= t^n(|\alpha_n|+|\beta_n|)+t^{n-1}(|\alpha_{n-1}|+|\beta_{n-1}|)\\&\quad+2(t^{2k}\alpha_{2k}+t^{2l-1}\alpha_{2l-1}+t^{2s}\beta_{2s}+t^{2q-1}\beta_{2q-1})\\&\quad+t(|\alpha_1|+|\beta_1|)-t^n(\alpha_n+\beta_n)\\&\quad-t^{n-1}(\alpha_{n-1}+\beta_{n-1})-t(\alpha_1+\beta_1)-(\alpha_0+\beta_0),\end{aligned}$$

$$\begin{aligned}M_2 &= t^{n+3}(|a_0|-\alpha_0-\beta_0)+t^{n+2}(|a_1|-\alpha_1-\beta_1)\\&\quad+(t^4+1)(t^{n-1-2k}\alpha_{2k}+t^{n-2l}\alpha_{2l-1}\\&\quad+t^{n-1-2s}\beta_{2s}+t^{n-2q}\beta_{2q-1})\\&\quad-(\alpha_{n-1}+\beta_{n-1})+|a_{n-1}|-t^{-1}(\alpha_n+\beta_n)\\&\quad+(t^4-1)\Bigg\{\sum_{j=0,j\text{even}}^{2k-2}t^{n-1-j}\alpha_j+\sum_{j=1,j\text{odd}}^{2l-3}t^{n-1-j}\alpha_j+\sum_{j=0,j\text{even}}^{2s-2}t^{n-1-j}\beta_j\\&\quad+\sum_{j=1,j\text{odd}}^{2q-3}t^{n-1-j}\beta_j-\sum_{j=2k+2,j\text{even}}^{2[n/2]}t^{n-1-j}\alpha_j-\sum_{j=2l+1,j\text{odd}}^{2[(n+1)/2]-1}t^{n-1-j}\alpha_j\\&\quad-\sum_{j=2s+2,j\text{even}}^{2[n/2]}t^{n-1-j}\beta_j-\sum_{j=2l+1,j\text{odd}}^{2[(n+1)/2]-1}t^{n-1-j}\beta_j\Bigg\}.\end{aligned}$$

Theorem E. *Let* $P(z)=\sum_{i=0}^{n}a_iz^i$ *be an nth-order polynomial such that* $|\arg a_i-\beta|\le\alpha\le\pi/2$, $i=0,1,2,\cdots,n$, *and for some* $t>0$ *and some nonnegative integer* k *and positive integer* l,

$$t^{2[n/2]}|a_{2[n/2]}|\le\cdots\le t^{2k+2}|a_{2k+2}|\le t^{2k}|a_{2k}|\ge\cdots\ge t^2|a_2|\ge|a_0|,$$

$$t^{2[n/2]}|a_{2[n/2]-1}| \leq \cdots \leq t^{2l}|a_{2l+1}| \leq t^{2l-2}|a_{2l-1}| \geq \cdots \geq t^2|a_3| \geq |a_1|.$$

Then $P(z)$ has all its zeros in the disk $R_1 \leq |z| \leq R_2$, where

$$R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\} \quad \textit{and} \quad R_2 = \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\},$$

and

$$\begin{aligned} M_1 &= t_n|a_n| + t_{n-1}|a_{n-1}| + t|a_1| \\ &\quad + \cos\alpha\Big\{2(t^{2k}|a_{2k}| + t^{2l-1}|a_{2l-1}|) - t^n|a_n| - t^{n-1}|a_{n-1}| - t|a_1| - |a_0|\Big\} \\ &\quad + \sin\alpha\Big\{t^n|a_n| + t^{n-1}|a_{n-1}| + t|a_1| + |a_0| + 2\sum_{j=2}^{n-2} t^j|a_j|\Big\}, \end{aligned}$$

$$\begin{aligned} M_2 &= t^{n+3}|a_0| + t^{n+2}|a_1| + |a_{n-1}| + \cos\alpha\Big\{(t^4-1)\Big(\sum_{j=0,j\text{even}}^{2k-2} t^{n-1-j}|a_j| \\ &\quad + \sum_{j=1,j\text{odd}}^{2l-3} t^{n-1-j}|a_j| - \sum_{j=2k+2,j\text{even}}^{2[n/2]} t^{n-1-j}|a_j| - \sum_{j=2l+1,j\text{odd}}^{2[(n+1)/2]-1} t^{n-1-j}|a_j|\Big) \\ &\quad +(t^4+1)(t^{n-1-2k}|a_{2k}| + t^{n-2l}|a_{2l-1}|) - t^{n+3}|a_0| - t^{n+}|a_1| - |a_{n-1}| \\ &\quad -t^{-1}|a_n|\Big\} + \sin\alpha\Big\{(t^4+1)\sum_{j=2}^{n-2} t^{n-1-j}|a_j| + t^{n-1}|a_0| + t^{n-2}|a_1| \\ &\quad +t^4|a_{n-1}| + t^3|a_n|\Big\}. \end{aligned}$$

The purpose of this paper is twofold. Firstly we extend Theorem B through Theorem E by considering more general coefficient conditions. Although the above mentioned results are nice, the expressions for the upper bounds are rather complicated in that they contain all the coefficients of polynomials. However much simpler expressions for the upper bounds in Theorem B through Theorem E can be obtained as special cases of the extended results. Secondly we investigate the zero-free regions of analytic functions with similar coefficient conditions.

2 Lemma

The following lemma will be used to prove Theorem 3.4 and Theorem 3.6 in the next section.

Lemma 2.1 *Consider two complex numbers a_0 and a_1. If $|a_0| \geq |a_1|$, and $|\arg a_i - \beta| \leq \alpha \leq \pi/2$, $i = 0, 1$, for some β,then*

$$|a_0 - a_1| \leq (|a_0| - |a_1|)\cos\alpha + (|a_0| + |a_1|)\sin\alpha.$$

3 Theorems and proofs

Theorem 3.1 *Consider an nth-order complex polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$ with $\mathrm{Re}\{a_i\} = \alpha_i$ and $\mathrm{Im}\{a_i\} = \beta_i$, $i = 0, 1, 2, \cdots, n$, and assume that for some k and r, and for some λ_1, λ_2 and $t > 0$,*

$$\lambda_1 t^n \alpha_n \leq t^{n-1}\alpha_{n-1} \leq \cdots \leq t^{k+1}\alpha_{k+1} \leq t^k \alpha_k \geq \cdots \geq t\alpha_1 \geq \alpha_0,$$

$$\lambda_2 t^n \beta_n \leq t^{n-1}\beta_{n-1} \leq \cdots \leq t^{r+1}\beta_{r+1} \leq t^r \beta_r \geq \cdots \geq t\beta_1 \geq \beta_0.$$

