Int. Journal of Math. Analysis, Vol. 5, 2011, no. 33, 1631 - 1664

Primal and Dual Evolution Macro-Hybrid Mixed Variational Inclusions

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Abstract

Primal and dual evolution macro-hybrid mixed variational inclusions in reflexive Banach spaces are analized. Existence and uniqueness results, and *a priori* estimates are established on the basis of composition duality principles. Applications to nonlinear diffusion control problems exemplify the theory.

Mathematics Subject Classification: 35A05, 35A15, 35K55, 35K90, 47J20, 49K24, 65N55

Keywords: Evolution variational inclusions, Macro-hybrid mixed variational problems, Composition duality methods, Primal-dual variational analysis, Set-valued analysis

1 Introduction

The purpose of this paper is to study qualitative properties of evolution variational inclusions in a setting of functional reflexive Banach spaces. Motivated by technological applications of nonlinear mechanics, the analysis is based on mixed variational formulations that permit the simultaneous computational treatment of primal and dual fields, as those of temperature- or concentration-flux and pressure-velocity in diffusive and fluid flows, and of velocity-stress for quasistatic solid deformations. Then, under the convention that the mixed variational modeling of physical systems assigns a primal character to the interactive field equation of balance or constitutivity, with the surrounding exterior, the evolution mixed problems may result of a primal or a dual classical type [15, 12, 20]. Furthermore, due to the importance of handling big and heterogeneous physical systems, with multi-scale and multi-constitutive behavior, it is convenient to perform the analysis for macro-hybrid mixed variational formulations, where by non-overlapping spatial decompositions, mixed subsystems are studied in parallel with dualized interface transmission conditions. Following the theory proposed in [5, 6] on evolution mixed variational inclusions, the strategy here in the analysis will be based on composition duality principles that determine primal-dual and dual-primal relations for the variational equations of macro-hybrid mixed evolution inclusion models.

Complementary to the primal and dual evolution analysis, we will also elaborate on the corresponding stationary analysis, determining conditions for compatibility and existence. As applications of the theory, distributed control nonlinear diffusion, primal and dual evolution problems will be considered.

The present analysis corresponds to an extension of our previous work [8], on linear evolution mixed variational problems, to subdifferential nonlinear evolution Cauchy problems.

2 Evolution Macro-Hybrid and Mixed Variational Inclusions

In this section, we introduce the evolution macro-hybrid mixed inclusion problems of the theory, and we established composition duality principles for their qualitative analysis. Here, we shall follow a local to global approach, conceiving global evolution physical processes as local multi-processes coexisting through interface continuity transmission conditions (see [6, 7], for a spatially global to local approach).

Hence, let us consider a dynamical system of a physical process evolving in a spatial bounded domain $\Omega \subset \Re^d$, $d \in \{1, 2, 3\}$, along a fixed time interval $(0, T)$, $T > 0$. To this system, for localization, we introduce non-overlapping decompositions in terms of disjoint and connected subdomains $\{\Omega_e\}$, $\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}_e$, with internal boundaries $\Gamma_e = \partial\Omega_e \cap \Omega$, $e = 1, 2, ..., E$, and interfaces $\Gamma_{ef} = \Gamma_e \cap \Gamma_f$, $1 \leq e < f \leq E$, assumed to be Lipschitz continuous. Thereby, for a localized mixed analysis, we further introduce primal and dual Ω-field spaces, $V(\Omega)$ and $Y^*(\Omega)$, reflexive Banach spaces decomposable in the sense

$$V(\Omega) = \left\{ \{v_e\} \in \boldsymbol{V}_{\{\Omega_e\}} \equiv \prod_{e=1}^{E} V(\Omega_e) : \ \{\pi_{\Gamma_e} v_e\} \in \boldsymbol{Q} \right\},$$
$$Y^*(\Omega) = \boldsymbol{Y}^*_{\{\Omega_e\}} \equiv \prod_{e=1}^{E} Y^*(\Omega_e), \tag{1}$$

where $[\pi_{\Gamma_e}]$ is the continuous linear internal boundary trace operator of the primal product space $\boldsymbol{V}_{\{\Omega_e\}}$, with values in $\boldsymbol{B}_{\{\Gamma_e\}} \equiv \prod_{e=1}^{E} B(\Gamma_e)$, satisfying the macro-hybrid compatibility condition

$$(\boldsymbol{C}_{[\boldsymbol{\pi_{\Gamma_e}}]})\ [\pi_{\Gamma_e}] \in \mathcal{L}(\boldsymbol{V}_{\{\Omega_e\}}, \boldsymbol{B}_{\{\Gamma_e\}}) \text{ is surjective.}$$

Also, $\boldsymbol{Q} \subset \boldsymbol{B}_{\{\Gamma_e\}}$ is the primal transmission admissibility subspace, with polar, the dual transmission admissibility subspace of the dual $\boldsymbol{B}^*_{\{\Gamma_e\}} \equiv \prod_{e=1}^{E} B^*(\Gamma_e)$,

$$\boldsymbol{Q}^* = \Big\{ \{\mu_e^*\} \in \boldsymbol{B}^*_{\{\Gamma_e\}} :\ \langle \{\mu_e^*\}, \{\mu_e\} \rangle_{B_{\{\Gamma_e\}}} = 0, \forall \{\mu_e\} \in \boldsymbol{Q} \Big\}. \tag{2}$$

Furthermore, for primal and dual Hilbert pivot spaces $H(\Omega)$ and $Z^*(\Omega)$, that is, $V(\Omega) \subset H(\Omega) \subset V^*(\Omega)$ and $Y^*(\Omega) \subset Z^*(\Omega) \subset Y(\Omega)$ with continuous and dense embeddings, we consider that $H(\Omega) = \boldsymbol{H}_{\{\Omega_e\}} \equiv \prod_{e=1}^{E} H(\Omega_e)$ and $Z^*(\Omega) = \boldsymbol{Z}^*_{\{\Omega_e\}} \equiv \prod_{e=1}^{E} Z^*(\Omega_e)$. On the other hand, we shall consider as operators for the dynamical system, two time independent monotone subdifferentials, $\partial F : V(\Omega) \to 2^{V^*(\Omega)}$ and $\partial G^* : Y^*(\Omega) \to 2^{Y(\Omega)}$, with proper convex and lower semicontinuous superpotentials, $F : V(\Omega) \to \Re \cup \{+\infty\}$ and $G^* : Y^*(\Omega) \to \Re \cup \{+\infty\}$, respectively. Moreover, as the local coupling operators of the mixed system, we introduce linear continuous local operators $\Lambda_e \in \mathcal{L}(V(\Omega_e), Y(\Omega_e))$, with transpose $\Lambda^T \in \mathcal{L}(Y^*(\Omega_e), V^*(\Omega_e))$, $e = 1, 2, ..., E$.

2.1 Primal evolution macro-hybrid mixed variational inclusion

Then, as a primal evolution macro-hybrid mixed variational model for the theory, we pose the localized problem,

$$(\boldsymbol{\mathcal{MH}}) \begin{cases} \text{Find } (\{u_e\}, \{p_e^*\}) \in \boldsymbol{\mathcal{W}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{Y}^*}_{\{\Omega_e\}} : \\[2mm] -\{\Lambda_e^T p_e^*\} - \{\pi_{\Gamma_e}^T \lambda_e^*\} \in \left\{ \dfrac{du_e}{dt} \right\} + \{\partial F_e(u_e)\} - \{f_e^*\}, & \text{in } \boldsymbol{\mathcal{V}^*}_{\{\Omega_e\}}, \\[2mm] \{\Lambda_e u_e\} \in \{\partial G_e^*(p_e^*)\} + \{g_e\}, & \text{in } \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}}, \\[2mm] \{u_e(0)\} = \{u_{0_e}\}, \end{cases}$$

synchronized by the internal boundary dual transmission problem

$$(\boldsymbol{\mathcal{T}}) \begin{cases} \text{Find } \{\lambda_e^*\} \in \boldsymbol{\mathcal{B}}^*_{\{\Gamma_e\}} : \\[2mm] \{\pi_{\Gamma_e} u_e\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda_e^*\}), \quad \text{in } \boldsymbol{\mathcal{B}}_{\{\Gamma_e\}}. \end{cases}$$

Here, the primal and dual evolution product spaces are the reflexive Banach spaces defined by $\boldsymbol{\mathcal{V}}_{\{\Omega_e\}} = L^p(0, T; \boldsymbol{V}_{\{\Omega_e\}})$, $2 \le p < \infty$, and $\boldsymbol{\mathcal{Y}}^*_{\{\Omega_e\}} = L^{p^*}(0, T; \boldsymbol{Y}^*_{\{\Omega_e\}})$, $p^* = p/(p-1)$, with topological duals $\boldsymbol{\mathcal{V}}^*_{\{\Omega_e\}} = L^{p^*}(0, T;$

$\boldsymbol{V}^*_{\{\Omega_e\}}$) and $\boldsymbol{\mathcal{Y}}_{\{\Omega_e\}} = L^p(0,T;\boldsymbol{Y}_{\{\Omega_e\}})$. Moreover, the solution primal space is given by $\boldsymbol{\mathcal{W}}_{\{\Omega_e\}} = \{\{v_e\} : \{v_e\} \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}}, \{dv_e/dt\} \in \boldsymbol{\mathcal{V}}^*_{\{\Omega_e\}}\}$, continuous and densely embedded in the vector Hilbert space $\boldsymbol{C}([0,T];\boldsymbol{H}_{\{\Omega_e\}})$ of time continuous vector functions, with initial values such that $\{\{v_e(0)\} : \{v_e\} \in \boldsymbol{\mathcal{W}}_{\{\Omega_e\}}\} = \boldsymbol{H}_{\{\Omega_e\}}$. Also, the macro-hybrid dual space is given by $\boldsymbol{\mathcal{B}}^*_{\{\Gamma_e\}} = L^{p^*}(0,T;\boldsymbol{B}^*_{\{\Gamma_e\}})$, with dual $\boldsymbol{\mathcal{B}}_{\{\Gamma_e\}} = L^p(0,T;\boldsymbol{B}_{\{\Gamma_e\}})$.