Then $P(z)$ has all its zeros in $R_1 \leq |z| \leq R_2$ where

$$R_1 = \frac{t|a_0|}{M_1} \quad \textit{and} \quad R_2 = \frac{M_2}{t^{n-1}|a_n|},$$

with

$$\begin{aligned} M_1 &= t^n|a_n| + t^n|(\lambda_1 - 1)\alpha_n| + t^n|(\lambda_2 - 1)\beta_n| + 2(t^k\alpha_k + t^r\beta_r) \\ &\quad - t^n(\lambda_1\alpha_n + \lambda_2\beta_n) - (\alpha_0 + \beta_0), \end{aligned}$$

and

$$\begin{aligned} M_2 &= t^n|(\lambda_1 - 1)\alpha_n| + t^n|(\lambda_2 - 1)\beta_n| + 2(t^k\alpha_k + t^r\beta_r) \\ &\quad - t^n(\lambda_1\alpha_n + \lambda_2\beta_n) - (\alpha_0 + \beta_0) + |a_0|. \end{aligned}$$

Proof. Firstly we consider the case where $t = 1$, i.e., assume that for some k and r, and for some λ_1, λ_2,

$$\lambda_1\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_{k+1} \leq \alpha_k \geq \cdots \geq \alpha_1 \geq \alpha_0,$$

$$\lambda_2\beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_{r+1} \leq \beta_r \geq \cdots \geq \beta_1 \geq \beta_0.$$

For the outer bound, consider a polynomial

$$\begin{aligned} \Phi(z) &= (1 - z)P(z) \\ &= -a_n z^{n+1} + a_0 + \sum_{j=0}^{n-1}(a_{n-j} - a_{n-j-1})z^{n-j} \\ &= -a_n z^{n+1} + a_0 - (\lambda_1 - 1)\alpha_n z^n - i(\lambda_2 - 1)\beta_n z^n + (\lambda_1\alpha_n - \alpha_{n-1})z^n \\ &\quad + \sum_{j=1}^{n-1}(\alpha_{n-j} - \alpha_{n-j-1})z^{n-j} \\ &\quad + i\left\{(\lambda_2\beta_n - \beta_{n-1})z^n + \sum_{j=1}^{n-1}(\beta_{n-j} - \beta_{n-j-1})z^{n-j}\right\}. \end{aligned}$$

If $|z| > 1$, then

$$\begin{aligned}
|\Phi(z)| &\geq |a_n||z|^{n+1} - |z|^n\Big\{|a_0||z|^{-n} + |(\lambda_1-1)\alpha_n| + |(\lambda_2-1)\beta_n| \\
&\quad +(\alpha_{n-1}-\lambda_1\alpha_n) + (\beta_{n-1}-\lambda_2\beta_n) + \sum_{j=1}^{n-k-1}\frac{(\alpha_{n-j-1}-\alpha_{n-j})}{|z|^j} \\
&\quad + \sum_{j=n-k}^{n-1}\frac{(\alpha_{n-j}-\alpha_{n-j-1})}{|z|^j} + \sum_{j=1}^{n-r-1}\frac{(\beta_{n-j-1}-\beta_{n-j})}{|z|^j} \\
&\quad + \sum_{j=n-r}^{n-1}\frac{(\beta_{n-j}-\beta_{n-j-1})}{|z|^j}\Big\} \\
&\geq |z|^n(|a_n||z| - M_{12}),
\end{aligned}$$

where

$$\begin{aligned}
M_{12} &= |(\lambda_1-1)\alpha_n| + |(\lambda_2-1)\beta_n| + 2(\alpha_k+\beta_r) - (\lambda_1\alpha_n+\lambda_2\beta_n) \\
&\quad -(\alpha_0+\beta_0) + |a_0|.
\end{aligned}$$

Then $\Phi(z)| > 0$ if $|z| > M_{12}/|a_n| = R_{12}$, and all the zeros of $P(z)$ with modulus greater than one lie in the disk $|z| \leq R_{12}$. It can be shown that $R_{12} \geq 1$. Consequently the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disk $|z| \leq R_{12}$.

For the inner bound, consider again the polynomial

$$\begin{aligned}
\Phi(z) &= (1-z)P(z) \\
&= a_0 + \phi(z),
\end{aligned}$$

where

$$\begin{aligned}
\phi(z) &= -a_n z^{n+1} + \sum_{j=0}^{n-1}(a_{n-j}-a_{n-j-1})z^{n-j} \\
&= -a_n z^{n+1} - (\lambda_1-1)\alpha_n z^n - i(\lambda_2-1)\beta_n z^n + (\lambda_1\alpha_n - \alpha_{n-1})z^n \\
&\quad + \sum_{j=1}^{n-1}(\alpha_{n-j}-\alpha_{n-j-1})z^{n-j} \\
&\quad + i\Big\{(\lambda_2\beta_n - \beta_{n-1})z^n + \sum_{j=1}^{n-1}(\beta_{n-j}-\beta_{n-j-1})z^{n-j}\Big\}.
\end{aligned}$$

If $|z| < 1$, then

$$|\phi(z)| \leq |a_n| + |(\lambda_1-1)\alpha_n| + |(\lambda_2-1)\beta_n| + (\alpha_{n-1}-\lambda_1\alpha_n)$$

$$+(\beta_{n-1}-\lambda_2\beta_n)+\sum_{j=1}^{n-k-1}(\alpha_{n-j-1}-\alpha_{n-j})+\sum_{j=n-k}^{n-1}(\alpha_{n-j}-\alpha_{n-j-1})$$
$$+\sum_{j=1}^{n-r-1}(\beta_{n-j-1}-\beta_{n-j})+\sum_{j=n-r}^{n-1}(\beta_{n-j}-\beta_{n-j-1})$$
$$=M_{11},$$

where

$$\begin{aligned} M_{11} &= |a_n|+|(\lambda_1-1)\alpha_n|+|(\lambda_2-1)\beta_n|+2(\alpha_k+\beta_r)-(\lambda_1\alpha_n+\lambda_2\beta_n) \\ &\quad -(\alpha_0+\beta_0). \end{aligned}$$

Since $\phi(0)=0$, it follows by Schwarz lemma that

$$|\phi(z)|\le M_{11}|z| \quad \text{for} \quad |z|<1.$$

Then for $|z|<1$,

$$\begin{aligned} \Phi(z) &\ge |a_0|-|\phi(z)| \\ &\ge |a_0|-M_{11}|z| \\ &> 0, \end{aligned}$$

if $|z|<|a_0|/M_{11}=R_{11}$. It can be shown that $R_{11}\le 1$. Hence if $t=1$, then all the zeros of $P(z)$ lie in the disk $R_{11}\le|z|\le R_{12}$.

The proof of the theorem is completed by applying the above results to $P(tz)$.

The proof of Theorem 3.2 below is similar to that of Theorem 3.1, and is omitted.