Next, in order to give a global interpretation to macro-hybrid problem ($\boldsymbol{\mathcal{MH}}$), we apply the following duality result, established in [6] (see [7], too).

Lemma 2.1 *Under compatibility condition* ($\boldsymbol{C}_{[\boldsymbol{\pi}_{\boldsymbol{\Gamma}_e}]}$), *the macro-hybrid compositional dualization*

$$\{\pi_{\Gamma_e} u_e\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda_e^*\}) \iff \{\pi_{\Gamma_e}^T \lambda_e^*\} \in \boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\}), \tag{3}$$

holds true.

Hence, according to Lemma 2.1, dualization of transmission problem ($\boldsymbol{\mathcal{T}}$) leads to the macro-hybridized primal evolution mixed problem

$$(\boldsymbol{\mathcal{M}}_{MH}) \begin{cases} \text{Find } (\{u_e\},\{p_e^*\}) \in \boldsymbol{\mathcal{W}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{Y}}^*_{\{\Omega_e\}} : \\ -\{\Lambda_e^T p_e^*\} \in \left\{\dfrac{du_e}{dt}\right\} + \{\partial F_e(u_e)\} + \boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\}) \\ \qquad\qquad -\{f_e^*\}, & \text{in } \boldsymbol{\mathcal{V}}^*_{\{\Omega_e\}}, \\ \{\Lambda_e u_e\} \in \{\partial G_e^*(p_e^*)\} + \{g_e\}, & \text{in } \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}}, \\ \{u_e(0)\} = \{u_{0_e}\}, \end{cases}$$

which in turn, recognizing the macro-hybrid term $\boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\})$ as the imposed primal transmission constraint of decomposition $(1)_1$, can be interpreted as a macro-hybridization of the global primal evolution mixed problem

$$(\boldsymbol{\mathcal{M}}) \begin{cases} \text{Given } f^* \in \mathcal{V}^*, g \in L^p(0,T;\mathcal{R}(\Lambda)) \text{ and } u_0 \in H(\Omega), \\ \text{find } (u,p^*) \in \mathcal{W} \times \mathcal{Y}^* : \\ -\Lambda^T p^* \in \dfrac{du}{dt} + \partial F(u) - f^*, & \text{in } \mathcal{V}^*, \\ \Lambda u \in \partial G^*(p^*) + g, & \text{in } \mathcal{Y}, \\ u(0) = u_0. \end{cases}$$

Here, the global functional framework corresponds to the primal evolution reflexive Banach space $\mathcal{V} = L^p(0,T;V(\Omega)) = \{v : [0,T] \to V(\Omega) | \ \|v\|_{\mathcal{V}} = [\int_0^T \|v(t)\|^p_{V(\Omega)} \, dt]^{1/p} < \infty\}$, $2 \leq p < \infty$, with topological dual $\mathcal{V}^* = L^{p^*}(0,T; V^*(\Omega))$, $p^* = p/(p-1)$, and the primal solution space $\mathcal{W} = \{v : v \in \mathcal{V}, dv/dt \in \mathcal{V}^*\} \subset C([0,T];H(\Omega))$, with the operator norm $\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|dv/dt\|_{\mathcal{V}^*}$. Moreover, the dual solution space is the reflexive Banach space $\mathcal{Y}^* = L^{p^*}(0,T;Y^*(\Omega))$, with dual $\mathcal{Y} = L^p(0,T;Y(\Omega))$. Also $\mathcal{R}(\Lambda) \subset Y(\Omega)$ is the range of the global coupling operator $\Lambda \in \mathcal{L}(V(\Omega), Y(\Omega))$, with transpose $\Lambda^T \in \mathcal{L}(Y^*(\Omega), V^*(\Omega))$.

Therefore, by construction, we can conclude that decomposed problem $(\boldsymbol{\mathcal{MH}})$, with dual synchronization $(\boldsymbol{\mathcal{T}})$, and macro-hybridized mixed problem $(\boldsymbol{\mathcal{M}}_{MH})$ as well as global mixed problem $(\boldsymbol{\mathcal{M}})$, are all equivalent in a solvability sense. This equivalent solvability structure will be of fundamental importance through the analysis.

2.2 Dual evolution macro-hybrid mixed variational inclusion

As a dual evolution macro-hybrid mixed variational model for the theory, we introduce the localized problem,

$$
(\boldsymbol{\mathcal{MH}}^*)
\begin{cases}
\text{Find } (\{u_e\},\{p_e^*\}) \in \mathcal{V}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{X}}^*_{\{\Omega_e\}} : \\[2ex]
-\{\Lambda_e^T p_e^*\} - \{\pi_{\Gamma_e}^T \lambda_e^*\} \in \{\partial F_e(u_e)\} - \{f_e^*\}, \quad \text{in } \mathcal{V}^*_{\{\Omega_e\}}, \\[2ex]
\{\Lambda_e u_e\} \in \left\{\dfrac{dp_e^*}{dt}\right\} + \{\partial G_e^*(p_e^*)\} + \{g_e\}, \qquad \text{in } \mathcal{Y}_{\{\Omega_e\}}, \\[2ex]
\{p_e^*(0)\} = \{p_{0_e}^*\},
\end{cases}
$$

synchronized, as in the primal evolution case, by the internal boundary dual transmission common problem $(\boldsymbol{\mathcal{T}})$. For this dual evolution model, the solution dual space is given by $\boldsymbol{\mathcal{X}}^*_{\{\Omega_e\}} = \{\{q_e^*\} : \{q_e^*\} \in \boldsymbol{\mathcal{Y}}^*_{\{\Omega_e\}}, \{dq_e^*/dt\} \in \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}}\}$, which is continuous and densely embedded in the vector Hilbert space of time continuous functions, $C([0,T];\boldsymbol{Z}^*_{\{\Omega_e\}})$, with initial values such that $\{\{q_e^*(0)\} : \{q_e^*\} \in \boldsymbol{\mathcal{X}}^*_{\{\Omega_e\}}\} = \boldsymbol{Z}^*_{\{\Omega_e\}}$.

Proceeding as for the primal case, a global interpretation to macro-hybrid problem $(\boldsymbol{\mathcal{MH}}^*)$ is obtained by applying the macro-hybrid compositional dualization of Lemma 2.1, (3), dualizing transmission problem $(\boldsymbol{\mathcal{T}})$, and concluding the macro-hybridized dual evolution mixed problem

$$(\boldsymbol{\mathcal{M}}^*_{MH}) \begin{cases} \text{Find } (\{u_e\}, \{p_e^*\}) \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{X}}^*_{\{\Omega_e\}} : \\ -\{\Lambda_e^T p_e^*\} \in \{\partial F_e(u_e)\} + \boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\}) - \{f_e^*\}, & \text{in } \boldsymbol{\mathcal{V}}^*_{\{\Omega_e\}}, \\ \{\Lambda_e u_e\} \in \left\{\dfrac{dp_e^*}{dt}\right\} + \{\partial G_e^*(p_e^*)\} + \{g_e\}, & \text{in } \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}}, \\ \{p_e^*(0)\} = \{p_{0_e}^*\}, \end{cases}$$

which can be interpreted as a macro-hybridization of the global dual evolution mixed problem

$$(\boldsymbol{\mathcal{M}}^*) \begin{cases} \text{Given } f^* \in L^p(0,T;\mathcal{R}(-\Lambda^T), g \in \mathcal{Y} \text{ and } p_0^* \in Z^*(\Omega), \\ \text{find } (u, p^*) \in \mathcal{V} \times \mathcal{X}^* : \\ -\Lambda^T p^* \in \partial F(u) - f^*, & \text{in } \mathcal{V}^*, \\ \Lambda u \in \dfrac{dp^*}{dt} + \partial G^*(p^*) + g, & \text{in } \mathcal{Y}, \\ p^*(0) = p_0^*. \end{cases}$$

Here, the global functional framework is given by the dual evolution reflexive Banach space $\mathcal{Y}^* = L^{p^*}(0,T;Y^*(\Omega))$, $2 \leq p^* < \infty$, with topological dual $\mathcal{Y} = L^p(0,T;Y(\Omega))$, $p = p^*/(p^*-1)$, with dual solution space $\mathcal{X}^* = \{q^* : q^* \in \mathcal{Y}^*, dq^*/dt \in \mathcal{Y}\} \subset C([0,T];Z^*(\Omega))$, endowed with the operator norm. Further, the primal solution space is the reflexive Banach space $\mathcal{V} = L^p(0,T;V(\Omega))$, with dual $\mathcal{V}^* = L^{p^*}(0,T;V^*(\Omega))$.