Theorem 3.2 *Consider an nth-order complex polynomial* $P(z)=\sum_{i=0}^{n}a_iz^i$ *with* $\mathrm{Re}\{a_i\}=\alpha_i$ *and* $\mathrm{Im}\{a_i\}=\beta_i$, $i=0,1,2,\cdots,n$, *and assume that for some* k, λ *and for some* $t>0$,

$$\lambda t^n\alpha_n\le t^{n-1}\alpha_{n-1}\le\cdots\le t^{k+1}\alpha_{k+1}\le t^k\alpha_k\ge\cdots\ge t\alpha_1\ge\alpha_0.$$

Then $P(z)$ *has all its zeros in* $R_1\le|z|\le R_2$ *where*

$$R_1=\frac{t|a_0|}{M_1} \quad \textit{and} \quad R_2=\frac{M_2}{t^{n-1}|a_n|},$$

with

$$M_1=t^n|a_n|+t^n|(\lambda-1)\alpha_n|+2t^k\alpha_k-\lambda t^n\alpha_n-\alpha_0+t^n|\beta_n|+|\beta_0|+2\sum_{j=1}^{n-1}t^j|\beta_j|,$$

and

$$M_2 = t^n|(\lambda-1)\alpha_n| + 2t^k\alpha_k - \lambda t^n\alpha_n - \alpha_0 + |a_0| + t^n|\beta_n| + |\beta_0| + 2\sum_{j=1}^{n-1} t^j|\beta_j|.$$

Theorem 3.3 *Consider an nth-order polynomial* $P(z) = \sum_{i=0}^{n} a_i z^i$ *with* $\mathrm{Re}\{a_i\} = \alpha_i$ *and* $\mathrm{Im}\{a_i\} = \beta_i$, $i = 0, 1, 2, \cdots, n$, *such that for some* $\lambda_1 \geq 1$, $\lambda_2 \geq 1$, $\lambda_3 \geq 1$, $\lambda_4 \geq 1$, $t > 0$ *and some nonnegative integers* k *and* s, *and positive integers* l *and* q,

$$\lambda_1 t^{2[n/2]}\alpha_{2[n/2]} \leq \cdots \leq t^{2k+2}\alpha_{2k+2} \leq t^{2k}\alpha_{2k} \geq \cdots \geq t^2\alpha_2 \geq \alpha_0,$$
$$\lambda_2 t^{2[n/2]}\alpha_{2[n/2]-1} \leq \cdots \leq t^{2l}\alpha_{2l+1} \leq t^{2l-2}\alpha_{2l-1} \geq \cdots \geq t^2\alpha_3 \geq \alpha_1,$$
$$\lambda_3 t^{2[n/2]}\beta_{2[n/2]} \leq \cdots \leq t^{2s+2}\beta_{2s+2} \leq t^{2s}\beta_{2s} \geq \cdots \geq t^2\beta_2 \geq \beta_0,$$
$$\lambda_4 t^{2[n/2]}\beta_{2[n/2]-1} \leq \cdots \leq t^{2q}\beta_{2q+1} \leq t^{2q-2}\beta_{2q-1} \geq \cdots \geq t^2\beta_3 \geq \beta_1.$$

If n *is even, then* $P(z)$ *has all its zeros in the disk* $R_{1e} \leq |z| \leq R_{2e}$, *where*

$$R_{1e} = \frac{t|a_0|}{M_{1e}} \quad \text{and} \quad R_{2e} = \frac{M_{2e}}{t^{n-1}|a_n|}.$$

Here

$$\begin{aligned} M_{1e} &= t^n|a_n| + t^{n-1}|a_{n-1}| + t|a_1| + t^n\Big\{|(\lambda_1-1)\alpha_n| + |(\lambda_3-1)\beta_n|\Big\} \\ &\quad + t^{n-1}\Big\{|(\lambda_2-1)\alpha_{n-1}| + |(\lambda_4-1)\beta_{n-1}|\Big\} + 2(t^{2k}\alpha_{2k} + t^{2l-1}\alpha_{2l-1} \\ &\quad + t^{2s}\beta_{2s} + t^{2q-1}\beta_{2q-1}) - t^n(\lambda_1\alpha_n + \lambda_3\beta_n) - t^{n-1}(\lambda_2\alpha_{n-1} + \lambda_4\beta_{n-1}) \\ &\quad - t(\alpha_1 + \beta_1) - (\alpha_0 + \beta_0), \end{aligned}$$

and

$$\begin{aligned} M_{2e} &= t^{n-1}|a_{n-1}| + t|a_1| + |a_0| + t^n\Big\{|(\lambda_1-1)\alpha_n| + |(\lambda_3-1)\beta_n|\Big\} \\ &\quad + t^{n-1}\Big\{|(\lambda_2-1)\alpha_{n-1}| + |(\lambda_4-1)\beta_{n-1}|\Big\} + 2(t^{2k}\alpha_{2k} + t^{2l-1}\alpha_{2l-1} \\ &\quad + t^{2s}\beta_{2s} + t^{2q-1}\beta_{2q-1}) - t^n(\lambda_1\alpha_n + \lambda_3\beta_n) - t^{n-1}(\lambda_2\alpha_{n-1} + \lambda_4\beta_{n-1}) \\ &\quad - t(\alpha_1 + \beta_1) - (\alpha_0 + \beta_0). \end{aligned}$$

If n *is odd, then* $P(z)$ *has all its zeros in the disk* $R_{1o} \leq |z| \leq R_{2o}$, *where*

$$R_{1o} = \frac{t|a_0|}{M_{1o}} \quad \text{and} \quad R_{2o} = \frac{M_{2o}}{t^{n-1}|a_n|}.$$

Here M_{1o} *and* M_{2o} *are respectively the same as* M_{1e} *and* M_{2e} *except that* λ_1, λ_2, λ_3 *and* λ_4 *are respectively replaced by* λ_2, λ_1, λ_4 *and* λ_3.