As for the primal evolution case, by construction, we have that decomposed problem $(\boldsymbol{\mathcal{MH}}^*)$, synchronized by dual transmission problem $(\boldsymbol{\mathcal{T}})$, and macro-hybridized mixed problem $(\boldsymbol{\mathcal{M}}^*_{MH})$ and global mixed problem $(\boldsymbol{\mathcal{M}}^*)$, are all equivalent in a solvability sense.

3 Evolution Duality Principles

We now proceed to establish duality principles for the primal and dual evolution macro-hybrid mixed variational problems $(\boldsymbol{\mathcal{MH}})$ and $(\boldsymbol{\mathcal{MH}}^*)$. Toward this end, we shall utilize the solvability equivalence of the local macro-hybrid problems with their corresponding global primal and dual evolution mixed problems $(\boldsymbol{\mathcal{M}})$ and $(\boldsymbol{\mathcal{M}}^*)$. Here, we shall follow the duality approach of paper [10], which has been inspired on the book of Le Tallec [18] (Chapter 4), as well as the classical duality approach applied in our previous study [6].

3.1 Primal evolution duality principles

We first define the primal admissibility set of evolution mixed problems ($\boldsymbol{\mathcal{M}}$), to which any primal component solution belongs, for a.e. $t \in (0,T)$,

$$\mathcal{S}_p(t) = \{v \in V(\Omega) : \Lambda v = p + g(\cdot,t), p \in \partial G^*(p^*) \subset Y(\Omega)\}. \tag{4}$$

Hence, introducing the coupling compatibility condition

$$(\boldsymbol{C_{\Lambda^T}})\ \Lambda^T \in \mathcal{L}(Y^*(\Omega), V^*(\Omega)) \text{ has a closed range, } \mathcal{R}(\Lambda^T),$$

we conclude the following dual compatibility condition of the problem.

Lemma 3.1 *Under condition ($\boldsymbol{C_{\Lambda^T}}$), the dual admissibility set of evolution mixed problem ($\boldsymbol{\mathcal{M}}$) is characterized by*

$$\partial I_{\mathcal{S}_p(t)}(v) = \begin{cases} \mathcal{R}(\Lambda^T), & v \in \mathcal{S}_p(t), \\ \\ \emptyset, & \textit{otherwise}, \end{cases} \tag{5}$$

the subdifferential of the indicator functional $I_{\mathcal{S}_p(t)}$.

Proof Considering that $\mathcal{D}(\partial I_{\mathcal{S}_p(t)}) = \mathcal{S}_p(t) = \mathcal{D}(I_{\mathcal{S}_p(t)})$, for $u \in \mathcal{S}_p(t)$,

$$\begin{aligned} \partial I_{\mathcal{S}_p(t)}(u) &= \{v^* \in V^*(\Omega) : 0 \geq \langle v^*, v - u\rangle_{V(\Omega)}, \forall\ v \in \mathcal{S}_p(t)\} \\ &= \{v^* \in V^*(\Omega) : 0 \geq \langle v^*, (\tilde{v} + u) - u\rangle_{V(\Omega)}, \forall\ \tilde{v} \in \mathcal{N}(\Lambda)\} \\ &= \mathcal{N}^\circ(\Lambda), \end{aligned}$$

where $\tilde{v} + u$ belongs to $\mathcal{S}_p(t)$, $\forall\ \tilde{v} \in \mathcal{N}(\Lambda)$, and $\mathcal{N}^\circ(\Lambda) \subset V^*(\Omega)$ is the polar subspace of the kernel $\mathcal{N}(\Lambda) \subset V(\Omega)$. Further, $\mathcal{N}^\circ(\Lambda) = \mathcal{R}(\Lambda^T)$, due to condition ($\boldsymbol{C_{\Lambda^T}}$) and the Closed Range Theorem.

Therefore, the following primal evolution duality principle for problem ($\boldsymbol{\mathcal{M}}$) is achieved.

Theorem 3.2 *Let condition ($\boldsymbol{C_{\Lambda^T}}$) be satisfied. Then primal evolution mixed problem ($\boldsymbol{\mathcal{M}}$) is solvable if, and only if, the primal evolution problem*

$$(\boldsymbol{\mathcal{P}}) \begin{cases} \textit{Find } u \in \mathcal{W} : \\ \\ 0 \in \dfrac{du}{dt} + \partial F(u) + \partial I_{\mathcal{S}_p}(u) - f^*, \quad \textit{in } \mathcal{V}^*, \\ \\ u(0) = u_0, \end{cases}$$

is solvable. That is, if $u \in \mathcal{W}$ *is a solution of primal problem* ($\boldsymbol{\mathcal{P}}$) *then there is an admissible dual function* $p^* \in \mathcal{Y}^*$*, with* $\Lambda^T p^* \in \partial I_{\mathcal{S}_p}(u)$*, such that* (u, p^*) *is a solution of mixed problem* ($\boldsymbol{\mathcal{M}}$) *and, conversely, if* $(u, p^*) \in \mathcal{W} \times \mathcal{Y}^*$ *is a solution of mixed problem* ($\boldsymbol{\mathcal{M}}$) *then primal admissible function* u *is a solution of primal problem* ($\boldsymbol{\mathcal{P}}$)*.*

Proof Solutions $u \in \mathcal{W}$ of primal evolution problem ($\boldsymbol{\mathcal{P}}$) necessarily belong to $\mathcal{S}_p(t)$ for a.e. $t \in (0, T)$ and, consequently, satisfy the dual equation of mixed problem ($\boldsymbol{\mathcal{M}}$), with $p \in \partial G^*(p^*)$ for some $p^* \in Y^*(\Omega)$. Moreover, primal equation of mixed problem ($\boldsymbol{\mathcal{M}}$) is further satisfied since $\Lambda^T p^* \in \partial I_{\mathcal{S}_p}(u)$. On the other hand, mixed solutions $(u, p^*) \in \mathcal{W} \times \mathcal{Y}^*$ of problem ($\boldsymbol{\mathcal{M}}$) are such that $u \in \mathcal{S}_p$ and, then, $\Lambda^T p^* \in \partial I_{\mathcal{S}_p}(u)$. Thus, u is a solution of primal problem ($\boldsymbol{\mathcal{P}}$).

From the equivalence of global primal evolution mixed problem ($\boldsymbol{\mathcal{M}}$) and primal evolution macro-hybrid mixed problem ($\boldsymbol{\mathcal{MH}}$), we have the corresponding macro-hybrid duality principle.

Corollary 3.3 *Under condition* ($\boldsymbol{C}_{\boldsymbol{\Lambda^T}}$)*, primal evolution macro-hybrid mixed problem* ($\boldsymbol{\mathcal{MH}}$) *has a solution if, and only if, primal evolution problem* ($\boldsymbol{\mathcal{P}}$) *has a solution.*

In relation with the composition duality methodology established in [6], we next give compositional interpretations of dual admissibility characterization (5).

Lemma 3.4 *The dual inclusion of problem* ($\boldsymbol{\mathcal{M}}$) *necessarily satisfies the compositional dualization*

$$\Lambda u \in \partial G^*(p^*) + g \implies \Lambda^T p^* \in \partial (G \circ \Lambda)(u - v_g), \tag{6}$$

where $v_g \in \mathcal{V}$ *is a fixed* Λ*-preimage of function* g*:* $\Lambda v_g = g$*.*

Proof By dualization $\Lambda u \in \partial G^*(p^*) + g \Leftrightarrow p^* \in \partial G(\Lambda(u - v_g))$. Then, the compositional dualization $p^* \in \partial G(\Lambda(u - v_g)) \Rightarrow \Lambda^T p^* \in \partial (G \circ \Lambda)(u - v_g)$ is valid since the variational inequality of the former subdifferential equation,

$$\Lambda(u - v_g) \in \mathcal{D}(G) : \; G(q) \geq G(\Lambda(u - v_g)) + \langle p^*, q - \Lambda(u - v_g)\rangle_{Y(\Omega)}, \; \forall q \in \mathcal{D}(G),$$

implies, taking variations $q = \Lambda v$, $v \in V(\Omega)$, the variational inequality of the latter,

$$u - v_g \in \mathcal{D}(G \circ \Lambda) : \; G \circ \Lambda(v) \geq G \circ \Lambda(u - v_g) + \langle \Lambda^T p^*, v - (u - v_g)\rangle_Y, \; \forall v \in \mathcal{D}(G \circ \Lambda).$$

Note that $\Lambda(u - v_g) \in \mathcal{D}(G) \Leftrightarrow u - v_g \in \mathcal{D}(G \circ \Lambda)$, and $\mathcal{D}(G \circ \Lambda) = \mathcal{D}(G_{|\mathcal{R}(\Lambda)})$.