Proof: For the case where n is even, assume that the coefficient conditions hold for $t = 1$, i.e.,

$$\lambda_1 t^n \alpha_n \leq \cdots \leq t^{2k+2}\alpha_{2k+2} \leq t^{2k}\alpha_{2k} \geq \cdots \geq t^2\alpha_2 \geq \alpha_0,$$

$$\lambda_2 t^n \alpha_{n-1} \leq \cdots \leq t^{2l}\alpha_{2l+1} \leq t^{2l-2}\alpha_{2l-1} \geq \cdots \geq t^2\alpha_3 \geq \alpha_1,$$

$$\lambda_3 t^n \beta_n \leq \cdots \leq t^{2s+2}\beta_{2s+2} \leq t^{2s}\beta_{2s} \geq \cdots \geq t^2\beta_2 \geq \beta_0,$$

$$\lambda_4 t^n \beta_{n-1} \leq \cdots \leq t^{2q}\beta_{2q+1} \leq t^{2q-2}\beta_{2q-1} \geq \cdots \geq t^2\beta_3 \geq \beta_1.$$

For the outer bound, consider a polynomial

$$\begin{aligned}
\Phi(z) &= (1-z^2)P(z) \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + a_1 z + a_0 + \sum_{j=2}^{n-2}(a_{n-j} - a_{n-j-2})z^{n-j} \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + a_1 z + a_0 - (\lambda_1 - 1)\alpha_n z^n - \\
&\quad (\lambda_2 - 1)\alpha_{n-1}z^{n-1} + (\lambda_1\alpha_n - \alpha_{n-2})z^n + (\lambda_2\alpha_{n-1} - \alpha_{n-3})z^{n-1} \\
&\quad + \sum_{j=2}^{n-2}(\alpha_{n-j} - \alpha_{n-j-2})z^{n-j} + i\Big\{ -(\lambda_3 - 1)\beta_n z^n - (\lambda_4 - 1)\beta_{n-1}z^{n-1} \\
&\quad + (\lambda_3\beta_n - \beta_{n-2})z^n + (\lambda_4\beta_{n-1} - \beta_{n-3})z^{n-1} \\
&\quad + \sum_{j=2}^{n-2}(\beta_{n-j} - \beta_{n-j-2})z^{n-j}\Big\}.
\end{aligned}$$

If $|z| > 1$,

$$\begin{aligned}
|\Phi(z)| \geq\ & |a_n||z|^{n+2} - |z|^{n+1}\Big\{|a_{n-1}| + \frac{|a_1|}{|z|^n} + \frac{|a_0|}{|z|^{n+1}} + \frac{|(\lambda_1 - 1)\alpha_n|}{|z|} \\
& + \frac{|(\lambda_2 - 1)\alpha_{n-1}|}{|z|^2} + \frac{|(\lambda_3 - 1)\beta_n|}{|z|} + \frac{|(\lambda_4 - 1)\beta_{n-1}|}{|z|^2} \\
& + \frac{(\alpha_{n-2} - \lambda_1\alpha_n)}{|z|} + \sum_{j=2,jeven}^{n-2k-2}\frac{(\alpha_{n-j-2} - \alpha_{n-j})}{|z|^{j+1}} \\
& + \sum_{j=n-2k,jeven}^{n-2}\frac{(\alpha_{n-j} - \alpha_{n-j-2})}{|z|^{j+1}} + \frac{(\alpha_{n-3} - \lambda_2\alpha_{n-1})}{|z|^2} \\
& + \sum_{j=3,jodd}^{n-2l-1}\frac{(\alpha_{n-j-2} - \alpha_{n-j})}{|z|^{j+1}} + \sum_{j=n-2l+1,jodd}^{n-2}\frac{(\alpha_{n-j} - \alpha_{n-j-2})}{|z|^{j+1}} \\
& + \frac{(\beta_{n-2} - \lambda_3\beta_n)}{|z|} + \sum_{j=2,jeven}^{n-2s-2}\frac{(\beta_{n-j-2} - \beta_{n-j})}{|z|^{j+1}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=n-2s,jeven}^{n-2}\frac{(\beta_{n-j}-\beta_{n-j-2})}{|z|^{j+1}}+\frac{(\beta_{n-3}-\lambda_4\beta_{n-1})}{|z|^2}\\
&+\sum_{j=3,jodd}^{n-2q-1}\frac{(\beta_{n-j-2}-\beta_{n-j})}{|z|^{j+1}}+\sum_{j=n-2q+1,jodd}^{n-2}\frac{(\beta_{n-j}-\beta_{n-j-2})}{|z|^{j+1}}\Big\}\\
\geq\ & |z|^{n+1}(|a_n||z|-\hat{M}_{2e}),
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{M}_{2e} &= |a_{n-1}|+|a_1|+|a_0|+|(\lambda_1-1)\alpha_n|+|(\lambda_3-1)\beta_n|\\
&\quad+|(\lambda_2-1)\alpha_{n-1}|+|(\lambda_4-1)\beta_{n-1}|+2(\alpha_{2k}+\alpha_{2l-1}+\beta_{2s}+\beta_{2q-1})\\
&\quad-(\lambda_1\alpha_n+\lambda_3\beta_n)-(\lambda_2\alpha_{n-1}+\lambda_4\beta_{n-1})+(\alpha_1+\beta_1)\\
&\quad-(\alpha_0+\beta_0).
\end{aligned}
$$

Then $|\Phi(z)|>0$ if $|z|>\hat{M}_{2e}/|a_n|=\hat{R}_{2e}$, and all the zeros of $P(z)$ with modulus greater than one lie in $|z|\leq\hat{R}_{2e}$. It can be shown that $\hat{M}_{2e}\geq|a_n|$. Consequently all the zeros of $P(z)$ with modulus less than or equal to one already lie in $|z|\leq\hat{R}_{2e}$.

For the inner bound, consider a function

$$
\begin{aligned}
\Phi(z) &= (1-z^2)P(z)\\
&= a_0+\phi(z),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(z) &= -a_nz^{n+2}-a_{n-1}z^{n+1}+a_1z+\sum_{j=2}^{n-2}(a_{n-j}-a_{n-j-2})z^{n-j}\\
&= -a_nz^{n+2}-a_{n-1}z^{n+1}+a_1z-(\lambda_1-1)\alpha_nz^n-(\lambda_2-1)\alpha_{n-1}z^{n-1}\\
&\quad+(\lambda_1\alpha_n-\alpha_{n-2})z^n+(\lambda_2\alpha_{n-1}-\alpha_{n-3})z^{n-1}\\
&\quad+\sum_{j=2}^{n-2}(\alpha_{n-j}-\alpha_{n-j-2})z^{n-j}+i\Big\{-(\lambda_3-1)\beta_nz^n\\
&\quad-(\lambda_4-1)\beta_{n-1}z^{n-1}+(\lambda_3\beta_n-\beta_{n-2})z^n+(\lambda_4\beta_{n-1}-\beta_{n-3})z^{n-1}\\
&\quad+\sum_{j=2}^{n-2}(\beta_{n-j}-\beta_{n-j-2})z^{n-j}\Big\}.
\end{aligned}
$$