Therefore, from Lemma 3.4 and Theorem 3.2, the following primal evolution composition duality principle for problem ($\boldsymbol{\mathcal{M}}$) is concluded.

Theorem 3.5 *Primal evolution mixed problem* $(\boldsymbol{\mathcal{M}})$ *is solvable, only if the primal evolution problem*

$$(\widetilde{\boldsymbol{\mathcal{P}}}) \begin{cases} \text{Find } u \in \mathcal{W}: \\ \\ 0 \in \dfrac{du}{dt} + \partial F(u) + \partial(G \circ \Lambda)(u - v_g) - f^*, \quad \text{in } \mathcal{V}^*, \\ \\ u(0) = u_0, \end{cases}$$

is solvable. That is, if $(u, p^*) \in \mathcal{W} \times \mathcal{Y}^*$ *is a solution of problem* $(\boldsymbol{\mathcal{M}})$ *then component* u *is a primal admissible solution of problem* $(\widetilde{\boldsymbol{\mathcal{P}}})$*. Moreover, compositional primal problem* $(\widetilde{\boldsymbol{\mathcal{P}}})$ *is solvable if primal evolution problem* $(\boldsymbol{\mathcal{P}})$ *is solvable.*

Also, taking into account the equivalence between global problem $(\boldsymbol{\mathcal{M}})$ and macro-hybrid problem $(\boldsymbol{\mathcal{MH}})$, we conclude the macro-hybrid composition duality principle.

Theorem 3.6 *Primal evolution macro-hybrid mixed problem* $(\boldsymbol{\mathcal{MH}})$ *has a solution, only if primal evolution problem* $(\widetilde{\boldsymbol{\mathcal{P}}})$ *has a solution.*

Furthermore, in our previous study [6], the classical primal evolution compatibility condition

$$(\boldsymbol{C_{G,\Lambda}}) \ \mathrm{int}\mathcal{D}(G) \cap \mathcal{R}(\Lambda) \neq \emptyset$$

was considered, under which the compositional operator equality

$$\partial(G \circ \Lambda) = \Lambda^T \partial G \circ \Lambda \tag{7}$$

holds true [15]. Here $\mathrm{int}\mathcal{D}(G)$ denotes the interior of the effective domain of the conjugate $G : Y \to \Re \cup \{+\infty\}$ of functional G^*. In [6], this compatibility condition was shown to be appropriate for distributed control mixed diffusion problems, exemplifying the theory. Moreover, assuming valid classical condition $(\boldsymbol{C_{G,\Lambda}})$, from the dual inclusion of problem $(\boldsymbol{\mathcal{M}})$, equivalently expressed by $p^* \in \partial G \circ \Lambda(u - v_g)$, the sufficiency of the principle of Theorem 3.5 is in addition readily obtained (see [6], Theorem 2.2). Hence, an alternative primal evolution duality principle is obtained.

Theorem 3.7 *Let compatibility condition* $(\boldsymbol{C_{G,\Lambda}})$ *be fulfilled. Then primal evolution macro-hybrid mixed problem* $(\boldsymbol{\mathcal{MH}})$*, equivalent to mixed problem* $(\boldsymbol{\mathcal{M}})$*, is solvable if, and only if, primal evolution problem* $(\widetilde{\boldsymbol{\mathcal{P}}})$ *is solvable. Moreover, under condition* $(\boldsymbol{C_{\Lambda^T}})$*, compositional primal problem* $(\widetilde{\boldsymbol{\mathcal{P}}})$ *is equivalent to primal evolution problem* $(\boldsymbol{\mathcal{P}})$*.*

3.2 Dual evolution duality principles

Following the same analysis procedure as for the primal evolution case, we start introducing the dual admissibility set of evolution mixed problems $(\boldsymbol{\mathcal{M}}^*)$, to which dual component solutions belong, for a.e. $t \in (0,T)$,

$$\mathcal{S}_{u^*}(t) = \{q^* \in Y^*(\Omega) : -\Lambda^T q^* = u^* - f^*(\cdot,t), u^* \in \partial F(u) \subset V^*(\Omega)\}. \quad (4^*)$$

Further, introducing the coupling compatibility condition

$$(\boldsymbol{C_{-\Lambda}}) - \Lambda \in \mathcal{L}(V(\Omega), Y(\Omega)) \text{ has a closed range, } \mathcal{R}(-\Lambda),$$

we have the following primal compatibility condition of the problem.

Lemma 3.1* *Under condition* $(\boldsymbol{C_{-\Lambda}})$*, the primal admissibility set of evolution mixed problems* $(\boldsymbol{\mathcal{M}}^*)$ *is characterized by*

$$\partial I_{\mathcal{S}_{u^*}(t)}(q^*) = \begin{cases} \mathcal{R}(-\Lambda), & q^* \in \mathcal{S}_{u^*}(t), \\ \\ \emptyset, & \textit{otherwise}, \end{cases} \quad (5^*)$$

the subdifferential of the indicator functional $I_{\mathcal{S}_{u^*}(t)}$*.*
Consequently, the following dual evolution duality principle for problem $(\boldsymbol{\mathcal{M}}^*)$ is concluded.

Theorem 3.2* *Under condition* $(\boldsymbol{C_{-\Lambda}})$*, dual evolution mixed problem* $(\boldsymbol{\mathcal{M}}^*)$ *has a solution if, and only if, the dual evolution problem*

$$(\boldsymbol{\mathcal{D}}) \begin{cases} \textit{Find } p^* \in \mathcal{X}^* : \\ \\ 0 \in \dfrac{dp^*}{dt} + \partial G^*(p^*) + \partial I_{\mathcal{S}_{u^*}}(p^*) + g, \quad \textit{in } \mathcal{Y}, \\ \\ p^*(0) = p_0^*, \end{cases}$$

has a solution. That is, if $p^* \in \mathcal{X}^*$ *is a solution of dual problem* $(\boldsymbol{\mathcal{D}})$ *then there is an admissible primal function* $u \in \mathcal{V}$*, with* $-\Lambda u \in \partial I_{\mathcal{S}_{u^*}}(p^*)$*, such that* (u, p^*) *is a solution of mixed problem* $(\boldsymbol{\mathcal{M}}^*)$ *and, conversely, if* $(u, p^*) \in \mathcal{V} \times \mathcal{X}^*$ *is a solution of mixed problem* $(\boldsymbol{\mathcal{M}}^*)$ *then dual admissible function* p^* *is a solution of dual problem* $(\boldsymbol{\mathcal{D}})$*.*

Taking into account the equivalence of global dual evolution mixed problem $(\boldsymbol{\mathcal{M}}^*)$ and dual evolution macro-hybrid mixed problem $(\boldsymbol{\mathcal{MH}}^*)$, the corresponding macro-hybrid duality principle is as follows.

Corollary 3.3* *Let condition* $(\boldsymbol{C_{-\Lambda}})$ *be fulfilled. Then dual evolution macro-hybrid mixed problem* $(\boldsymbol{\mathcal{MH}^*})$ *is solvable if, and only if, dual evolution problem* $(\boldsymbol{\mathcal{D}})$ *is solvable.*

In relation with our previous work [6], compositional interpretations of primal admissibility characterization (5*) are the following.

Lemma 3.4* *The primal inclusion of problem* $(\boldsymbol{\mathcal{M}^*})$ *necessarily satisfies the compositional dualization*

$$-\Lambda^T p^* \in \partial F(u) - f^* \implies -\Lambda u \in \partial(F^* \circ (-\Lambda^T))(p^* + r_{f^*}), \qquad (6^*)$$

where $r_{f^*} \in \mathcal{Y}^*$ *is a fixed* $-\Lambda^T$*-preimage of function* f^**:* $-\Lambda^T r_{f^*} = f^*$*.*

In conclusion, from Lemma 3.4* and Theorem 3.2*, the following dual evolution composition duality principle for problem $(\boldsymbol{\mathcal{M}^*})$ is achieved.

Theorem 3.5* *Dual evolution mixed problem* $(\boldsymbol{\mathcal{M}^*})$ *has a solution, only if the dual evolution problem*

$$(\widetilde{\boldsymbol{\mathcal{D}}})\begin{cases} \text{Find } p^* \in \mathcal{X}^* : \\[2ex] 0 \in \dfrac{dp^*}{dt} + \partial G^*(p^*) + \partial(F^* \circ (-\Lambda^T))(p^* + r_{f^*}) + g, \quad \text{in } \mathcal{Y}, \\[2ex] p^*(0) = p_0^*, \end{cases}$$

has a solution. That is, if $(u, p^*) \in \mathcal{V} \times \mathcal{X}^*$ *is a solution of problem* $(\boldsymbol{\mathcal{M}^*})$ *then component* p^* *is a dual admissible solution of problem* $(\widetilde{\boldsymbol{\mathcal{D}}})$*. Moreover, compositional dual problem* $(\widetilde{\boldsymbol{\mathcal{D}}})$ *has a solution if dual evolution problem* $(\boldsymbol{\mathcal{D}})$ *has a solution.*

Further, the equivalence of global problem $(\boldsymbol{\mathcal{M}^*})$ and macro-hybrid problem $(\boldsymbol{\mathcal{MH}^*})$ leads to the macro-hybrid composition duality principle.