If $|z|<1$, then

$$
\begin{aligned}
|\phi(z)| &\leq |a_n|+|a_{n-1}|+|a_1|+|(\lambda_1-1)\alpha_n|+|(\lambda_2-1)\alpha_{n-1}|+|(\lambda_3-1)\beta_n|\\
&\quad+|(\lambda_4-1)\beta_{n-1}|+(\alpha_{n-2}-\lambda_1\alpha_n)+\sum_{j=2,jeven}^{n-2k-2}(\alpha_{n-j-2}-\alpha_{n-j})
\end{aligned}
$$

$$+\sum_{j=n-2k,jeven}^{n-2}(\alpha_{n-j}-\alpha_{n-j-2})+(\alpha_{n-3}-\lambda_2\alpha_{n-1})$$
$$+\sum_{j=3,jodd}^{n-2l-1}(\alpha_{n-j-2}-\alpha_{n-j})+\sum_{j=n-2l+1,jodd}^{n-2}(\alpha_{n-j}-\alpha_{n-j-2})$$
$$+(\beta_{n-2}-\lambda_3\beta_n)+\sum_{j=2,jeven}^{n-2s-2}(\beta_{n-j-2}-\beta_{n-j})$$
$$+\sum_{j=n-2s,jeven}^{n-2}(\beta_{n-j}-\beta_{n-j-2})+(\beta_{n-3}-\lambda_4\beta_{n-1})$$
$$+\sum_{j=3,jodd}^{n-2q-1}(\beta_{n-j-2}-\beta_{n-j})+\sum_{j=n-2q+1,jodd}^{n-2}(\beta_{n-j}-\beta_{n-j-2})$$
$$=\hat{M}_{1e},$$

where

$$\begin{aligned}\hat{M}_{1e} &= |a_n|+|a_{n-1}|+|a_1|+|(\lambda_1-1)\alpha_n|+|(\lambda_3-1)\beta_n| \\ &+|(\lambda_2-1)\alpha_{n-1}|+|(\lambda_4-1)\beta_{n-1}|+2(\alpha_{2k}+\alpha_{2l-1}+\beta_{2s}+\beta_{2q-1}) \\ &-(\lambda_1\alpha_n+\lambda_3\beta_n)-(\lambda_2\alpha_{n-1}+\lambda_4\beta_{n-1})+(\alpha_1+\beta_1)-(\alpha_0+\beta_0).\end{aligned}$$

Since $\phi(0)=0$, it follows by Schwarz lemma that

$$|\phi(z)|\le \hat{M}_{1e}|z| \quad \text{for} \quad |z|<1.$$

Then, for $|z|<1$,

$$\begin{aligned}|\Phi(z)| &\ge |a_0|-|\phi(z)| \\ &\ge |a_0|-\hat{M}_{1e}|z| \\ &> 0,\end{aligned}$$

if $|z|<|a_0|/\hat{M}_{1e}=\hat{R}_{1e}$. Hence $P(z)$ does not vanish in $|z|<\hat{R}_{1e}$. It can be shown that $\hat{M}_{1e}\le|a_0|$. Hence, if $t=1$, then $P(z)$ has all its zeros in the disk $\hat{R}_{1e}\le|z|\le\hat{R}_{2e}$. The proof of theorem for the case where n is even is completed by applying the above results to $P(tz)$.

The proof of the theorem can be proceeded similarly for the case where n is odd, and is omitted.

Theorem 3.4 *Let $P(z)=\sum_{i=0}^{n}a_iz^i$ be an nth-order polynomial such that $|\arg a_i-\beta|\le\alpha\le\pi/2$, $i=0,1,2,\cdots,n$, and for some $\lambda_1>0$, $\lambda_2>0$, $t>0$ and some nonnegative integer k and positive integer l,*

$$\lambda_1t^{2[n/2]}|a_{2[n/2]}|\le\cdots\le t^{2k+2}|a_{2k+2}|\le t^{2k}|a_{2k}|\ge\cdots\ge t^2|a_2|\ge|a_0|,$$

$$\lambda_2 t^{2[n/2]}|a_{2[n/2]-1}| \le \cdots \le t^{2l}|a_{2l+1}| \le t^{2l-2}|a_{2l-1}| \ge \cdots \ge t^2|a_3| \ge |a_1|.$$

If n is even, then $P(z)$ has all its zeros in the disk $R_{1e} \le |z| \le R_{2e}$, where

$$R_{1e} = \frac{t|a_0|}{M_{1e}} \quad \textit{and} \quad R_{2e} = \frac{M_{2e}}{t^{n-1}|a_n|},$$

and

$$\begin{aligned} M_{1e} &= t_n|a_n| + t_{n-1}|a_{n-1}| + t|a_1| + t^n|(\lambda_1 - 1)a_n| + t^{n-1}|(\lambda_2 - 1)a_{n-1}| \\ &\quad + \cos\alpha\Big\{2t^{2k}|a_{2k}| + 2t^{2l-1}|a_{2l-1}| - t_n\lambda_1|a_n| - t_{n-1}\lambda_2|a_{n-1}| - t|a_1| \\ &\quad -|a_0|\Big\} + \sin\alpha\Big\{\lambda_1 t^n|a_n| + \lambda_2 t^{n-1}|a_{n-1}| + t|a_1| + |a_0| + 2\sum_{j=2}^{n-2} t^j|a_j|\Big\}, \end{aligned}$$

$$\begin{aligned} M_{2e} &= t_{n-1}|a_{n-1}| + t|a_1| + |a_0| + t^n|(\lambda_1 - 1)a_n| + t^{n-1}|(\lambda_2 - 1)a_{n-1}| \\ &\quad + \cos\alpha\Big\{2t^{2k}|a_{2k}| + 2t^{2l-1}|a_{2l-1}| - t_n\lambda_1|a_n| - t_{n-1}\lambda_2|a_{n-1}| - t|a_1| \\ &\quad -|a_0|\Big\} + \sin\alpha\Big\{\lambda_1 t^n|a_n| + \lambda_2 t^{n-1}|a_{n-1}| + t|a_1| + |a_0| + 2\sum_{j=2}^{n-2} t^j|a_j|\Big\}. \end{aligned}$$

If n is odd, then $P(z)$ has all its zeros in the disk $R_{1o} \le |z| \le R_{2o}$, where

$$R_{1o} = \frac{t|a_0|}{M_{1o}} \quad \textit{and} \quad R_{2o} = \frac{M_{2o}}{t^{n-1}|a_n|}.$$

Here M_{1o} and M_{2o} are respectively the same as M_{1e} and M_{2e} except that λ_1 and λ_2 are respectively replaced by λ_2 and λ_1.