Theorem 3.6* *Dual evolution macro-hybrid mixed problem* $(\boldsymbol{\mathcal{MH}^*})$ *is solvable, only if dual evolution problem* $(\widetilde{\boldsymbol{\mathcal{D}}})$ *is solvable.*

On the other hand, as in [6], considering the classical dual evolution compatibility condition

$$(\boldsymbol{C_{F^*,-\Lambda^T}})\ \mathrm{int}\mathcal{D}(F^*) \cap \mathcal{R}(-\Lambda^T) \neq \emptyset,$$

the compositional operator equality

$$\partial(F^* \circ (-\Lambda^T)) = \Lambda \partial F^* \circ (-\Lambda^T) \qquad (7^*)$$

is valid, where $\text{int}\mathcal{D}(F^*)$ is the interior of the effective domain of the conjugate $F^* : V^* \to \Re \cup \{+\infty\}$ of functional F. As we mentioned for the primal evolution case, this compatibility condition turned out to be appropriate for distributed control mixed diffusion problems, now in a dual evolution sense. Hence, under classical condition $(\boldsymbol{C}_{\boldsymbol{F^*},-\boldsymbol{\Lambda^T}})$, the primal inclusion of problem $(\boldsymbol{\mathcal{M}}^*)$ is expressed equivalently by $u \in \partial F^* \circ (-\Lambda^T)(p^* + r^*_{f^*})$, and the sufficiency of the principle of Theorem 3.5* is additionally obtained (see [6], Theorem 2.2*). Thereby, an alternative dual evolution duality principle is the following.

Theorem 3.7* *Under classical compatibility condition* $(\boldsymbol{C}_{\boldsymbol{F^*},-\boldsymbol{\Lambda^T}})$*, dual evolution macro-hybrid mixed problem* $(\boldsymbol{\mathcal{MH}}^*)$*, equivalent to mixed problem* $(\boldsymbol{\mathcal{M}}^*)$*, has a solution if, and only if, dual evolution problem* $(\widetilde{\boldsymbol{\mathcal{D}}})$ *has a solution. Moreover, due to condition* $(\boldsymbol{C}_{-\boldsymbol{\Lambda}})$*, compositional dual problem* $(\widetilde{\boldsymbol{\mathcal{D}}})$ *is equivalent to dual evolution problem* $(\boldsymbol{\mathcal{D}})$*.*

4 Evolution Macro-hybrid Mixed Well-Posedness Analysis

On the basis of the evolution duality principles, established in Theorems 3.2 and 3.7 for the primal evolution case, and in Theorems 3.2* and 3.7* for the dual case, we next perform a well-posedness analysis of primal and dual mixed evolution problems $(\boldsymbol{\mathcal{M}})$ and $(\boldsymbol{\mathcal{M}}^*)$, respectively. The evolution existence results will follow from Akagi and Ôtani paper [2], based on Hilbert approximations.

4.1 Primal evolution analysis

Hence, according to the evolution duality principles, we determine the well-posednedss of primal evolution mixed problem $(\boldsymbol{\mathcal{M}})$, equivalent to macro-hybrid mixed problem $(\boldsymbol{\mathcal{MH}})$, through the analysis of primal evolution problems $(\boldsymbol{\mathcal{P}})$ and $(\widetilde{\boldsymbol{\mathcal{P}}})$ (see Theorems 3.2, 3.5 and 3.7). Toward this end, and in preparation for applying Akagi and Ôtani existence theorem, we introduce the composition primal superpotential

$$\widetilde{G}_g(v) = G \circ \Lambda(v - v_g), \ v \in V(\Omega), \tag{8}$$

v_g being the fixed Λ-preimage of Lemma 3.4, for which the effective domain and subdifferential relations

$$\mathcal{D}(\widetilde{G}_g) = \mathcal{D}(G \circ \Lambda) + v_g,$$
$$\partial \widetilde{G}_g(v) = \partial (G \circ \Lambda)(v - v_g), \tag{9}$$

hold. Then, assuming the Moreau-Rockafellar-Robinson condition (see [16])

$$(\boldsymbol{C}_{\boldsymbol{F},\widetilde{\boldsymbol{G}}_{\boldsymbol{g}}}) \ \mathrm{int}\mathcal{D}(F) \cap \mathcal{D}(\widetilde{G}_g) \neq \emptyset,$$

the primal subdifferential sum rule is satisfied,

$$\partial \varphi \equiv \partial (F + \widetilde{G}_g) = \partial F + \partial \widetilde{G}_g. \tag{10}$$

Therefore, primal evolution problem $(\widetilde{\boldsymbol{\mathcal{P}}})$ can be written in the classical subdifferential form

$$(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}}) \begin{cases} \text{Find } u \in \mathcal{W}: \\ 0 \in \dfrac{du}{dt} + \partial \varphi(u) - f^*, \quad \text{in } \mathcal{V}^*, \\ u(0) = u_0, \end{cases}$$

to which the following existence theorem applies in accordance with [2].

Theorem 4.1 *Let the coercivity and boundedness conditions*

$$(\boldsymbol{C1}_{\boldsymbol{\varphi}}) \ \|v\|^p_{V(\Omega)} - C_1 \|v\|^2_{H(\Omega)} - C_2 \leq C_3 \varphi(v), \forall v \in \mathcal{D}(\varphi), 2 \leq p < \infty,$$

$$(\boldsymbol{C2}_{\boldsymbol{\varphi}}) \ \|v^*\|^{p^*}_{V^*(\Omega)} \leq \ell(\|v\|_{H(\Omega)})\{\varphi(v) + 1\}, \forall v^* \in \partial \varphi(v),$$

ℓ a non-decreasing real function, be fulfilled. Then primal Cauchy problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}})$ possesses a unique solution.

Remark 4.2 The uniqueness of Theorem 4.1 follows as usual from the monotonicity of the primal subdifferential $\partial \varphi$, utilizing the time integration by parts formula, for $s < \tau$, $u, v \in \mathcal{W}$,

$$\int_s^\tau \left\langle \frac{du}{dt}(t), v(t) \right\rangle_{V(\Omega)} dt$$
$$= (u(\tau), v(\tau))_{H(\Omega)} - (u(s), v(s))_{H(\Omega)} - \int_s^\tau \left\langle \frac{dv}{dt}(t), u(t) \right\rangle_{V(\Omega)} dt. \tag{11}$$

The existence result is based on Hilbert approximations defined by the classical, uniquely solvable, Cauchy problems [11]

$$(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}_{\boldsymbol{\mathcal{H}}}}})_n \begin{cases} \text{Find } u_n \in \mathcal{U} : \\ 0 \in \dfrac{du_n}{dt} + \partial\varphi_H(u_n) - f_n^*, \quad \text{in } \mathcal{H}, \\ u_n(0) = u_{0_n}, \end{cases}$$

where $\mathcal{U} = \{v : v \in \mathcal{V}, dv/dt \in \mathcal{H} = L^2(0,T; H(\Omega))\} \subset \mathcal{W}$ is a reflexive Banach space, $\varphi_H : H(\Omega) \to \Re \cup \{+\infty\}$ is such that $\varphi_H = \varphi$ in $V(\Omega)$ and $+\infty$ in $H(\Omega) \backslash V(\Omega)$, a proper lower semicontinuous functional whose $\partial\varphi_H \subset \partial\varphi$. Further, $f_n^* \to f^* \in \mathcal{V}^*$ and $u_{0_n} \to u_0 \in H(\Omega)$ are strongly convergent sequences as $n \to +\infty$. Then, under conditions $(\boldsymbol{C1_\varphi})$ and $(\boldsymbol{C2_\varphi})$, the sequence of problems $\{(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}_{\boldsymbol{\mathcal{H}}}}})_n\}$ converge weakly to problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}})$, classical subdifferential version of primal evolution problem $(\widetilde{\boldsymbol{\mathcal{P}}})$ (see [2], Theorem 3.2).

Therefore, from Theorems 3.7 and 4.1, the evolution macro-hybrid mixed solvability of the theory is achieved.

Theorem 4.3 *Let compatibility condition* $(\boldsymbol{C_{G,\Lambda}})$*, and functional conditions* $(\boldsymbol{C1}_\varphi)$ *and* $(\boldsymbol{C2}_\varphi)$*, be fulfilled. Then primal evolution macro-hybrid mixed problem* $(\boldsymbol{\mathcal{MH}})$ *is uniquely solvable.*

Furthermore, a priori bounds for the solutions of primal evolution problems $(\boldsymbol{\mathcal{M}})$ and $(\boldsymbol{\mathcal{MH}})$ can be established assuming the following operator versions of functional coercivity and boundedness conditions $(\boldsymbol{C1}_\varphi)$ and $(\boldsymbol{C2}_\varphi)$:

$$(\boldsymbol{C1_{\partial\varphi}})\ \exists\ \alpha > 0 : \ \langle v^*, v\rangle_{V(\Omega)} \geq \alpha \|v\|^p_{V(\Omega)}, \ v^* \in \partial\varphi(v), \forall v \in V(\Omega),$$

$$(\boldsymbol{C2_{\partial\varphi}})\ \|v^*\|_{V^*(\Omega)} \leq \beta(\|v\|_{V(\Omega)}), \ v^* \in \partial\varphi(v), \forall v \in V(\Omega),$$

where β is an strictly increasing continuous function from $\Re^+ \to \Re^+$. Also, we shall assume the dual coercivity condition

$$(\boldsymbol{C1_{\partial G^*}})\ \exists\ \beta^* > 0 : \ \langle q^*, q\rangle_{Y(\Omega)} \geq \beta^* \|q^*\|^{p^*}_{Y^*(\Omega)}, \ q \in \partial G^*(q^*), \forall q^* \in Y^*(\Omega),$$

and denote by m_Λ the boundedness constant of the coupling operator $\Lambda \in \mathcal{L}(V(\Omega), Y(\Omega))$.

Theorem 4.4 *Let conditions* $(\boldsymbol{C_{G,\Lambda}})$, $(\boldsymbol{C1_{\partial\varphi}})$, $(\boldsymbol{C2_{\partial\varphi}})$ *and* $(\boldsymbol{C1_{\partial G^*}})$ *be satisfied, and let* $(u,p^*) \in \mathcal{W} \times \mathcal{Y}^*$ *be a solution of primal evolution mixed problem* $(\boldsymbol{\mathcal{M}})$. *Then the estimates*

$$\|u(\tau)\|_{H(\Omega)} \leq \|u_0\|_{H(\Omega)} + c_1\Big\{ \int_0^\tau \|f^*(t)\|^{p^*}_{V^*(\Omega)}\, dt\Big\}^{1/2}, \quad \forall \tau \in [0,T], \tag{12}$$

$$\|u\|_{\mathcal{V}} \leq c_2(\alpha)\|u_0\|^{2/p}_{H(\Omega)} + c_3(\alpha)\|f^*\|^{p^*/p}_{\mathcal{V}^*}, \tag{13}$$

$$\Big\|\frac{du}{dt}\Big\|_{\mathcal{V}^*} \leq \beta(c_2(\alpha))\|u_0\|^{2/p}_{H(\Omega)} + c_3(\alpha)\|f^*\|^{p^*/p}_{\mathcal{V}^*} + \|f^*\|_{\mathcal{V}^*}, \tag{14}$$

and

$$\|p^*\|_{\mathcal{Y}^*} \leq \frac{m_\Lambda c_2(\alpha)}{\beta^*}\|u_0\|^{2/p}_{H(\Omega)} + \frac{m_\Lambda c_3(\alpha)}{\beta^*}\|f^*\|^{p^*/p}_{\mathcal{V}^*} + \frac{1}{\beta^*}\|g\|_{\mathcal{Y}}, \tag{15}$$

hold true. Consequently, $u \in L^\infty(0,T;H(\Omega))$ *and the mapping* $(u_0, f^*, g) \in H(\Omega) \times \mathcal{V}^* \times \mathcal{Y} \mapsto (u,p^*) \in \mathcal{W} \times \mathcal{Y}^*$ *is bounded.*

Proof Let $u \in \mathcal{W}$ be solution of primal problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}})$ and $\tau \in [0,T]$. Then, appling integration by parts formula (11) and coercivity condition $(\boldsymbol{C1_{\partial\varphi}})$, for $u^* \in \partial\varphi(u)$, it follows that

$$\int_0^\tau \langle f^*(t), u(t)\rangle_{V(\Omega)}\, dt = \int_0^\tau \Big\langle \frac{du}{dt}(t), u(t)\Big\rangle_{V(\Omega)}\, dt + \int_0^\tau \langle u^*(t), u(t)\rangle_{V(\Omega)}\, dt$$

$$\geq \frac{1}{2}\|u(\tau)\|^2_{H(\Omega)} - \frac{1}{2}\|u_0\|^2_{H(\Omega)} + \alpha \int_0^\tau \|u(t)\|^p_{V(\Omega)}\, dt,$$

and applying Young's inequality, with constant $b > 0$,

$$\begin{aligned}\int_0^\tau \langle f^*(t), u(t)\rangle_{V(\Omega)}\, dt &\leq \int_0^\tau \|f^*(t)\|_{V^*(\Omega)}\|u(t)\|_{V(\Omega)}\, dt \\ &\leq \frac{1}{p^* b^{p^*}} \int_0^\tau \|f^*(t)\|^{p^*}_{V^*(\Omega)}\, dt + \frac{b^p}{p}\int_0^\tau \|u(t)\|^p_{V(\Omega)}\, dt.\end{aligned}$$

Hence, choosing $b > 0$ such that $c(\alpha) = \alpha - b^p/p > 0$, we obtain the estimate

$$\frac{1}{2}\|u(\tau)\|^2_{H(\Omega)} + c(\alpha)\int_0^\tau \|u(t)\|^p_{V(\Omega)}\, dt \leq \frac{1}{2}\|u_0\|^2_{H(\Omega)} + \frac{1}{p^* b^{p^*}}\int_0^\tau \|f^*(t)\|^{p^*}_{V^*(\Omega)}\, dt, \tag{16}$$

from which (12) and (13) are concluded. Estimate (14) is then derived from the variational equation of problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}})$ applying condition $(\boldsymbol{C2_{\partial\varphi}})$, as well as estimate (15) from the dual equation of problem ($\boldsymbol{\mathcal{M}}$) using dual coercivity condition ($\boldsymbol{C1_{\partial G^*}}$).

Next, we conclude this primal evolution analysis with the corresponding macro-hybrid a priori bounds for evolution problem ($\boldsymbol{\mathcal{MH}}$). Let m_{Λ^T} denote the boundedness constant of transpose coupling operator $\Lambda^T \in \mathcal{L}(\mathcal{Y}^*, \mathcal{V}^*)$. Further, taking into account the lower boundedness of the transpose primal trace operator $[\pi^T_{\Gamma_e}] \in \mathcal{L}(\mathcal{B}^*_{\{\Gamma_e\}}, \mathcal{V}^*_{\{\Omega_e\}})$, equivalent to macro-hybrid compatibility condition($\boldsymbol{C_{[\pi_{\Gamma_e}]}}$) [22], let $\beta_{\pi^T_\Gamma}$ denote its lower boundedness constant.

Theorem 4.5 *Let conditions of Theorem 4.4, as well as macro-hybrid condition* ($\boldsymbol{C_{[\pi_{\Gamma_e}]}}$)*, be satisfied. Then the solutions of primal evolution macro-hybrid mixed problem* ($\boldsymbol{\mathcal{MH}}$), $(\{u_e\}, \{p^*_e\}, \{\lambda^*_e\}) \in \boldsymbol{\mathcal{W}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{Y}}^*_{\{\Omega_e\}} \times \boldsymbol{\mathcal{B}}^*_{\{\Gamma_e\}}$*, are such that estimates* $(12) - (15)$ *hold true, in the corresponding product sense, with the macro-hybrid a priori bound*

$$\begin{aligned}\|\{\lambda^*_e\}\|_{\mathcal{B}^*_{\{\Omega_e\}}} \leq \frac{1}{\beta_{\pi^T_\Gamma}}\Big(\frac{m_{\Lambda^T} m_\Lambda}{\beta^*} + \beta(\|u\|_{V(\Omega)}) + m_\Lambda\Big) c_2(\alpha)\|u_0\|^{2/p}_{H(\Omega)} \\ + \frac{1}{\beta_{\pi^T_\Gamma}}\Big(\Big(\frac{m_{\Lambda^T} m_\Lambda}{\beta^*} + \beta(\|u\|_{V(\Omega)}) + m_\Lambda\Big) c_3(\alpha) + 1\Big)\|f^*\|^{p^*/p}_{\mathcal{V}^*} \\ + \frac{1}{\beta_{\pi^T_\Gamma}}\|f^*\|^{p^*/p}_{\mathcal{V}^*} + \frac{m_{\Lambda^T}}{\beta_{\pi^T_\Gamma}\beta^*}\|g\|_{\mathcal{Y}}.\end{aligned} \tag{17}$$

Consequently, $\{u_e\} \in L^\infty(0,T;H(\Omega))$ *and the mapping* $(u_0, f^*) \in H(\Omega) \times \mathcal{V}^* \mapsto (\{u_e\}, \{p^*_e\}, \{\lambda^*_e\}) \in \boldsymbol{\mathcal{W}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{Y}}^*_{\{\Omega_e\}} \times \boldsymbol{\mathcal{B}}^*_{\{\Gamma_e\}}$ *is bounded.*

Proof Estimate (17) follows from the local primal equations of problem ($\boldsymbol{\mathcal{MH}}$), considering the lower boundedness of the transpose operator $[\pi^T_{\Gamma_e}]$, equivalent to surjectivity condition ($\boldsymbol{C_{[\pi_{\Gamma_e}]}}$).