Proof: For the case where n is even, assume that the coefficient conditions hold for $t = 1$, i.e.,

$$\lambda_1|a_n| \le \cdots \le |a_{2k+2}| \le |a_{2k}| \ge \cdots \ge |a_2| \ge |a_0|,$$

$$\lambda_2|a_{n-1}| \le \cdots \le |a_{2l+1}| \le |a_{2l-1}| \ge \cdots \ge |a_3| \ge |a_1|.$$

For the outer bound, consider a polynomial

$$\begin{aligned} \Phi(z) &= (1 - z^2)P(z) \\ &= -a_n z^{n+2} - a_{n-1}z^{n+1} + a_1 z + a_0 + \sum_{j=2}^{n-2}(a_{n-j} - a_{n-j-2})z^{n-j} \\ &= -a_n z^{n+2} - a_{n-1}z^{n+1} + a_1 z + a_0 - (\lambda_1 - 1)\alpha_n z^n - (\lambda_2 - 1)\alpha_{n-1}z^{n-1} \\ &\quad +(\lambda_1 a_n - a_{n-2})z^n + (\lambda_2 a_{n-1} - a_{n-3})z^{n-1} + \sum_{j=2}^{n-2}(a_{n-j} - a_{n-j-2})z^{n-j}. \end{aligned}$$

If $|z| > 1$,

$$\begin{aligned}
|\Phi(z)| \quad \geq \quad & |a_n||z|^{n+2} - |z|^{n+1}\Bigg\{|a_{n-1}| + \frac{|a_1|}{|z|^n} + \frac{|a_0|}{|z|^{n+1}} + \frac{|(\lambda_1 - 1)a_n|}{|z|} \\
& + \frac{|(\lambda_2 - 1)a_{n-1}|}{|z|^2} + \frac{|a_{n-2} - \lambda_1 a_n)|}{|z|} + \sum_{j=2, jeven}^{n-2k-2} \frac{|a_{n-j-2} - a_{n-j}|}{|z|^{j+1}} \\
& + \sum_{j=n-2k, jeven}^{n-2} \frac{|a_{n-j} - a_{n-j-2}|}{|z|^{j+1}} + \frac{|a_{n-3} - \lambda_2 a_{n-1})|}{|z|} \\
& + \sum_{j=3, jodd}^{n-2l-1} \frac{|a_{n-j-2} - a_{n-j}|}{|z|^{j+1}} + \sum_{j=n-2l+1, jodd}^{n-2} \frac{|a_{n-j} - a_{n-j-2}|}{|z|^{j+1}}.
\end{aligned}$$

Using Lemma 2.1, it can be shown that

$$|\Phi(z)| \geq |a_n||z|^{n+2} - \hat{M}_{2e}|z|^{n+1},$$

where

$$\begin{aligned}
\hat{M}_{2e} \quad = \quad & |a_{n-1}| + |a_1| + |a_0| + |(\lambda_1 - 1)a_n| + |(\lambda_2 - 1)a_{n-1}| \\
& + \cos\alpha\Big\{2|a_{2k}| + 2|a_{2l-1}| - \lambda_1|a_n| - \lambda_2|a_{n-1}| - |a_1| - |a_0|\Big\} \\
& + \sin\alpha\Bigg\{\lambda_1|a_n| + \lambda_2|a_{n-1}| + |a_1| + |a_0| + 2\sum_{j=2}^{n-2}|a_j|\Bigg\}.
\end{aligned}$$

Then $|\Phi(z)| > 0$ if $|z| > \hat{M}_{2e}/|a_n| = \hat{R}_{2e}$, and all the zeros of $P(z)$ with modulus greater than one lie in $|z| \leq \hat{R}_{2e}$. It can be shown that $\hat{M}_{2e} \geq |a_n|$. Consequently all the zeros of $P(z)$ with modulus less than or equal to one already lie in $|z| \leq \hat{R}_{2e}$.

For the inner bound, consider a function

$$\begin{aligned}
\Phi(z) \quad &= \quad (1 - z^2)P(z) \\
&= \quad a_0 + \phi(z),
\end{aligned}$$

where

$$\begin{aligned}
\phi(z) \quad = \quad & -a_n z^{n+2} - a_{n-1}z^{n+1} + a_1 z + \sum_{j=2}^{n-2}(a_{n-j} - a_{n-j-2})z^{n-j} \\
= \quad & -a_n z^{n+2} - a_{n-1}z^{n+1} + a_1 z - (\lambda_1 - 1)\alpha_n z^n - (\lambda_2 - 1)\alpha_{n-1}z^{n-1} \\
& +(\lambda_1 a_n - a_{n-2})z^n + (\lambda_2 a_{n-1} - a_{n-3})z^{n-1} + \sum_{j=2}^{n-2}(a_{n-j} - a_{n-j-2})z^{n-j}.
\end{aligned}$$

If $|z| < 1$,

$$\begin{aligned}|\phi(z)| \leq\ & |a_n| + |a_{n-1}| + |a_1| + |(\lambda_1 - 1)a_n| + |(\lambda_2 - 1)a_{n-1}| \\ & + |a_{n-2} - \lambda_1 a_n)| + \sum_{j=2, jeven}^{n-2k-2} |a_{n-j-2} - a_{n-j}| \\ & + \sum_{j=n-2k, jeven}^{n-2} |a_{n-j} - a_{n-j-2}| + |a_{n-3} - \lambda_2 a_{n-1})| \\ & + \sum_{j=3, jodd}^{n-2l-1} |a_{n-j-2} - a_{n-j}| + \sum_{j=n-2l+1, jodd}^{n-2} |a_{n-j} - a_{n-j-2}|.\end{aligned}$$

Again using Lemma 2.1, we can show that

$$|\phi(z)| \leq \hat{M}_{1e}|z| \quad \text{for} \quad |z| < 1,$$

where

$$\begin{aligned}\hat{M}_{1e} =\ & |a_n| + |a_{n-1}| + |a_1| + |(\lambda_1 - 1)a_n| + |(\lambda_2 - 1)a_{n-1}| \\ & + \cos\alpha \Big\{ 2|a_{2k}| + 2|a_{2l-1}| - \lambda_1|a_n| - \lambda_2|a_{n-1}| - |a_1| - |a_0| \Big\} \\ & + \sin\alpha \Big\{ \lambda_1|a_n| + \lambda_2|a_{n-1}| + |a_1| + |a_0| + 2\sum_{j=2}^{n-2} |a_j| \Big\}.\end{aligned}$$

Since $\phi(0) = 0$, it follows by Schwarz lemma that

$$|\phi(z)| \leq \hat{M}_{1e}|z| \quad \text{for} \quad |z| < 1.$$

Then, for $|z| < 1$,

$$\begin{aligned}|\Phi(z)| &\geq |a_0| - |\phi(z)| \\ &\geq |a_0| - \hat{M}_{1e}|z| \\ &> 0\end{aligned}$$

if $|z| < |a_0|/\hat{M}_{1e} = \hat{R}_{1e}$. It can be shown that $\hat{M}_{1e} \geq |a_0|$. Hence $P(z)$ does not vanish in $|z| < \hat{R}_{1e}$. Consequently, if $t = 1$, $P(z)$ has all its zeros in the disk $\hat{R}_{1e} \leq |z| \leq \hat{R}_{2e}$. The proof of theorem for the case where n is even is completed by applying the above results to $P(tz)$. The case where n is odd can be proved similarly, and is omitted.