4.2 Dual evolution analyisis

Next, we continue with the analysis of dual evolution mixed problem ($\boldsymbol{\mathcal{M}^*}$), equivalent to macro-hybrid mixed problem ($\boldsymbol{\mathcal{MH}^*}$), in terms of dual evolution problems ($\boldsymbol{\mathcal{D}^*}$) and ($\widetilde{\boldsymbol{\mathcal{D}^*}}$) (see Theorems 3.2*, 3.5* and 3.7*). For the application of Akagi and Ôtani existence theorem, we introduce the composition dual superpotential

$$\widetilde{F}^*_{f^*}(q^*) = F^* \circ (-\Lambda^T)(q^* + r_{f^*}),\ q^* \in Y^*(\Omega), \tag{8*}$$

where r_{f^*} is the fixed $-\Lambda^T$-preimage of Lemma 3.4*, and whose effective domain and subdifferential are such that

$$\mathcal{D}(\widetilde{F}^*_{f^*}) = \mathcal{D}(F^* \circ (-\Lambda^T)) - r_{f^*},$$
$$\partial \widetilde{F}^*_{f^*}(q^*) = \partial(F^* \circ (-\Lambda^T))(q^* + r_{f^*}). \tag{9*}$$

Hence, assuming the Moreau-Rockafellar-Robinson condition [16]

$$(\boldsymbol{C}_{\boldsymbol{G^*},\widetilde{\boldsymbol{F}}^*_{\boldsymbol{f^*}}})\ \mathrm{int}\mathcal{D}(G^*) \cap \mathcal{D}(\widetilde{F}^*_{f^*}) \neq \emptyset,$$

the dual subdifferential sum rule is guaranteed:

$$\partial\varphi^* \equiv \partial(G^* + \widetilde{F}^*_{f^*}) = \partial G^* + \partial \widetilde{F}^*_{f^*}. \tag{10*}$$

Thereby, dual evolution problem $(\widetilde{\boldsymbol{\mathcal{D}}})$ is expressed in a classical subdifferential form by

$$(\widetilde{\widetilde{\boldsymbol{\mathcal{D}}}}) \begin{cases} \text{Find } p^* \in \mathcal{X}^* : \\ 0 \in \dfrac{dp^*}{dt} + \partial\varphi^*(p^*) + g, \quad \text{in } \mathcal{Y}, \\ u(0) = u_0, \end{cases}$$

and, according to [2], the following dual existence theorem applies.

Theorem 4.1* *Let the coercivity and boundedness conditions*

$$(\boldsymbol{C1}_{\boldsymbol{\varphi^*}})\ \|q^*\|^{p^*}_{Y^*(\Omega)} - C^*_1\|q^*\|^2_{Z(\Omega)} - C^*_2 \leq C^*_3\varphi^*(q^*), \forall q^* \in \mathcal{D}(\varphi^*), 2 \leq p^* < \infty,$$

$$(\boldsymbol{C2}_{\boldsymbol{\varphi^*}})\ \|q\|^p_{Y(\Omega)} \leq \ell^*(\|q^*\|_{Z(\Omega)})\{\varphi^*(q^*) + 1\}, \forall q \in \partial\varphi^*(q^*),$$

where ℓ^* is a non-decreasing real function, be satisfied. Then dual Cauchy problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{D}}}})$ is uniquely solvable.

Remark 4.2* The uniqueness of Theorem 4.1* follows from the monotonicity of the dual subdifferential $\partial\varphi^*$, applying the corresponding $\mathcal{X}$-time integration by parts formula. As for the primal evolution case, the existence result is based on Hilbert approximations defined by classical, uniquely solvable, dual Cauchy problems [11]

$$(\widetilde{\widetilde{\boldsymbol{\mathcal{D}}}}_{\boldsymbol{\mathcal{Z}}})_n \begin{cases} \text{Find } p^*_n \in \mathcal{U}^* : \\ 0 \in \dfrac{dp^*_n}{dt} + \partial\varphi^*_Z(p^*_n) + g_n, \quad \text{in } \mathcal{Z}, \\ p^*_n(0) = p^*_{0_n}. \end{cases}$$

Here, $\mathcal{U}^* = \{q^* : q^* \in \mathcal{Y}^*, dq^*/dt \in \mathcal{Z} = L^2(0,T;Z(\Omega))\} \subset \mathcal{X}$, a reflexive Banach space, $\varphi_Z^* : Z(\Omega) \to \Re \cup \{+\infty\}$ is such that $\varphi_Z^* = \varphi^*$ in $Y^*(\Omega)$ and $+\infty$ in $Z(\Omega)\backslash Y^*(\Omega)$, a proper lower semicontinuous functional for which $\partial\varphi_Z^* \subset \partial\varphi^*$. Also, $g_n \to g \in \mathcal{Y}$ as well as $p_{0_n}^* \to p_0^* \in Z(\Omega)$, are strongly convergent sequences for $n \to +\infty$. Moreover, under conditions $(\boldsymbol{C1_{\varphi^*}})$ and $(\boldsymbol{C2_{\varphi^*}})$, the sequence of dual problems $\{(\widetilde{\widetilde{\boldsymbol{\mathcal{D}}_{\mathcal{Z}}}})_n\}$ converge weakly to problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{D}}}})$, subdifferential version of dual evolution problem $(\widetilde{\boldsymbol{\mathcal{D}}})$ [2].

Consequently, from Theorems 3.7* and 4.1*, the dual evolution macro-hybrid mixed solvability of the theory is also achieved.

Theorem 4.3* *Let compatibility condition* $(\boldsymbol{C_{F^*,-\Lambda^T}})$*, and dual functional conditions* $(\boldsymbol{C1_{\varphi^*}})$ *and* $(\boldsymbol{C2_{\varphi^*}})$*, be satisfied. Then dual evolution macro-hybrid mixed problem* $(\boldsymbol{\mathcal{MH}^*})$ *possesses a unique solution.*

Moreover, proceeding as for the primal evolution case, a priori bounds for dual evolution problems $(\boldsymbol{\mathcal{M}^*})$ and $(\boldsymbol{\mathcal{MH}^*})$ can be derived in terms of operator versions of dual functional coercivity and boundedness conditions $(\boldsymbol{C1_{\varphi^*}})$ and $(\boldsymbol{C2_{\varphi^*}})$:

$$(\boldsymbol{C1_{\partial\varphi^*}})\ \exists\ \alpha^* > 0 :\ \langle q^*, q\rangle_{Y(\Omega)} \geq \alpha^* \|q^*\|_{Y^*(\Omega)}^{p^*},\ q \in \partial\varphi^*(q^*), \forall q^* \in Y^*(\Omega),$$

$$(\boldsymbol{C2_{\partial\varphi^*}})\ \|q\|_{Y(\Omega)} \leq \beta^*(\|q^*\|_{Y^*(\Omega)}),\ q \in \partial\varphi^*(q^*), \forall q^* \in Y^*(\Omega).$$

Here, β^* is an strictly increasing continuous function from $\Re^+ \to \Re^+$. Further, we shall assume the primal coercivity condition

$$(\boldsymbol{C1_{\partial F}})\ \exists\ \beta > 0 :\ \langle v^*, v\rangle_{V(\Omega)} \geq \beta \|v\|_{V(\Omega)}^{p},\ v^* \in \partial F(v), \forall v \in V(\Omega),$$

and denote by $m_{-\Lambda^T}^*$ the boundedness constant of the coupling operator $-\Lambda^T \in \mathcal{L}(Y^*(\Omega), V^*(\Omega))$.

Theorem 4.4* *Let conditions* $(\boldsymbol{C_{F^*,-\Lambda^T}})$, $(\boldsymbol{C1_{\partial\varphi^*}})$ *and* $(\boldsymbol{C2_{\partial\varphi^*}})$ *be satisfied, and let* $(u, p^*) \in \mathcal{V} \times \mathcal{X}^*$ *be a solution of dual evolution mixed problem* $(\boldsymbol{\mathcal{M}^*})$*. Then the estimates*

$$\|p^*(\tau)\|_{Z(\Omega)} \leq \|p_0^*\|_{Z(\Omega)} + c_1^* \Big\{ \int_0^\tau \|g(t)\|_{Y(\Omega)}^p\, dt \Big\}^{1/2}, \quad \forall \tau \in [0,T], \tag{12*}$$

$$\|p^*\|_{\mathcal{Y}^*} \leq c_2^*(\alpha^*) \|p_0^*\|_{Z(\Omega)}^{2/p^*} + c_3^*(\alpha^*) \|g\|_{\mathcal{Y}}^{p/p^*}, \tag{13*}$$

$$\Big\| \frac{dp^*}{dt} \Big\|_{\mathcal{Y}} \leq \beta^*(c_2^*(\alpha^*)) \|p_0^*\|_{Z(\Omega)}^{2/p^*} + c_3^*(\alpha^*) \|g\|_{\mathcal{Y}}^{p/p^*} + \|g\|_{\mathcal{Y}}, \tag{14*}$$

and

$$\|u\|_{\mathcal{V}} \leq \frac{m_{-\Lambda^T} c_2^*(\alpha^*)}{\beta} \|p_0^*\|_{Z(\Omega)}^{2/p^*} + \frac{m_{-\Lambda^T} c_3^*(\alpha^*)}{\beta} \|g\|_{\mathcal{Y}}^{p/p^*} + \frac{1}{\beta} \|f^*\|_{\mathcal{V}^*}, \qquad (15^*)$$

hold true. Consequently, $p^* \in L^\infty(0,T; Z(\Omega))$ *and the mapping* $(p_0^*, g, f^*) \in Z(\Omega) \times \mathcal{Y} \times \mathcal{V}^* \mapsto (u, p^*) \in \mathcal{V} \times \mathcal{X}$ *is bounded.*

Now, we complete the dual evolution analysis with the macro-hybrid a priori bounds for evolution problem $(\boldsymbol{\mathcal{MH}^*})$.