Theorem 3.5 *Let* $P(z) = \sum_{i=0}^{\infty} a_i z^i (a_0 \neq 0)$ *be analytic in* $|z| < t$. *Let* $\text{Re}\{a_i\} = \alpha_i$ *and* $\text{Im}\{a_i\} = \beta_i$, $i = 0, 1, 2, \cdots$, *and assume that for some* $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$, *and for some* k, l, s *and* q,

$$\lambda_1\alpha_0 \leq t^2\alpha_2 \leq \cdots \leq t^{2k}\alpha_{2k} \geq t^{2k+2}\alpha_{2k+2} \geq \cdots,$$

$$\lambda_2\alpha_1 \le t^2\alpha_3 \le \cdots \le t^{2l-2}\alpha_{2l-1} \ge t^{2l}\alpha_{2l+1} \ge \cdots,$$
$$\lambda_3\beta_0 \le t^2\beta_2 \le \cdots \le t^{2s}\beta_{2s} \ge t^{2s+2}\beta_{2s+2} \ge \cdots,$$
$$\lambda_4\beta_1 \le t^2\beta_3 \le \cdots \le t^{2q-2}\beta_{2q-1} \ge t^{2q}\beta_{2q+1} \ge \cdots.$$

Then $P(z)$ does not vanish in

$$|z+z_p| < \frac{M|a_0|}{1-M^2|a_1|^2},$$

where $M = t/M_0$ with

$$\begin{aligned} M_0 &= 2(t^{2k}\alpha_{2k} + t^{2l-1}\alpha_{2l-1} + t^{2s}\beta_{2s} + t^{2q-1}\beta_{2q-1} \\ &\quad + t\Big\{|(\lambda_2-1)\alpha_1| + |(\lambda_4-1)\beta_1|\Big\} + \Big\{|(\lambda_1-1)\alpha_0| + |(\lambda_3-1)\beta_0|\Big\} \\ &\quad - t(\lambda_2\alpha_1 + \lambda_4\beta_1) - (\lambda_1\alpha_0 + \lambda_3\beta_0), \end{aligned}$$

and

$$z_p = \left(-\frac{M^2(\alpha_0\alpha_1+\beta_0\beta_1)}{1-M^2|a_1|^2}, \frac{M^2(\alpha_0\beta_1-\alpha_1\beta_0)}{1-M^2|a_1|^2}\right).$$

Proof. Consider a polynomial

$$\begin{aligned} \Phi(z) &= (z^2-t^2)P(z) \\ &= -t^2a_0 - t^2a_1z + \phi(z), \end{aligned}$$

where

$$\begin{aligned} \phi(z) &= \sum_{j=0}^{\infty}(a_j - t^2a_{j+2})z^{j+2} \\ &= \sum_{j=0}^{\infty}(\alpha_j - t^2\alpha_{j+2})z^{j+2} + i\Big\{\sum_{j=0}^{\infty}(\beta_j - t^2\beta_{j+2})z^{j+2}\Big\} \\ &= -(\lambda_1-1)\alpha_0z^2 - (\lambda_2-1)\alpha_1z^3 - i(\lambda_3-1)\beta_0z^2 - i(\lambda_4-1)\beta_1z^3 \\ &\quad + (\lambda_1\alpha_0 - t^2\alpha_2)z^2 + (\lambda_2\alpha_1 - t^2\alpha_3)z^3 + \sum_{j=2}^{\infty}(\alpha_j - t^2\alpha_{j+2})z^{j+2} \\ &\quad + i\Big\{(\lambda_3\beta_0 - t^2\beta_2)z^2 + (\lambda_4\beta_1 - t^2\beta_3)z^3 + \sum_{j=2}^{\infty}(\beta_j - t^2\beta_{j+2})z^{j+2}\Big\}. \end{aligned}$$

For $|z| < t$, we have

$$\begin{aligned} |\phi(z)| &\le t^2|(\lambda_1-1)\alpha_0| + t^3|(\lambda_2-1)\alpha_1| + t^2|(\lambda_3-1)\beta_0| + t^3|(\lambda_4-1)\beta_1| \\ &\quad + (t^2\alpha_2 - \lambda_1\alpha_0)t^2 + \sum_{j=2, j even}^{2k-2}(t^2\alpha_{j+2} - \alpha_j)t^{j+2} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=2k,jeven}^{\infty}(\alpha_j-t^2\alpha_{j+2})t^{j+2}+(t^2\alpha_3-\lambda_2\alpha_1)t^3\\
&+\sum_{j=3,jodd}^{2l-3}(t^2\alpha_{j+2}-\alpha_j)t^{j+2}+\sum_{j=2l-1,jodd}^{\infty}(\alpha_j-t^2\alpha_{j+2})t^{j+2}\\
&+(t^2\beta_2-\lambda_3\beta_0)t^2+\sum_{j=2,jeven}^{2s-2}(t^2\beta_{j+2}-\beta_j)t^{j+2}\\
&+\sum_{j=2s,jeven}^{\infty}(\beta_j-t^2\beta_{j+2})t^{j+2}+(t^2\beta_3-\lambda_4\beta_1)t^3\\
&+\sum_{j=3,jodd}^{2q-3}(t^2\beta_{j+2}-\beta_j)t^{j+2}+\sum_{j=2q-1,jodd}^{\infty}(\beta_j-t^2\beta_{j+2})t^{j+2}\\
&=tM_1,
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 \;=\;& 2t(t^{2k}\alpha_{2k}+t^{2l-1}\alpha_{2l-1}+t^{2s}\beta_{2s}+t^{2q-1}\beta_{2q-1}\\
&+t^2\Big\{|(\lambda_2-1)\alpha_1|+|(\lambda_4-1)\beta_1|\Big\}+t\Big\{|(\lambda_1-1)\alpha_0|+|(\lambda_3-1)\beta_0|\Big\}\\
&-t^2(\lambda_2\alpha_1+\lambda_4\beta_1)-t(\lambda_1\alpha_0+\lambda_3\beta_0).
\end{aligned}
$$

Since $\phi(0)=0$, it follows by Schwarz lemma that

$$|\phi(z)|\le M_1|z| \quad \text{for} \quad |z|<t.$$

Then for $|z|<t$,

$$
\begin{aligned}
\Phi(z) &\ge t^2(|a_0+a_1z|-|\phi(z)|\\
&\ge t^2(|a_0+a_1z|-M_1|z|\\
&> 0,
\end{aligned}
$$

if

$$|z|<M|a_0+a_1z|,$$

where $M=t/M_0$ with

$$
\begin{aligned}
M_0 \;=\;& 2(t^{2k}\alpha_{2k}+t^{2l-1}\alpha_{2l-1}+t^{2s}\beta_{2s}+t^{2q-1}\beta_{2q-1}\\
&+t\Big\{|(\lambda_2-1)\alpha_1|+|(\lambda_4-1)\beta_1|\Big\}+\Big\{|(\lambda_1-1)\alpha_0|+|(\lambda_3-1)\beta_0|\Big\}\\
&-t(\lambda_2\alpha_1+\lambda_4\beta_1)-(\lambda_1\alpha_0+\lambda_3\beta_0).
\end{aligned}
$$

It is easy to show that the region defined by

$$\{z:|z|<M|a_0+a_1z|\},$$

is the disk

$$\left\{z : |z + z_p| < \frac{M|a_0|}{1 - M^2|a_1|^2}\right\},$$

where

$$z_p = \left(- \frac{M^2(\alpha_0\alpha_1 + \beta_0\beta_1)}{1 - M^2|a_1|^2}, \frac{M^2(\alpha_0\beta_1 - \alpha_1\beta_0)}{1 - M^2|a_1|^2}\right).$$

It also can be shown that the disk given above is contained in the disk $|z| < t$, and the proof is completed.

Theorem 3.6 *Let* $P(z) = \sum_{i=0}^{\infty} a_i z^i (\not\equiv 0)$ *be analytic in* $|z| < t$. *If for some* $\lambda_1 > 0$ *and* $\lambda_2 > 0$, *and for some* k *and* l,

$$\lambda_1|a_0| \le t^2|a_2| \le \cdots \le t^{2k}|a_{2k}| \ge t^{2k+2}|a_{2k+2}| \ge \cdots,$$
$$\lambda_2|a_1| \le t^2|a_3| \le \cdots \le t^{2l-2}|a_{2l-1}| \ge t^{2l}|a_{2l+1}| \ge \cdots,$$

and for some β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \quad j = 0, 1, 2, \cdots,$$

then $P(z)$ *does not vanish in*

$$|z + z_p| < \frac{M|a_0|}{1 - M^2|a_1|^2},$$

where $M = t/M_0$ *with*

$$\begin{aligned} M_0 &= t|(\lambda_2 - 1)a_1| + |(\lambda_1 - 1)a_0| + 2(t^{2k}|a_{2k}| + t^{2l-1}|a_{2l-1}|)\cos\alpha \\ &\quad +(t\lambda_2|a_1| + \lambda_1|a_0|)(\sin\alpha - \cos\alpha) + 2\sum_{j=2}^{\infty} t^j|a_j|\sin\alpha, \end{aligned}$$

and

$$z_p = \left(- \frac{M^2(\alpha_0\alpha_1 + \beta_0\beta_1)}{1 - M^2|a_1|^2}, \frac{M^2(\alpha_0\beta_1 - \alpha_1\beta_0)}{1 - M^2|a_1|^2}\right),$$

with $\mathrm{Re}\{a_j\} = \alpha_j$ *and* $\mathrm{Im}\{a_j\} = \beta_j$, $j = 0, 1$.

Proof. Consider a polynomial

$$\begin{aligned} \Phi(z) &= (z^2 - t^2)P(z) \\ &= -t^2a_0 - t^2a_1z + \phi(z), \end{aligned}$$

where

$$\begin{aligned} \phi(z) &= \sum_{j=0}^{\infty}(a_j - t^2a_{j+2})z^{j+2} \\ &= -(\lambda_1 - 1)a_0z^2 - (\lambda_2 - 1)a_1z^3 + (\lambda_1a_0 - t^2a_2)z^2 \\ &\quad +(\lambda_2a_1 - t^2a_3)z^3 + \sum_{j=2}^{\infty}(a_j - t^2a_{j+2})z^{j+2}. \end{aligned}$$

If $|z| < t$, then

$$\begin{aligned} |\phi(z)| &\leq t^2|(\lambda_1 - 1)a_0| + t^3|(\lambda_2 - 1)a_1| + |(t^2 a_2 - \lambda_1 a_0)|t^2 \\ &+ \sum_{j=2, jeven}^{2k-2} |(t^2 a_{j+2} - a_j)|t^{j+2} + \sum_{j=2k, jeven}^{\infty} |(a_j - t^2 a_{j+2})|t^{j+2} \\ &+ |(t^2 a_3 - \lambda_2 a_1)|t^3 + \sum_{j=3, jodd}^{2l-3} |(t^2 a_{j+2} - a_j)|t^{j+2} \\ &+ \sum_{j=2l-1, jodd}^{\infty} |(a_j - t^2 a_{j+2})|t^{j+2}. \end{aligned}$$

Using Lemma 2.1, we can show that

$$|\phi(z)| \leq tM_1 \quad \text{for} \quad |z| < t,$$

where

$$\begin{aligned} M_1 &= t^2|(\lambda_2 - 1)a_1| + t|(\lambda_1 - 1)a_0| + 2t(t^{2k}|a_{2k}| + t^{2l-1}|a_{2l-1}|)\cos\alpha \\ &+ (t^2\lambda_2|a_1| + t\lambda_1|a_0|)(\sin\alpha - \cos\alpha) + 2t\sum_{j=2}^{\infty} t^j|a_j|\sin\alpha. \end{aligned}$$

Since $\phi(0) = 0$, it follows by Schwarz lemma that

$$|\phi(z)| \leq M_1|z| \quad \text{for} \quad |z| < t.$$

Then for $|z| < t$,

$$\begin{aligned} |\Phi(z)| &\geq t^2(|a_0 + a_1 z| - |\phi(z)| \\ &\geq t^2(|a_0 + a_1 z| - M_1|z| \\ &> 0, \end{aligned}$$

if

$$|z| < M|a_0 + a_1 z|,$$

where $M = t/M_0$ with

$$\begin{aligned} M_0 &= t|(\lambda_2 - 1)a_1| + |(\lambda_1 - 1)a_0| + 2(t^{2k}|a_{2k}| + t^{2l-1}|a_{2l-1}|)\cos\alpha \\ &+ (t\lambda_2|a_1| + \lambda_1|a_0|)(\sin\alpha - \cos\alpha) + 2\sum_{j=2}^{\infty} t^j|a_j|\sin\alpha. \end{aligned}$$

It is easy to show that the region defined by

$$\{z : |z| < M|a_0 + a_1 z|\},$$

is the disk

$$\left\{ z : |z + z_p| < \frac{M|a_0|}{1 - M^2|a_1|^2} \right\},$$

where

$$z_p = \left(-\frac{M^2(\alpha_0\alpha_1 + \beta_0\beta_1)}{1 - M^2|a_1|^2}, \frac{M^2(\alpha_0\beta_1 - \alpha_1\beta_0)}{1 - M^2|a_1|^2} \right),$$

with $\text{Re}\{a_j\} = \alpha_j$ and $\text{Im}\{a_j\} = \beta_j$, $j = 0, 1$. It also can be shown that the disk given above is contained in the disk $|z| < t$, and the proof is completed.

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Received: February, 2011