Theorem 4.5* *Let conditions of Theorem 4.4***, as well as macro-hybrid condition* $(\boldsymbol{C}_{[\boldsymbol{\pi}_{\boldsymbol{\Gamma}_e}]})$*, be satisfied. Then the solutions of dual evolution macro-hybrid mixed problem* $(\boldsymbol{\mathcal{MH}^*})$*,* $(\{u_e\}, \{p_e^*\}, \{\lambda_e^*\}) \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{Y}}^*_{\{\Omega_e\}} \times \boldsymbol{\mathcal{B}}^*_{\{\Gamma_e\}}$*, are such that estimates* (12^*)*-*(15^*) *hold true, in the corresponding product sense, with the macro-hybrid a priori bound*

$$\begin{aligned} \|\{\lambda_e^*\}\|_{\mathcal{B}^*_{\{\Omega_e\}}} \leq{}& \frac{1}{\beta_{\pi_\Gamma^T}} \Big(\frac{m_{-\Lambda} m_{-\Lambda^T}}{\beta} + \beta^* (\|p^*\|_{Y^*(\Omega)}) + m_{-\Lambda^T} \Big) c_2^*(\alpha^*) \|p_0^*\|_{Z(\Omega)}^{2/p^*} \\ &+ \frac{1}{\beta_{\pi_\Gamma^T}} \Big(\Big(\frac{m_{-\Lambda} m_{-\Lambda^T}}{\beta} + \beta^* (\|p^*\|_{Y^*(\Omega)}) + m_{-\Lambda^*} \Big) c_3^*(\alpha^*) + 1 \Big) \|g\|_{\mathcal{Y}}^{p/p^*} \qquad (17^*) \\ &+ \frac{1}{\beta_{\pi_\Gamma^T}} \|g\|_{\mathcal{Y}}^{p/p^*} + \frac{m_{-\Lambda}}{\beta_{\pi_\Gamma^T} \beta} \|f^*\|_{\mathcal{V}^*}. \end{aligned}$$

Consequently, $\{p_e^*\} \in L^\infty(0,T; Z(\Omega))$ *and the mapping* $(p_0^*, g) \in Z(\Omega) \times \mathcal{Y} \mapsto (\{u_e\}, \{p_e^*\}, \{\lambda_e^*\}) \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{X}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{B}}^*_{\{\Gamma_e\}}$ *is bounded.*

Proof Estimate (17^*) is concluded from the local dual equations of problem $(\boldsymbol{\mathcal{MH}^*})$, taking into account the lower boundedness of the transpose operator $[\pi_{\Gamma_e}^T]$, equivalent to macro-hybrid condition $(\boldsymbol{C}_{[\boldsymbol{\pi}_{\boldsymbol{\Gamma}_e}]})$.

5 Stationary Macro-Hybrid Mixed Analysis

In this section, we analyze the stationary variational macro-hybrid mixed system, associated to primal and dual evolution problems $(\boldsymbol{\mathcal{MH}})$ and $(\boldsymbol{\mathcal{MH}^*})$, which reads as follows.

$$(\boldsymbol{MH}) \begin{cases} \text{Find } (\{u_{e_s}\}, \{p_{e_s}^*\}) \in \boldsymbol{V}_{\{\Omega_e\}} \times \boldsymbol{Y^*}_{\{\Omega_e\}} : \\ -\{\Lambda_e^T p_{e_s}^*\} - \{\pi_{\Gamma_e}^T \lambda_{e_s}^*\} \in \{\partial F_e(u_{e_s})\} - \{f_{e_s}^*\}, & \text{in } \boldsymbol{V^*}_{\{\Omega_e\}}, \\ \{\Lambda_e u_{e_s}\} \in \{\partial G_e^*(p_{e_s}^*)\} + \{g_{e_s}\}, & \text{in } \boldsymbol{Y}_{\{\Omega_e\}}, \end{cases}$$

synchronized by the internal boundary dual transmission problem

$$(\boldsymbol{T})\begin{cases}\text{Find } \{\lambda^*_{e_s}\} \in \boldsymbol{B}^*_{\{\Gamma_e\}} : \\ \{\pi_{\Gamma_e} u_{e_s}\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda^*_{e_s}\}), \quad \text{in } \boldsymbol{B}_{\{\Gamma_e\}}.\end{cases}$$

This stationary macro-hybrid mixed problem, in accordance with the macro-hybrid compositional dualization, (3), of Lemma 2.1, has the macro-hybridized version

$$(\boldsymbol{M_{MH}})\begin{cases}\text{Find } (\{u_{e_s}\}, \{p^*_{e_s}\}) \in \boldsymbol{V}_{\{\Omega_e\}} \times \boldsymbol{Y^*}_{\{\Omega_e\}} : \\ -\{\Lambda_e^T p^*_{e_s}\} \in \{\partial F_e(u_{e_s})\} + \boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\}) - \{f^*_{e_s}\}, & \text{in } \boldsymbol{V^*}_{\{\Omega_e\}}, \\ \{\Lambda_e u_{e_s}\} \in \{\partial G^*_e(p^*_{e_s})\} + \{g_{e_s}\}, & \text{in } \boldsymbol{Y}_{\{\Omega_e\}},\end{cases}$$

which, in fact, corresponds to the stationary problem of macro-hybridized primal and dual evolution mixed problems ($\boldsymbol{\mathcal{M}_{MH}}$) and ($\boldsymbol{\mathcal{M}^*_{MH}}$). Moreover, as for the evolution problems, regarding the macro-hybrid term $\boldsymbol{\partial}(I_Q \circ [\pi_{\Gamma_e}])(\{u_e\})$ as the primal transmission constraint of decomposition $(1)_1$, problem ($\boldsymbol{M_{MH}}$) turns out to be the macro-hybridization of the mixed problem

$$(\boldsymbol{M})\begin{cases}\text{Find } (u_s, p^*_s) \in V(\Omega) \times Y^*(\Omega) : \\ -\Lambda^T p^*_s \in \partial F(u_s) - f^*_s, & \text{in } V^*(\Omega), \\ \Lambda u_s \in \partial G^*(p^*_s) + g_s, & \text{in } Y(\Omega),\end{cases}$$

the common stationary problem of the global primal and dual evolution mixed problems ($\boldsymbol{\mathcal{M}}$) and ($\boldsymbol{\mathcal{M}^*}$).

5.1 Primal stationary analysis

For the primal analysis of stationary macro-hybrid mixed problem ($\boldsymbol{MH}$)-($\boldsymbol{T}$), as associated to primal evolution problem ($\boldsymbol{\mathcal{MH}}$), we consider the classical primal compatibility condition ($\boldsymbol{C_{G,\Lambda}}$) of Section 3 that guarantees compositional operator equality (7); that is $\partial(G \circ \Lambda) = \Lambda^T \partial G \circ \Lambda$. Then the following primal stationary composition duality principle is readily obtained.

Theorem 5.1 *Under compatibility condition* ($\boldsymbol{C_{G,\Lambda}}$)*, stationary problem* ($\boldsymbol{M}$) *has a solution in a primal mixed sense if, and only if, primal stationary problem*

$$(\boldsymbol{P})\begin{cases}\textit{Find } u_s \in V(\Omega) : \\ 0 \in \partial F(u_s) + \partial(G \circ \Lambda)(u_s - v_{g_s}) - f^*_s, \quad \textit{in } V^*(\Omega),\end{cases}$$

equivalently expressed by

$$(\widetilde{\widetilde{\boldsymbol{P}}})\begin{cases} \textit{Find } u_s \in V(\Omega) : \\ 0 \in \partial\varphi(u_s) - f_s^*, \quad \textit{in } V^*(\Omega), \end{cases}$$

has a solution, where v_{g_s} *is a* Λ*-preimage of function* g_s*:* $\Lambda v_{g_s} = g_s$*. That is, if* $(u_s, p_s^*) \in V(\Omega) \times Y^*(\Omega)$ *is a solution of mixed problem* $(\boldsymbol{M})$ *then* u_s *is a solution of primal problem* $(\boldsymbol{P})$ *and, conversely, if* $u_s \in V(\Omega)$ *is a solution of primal problem* $(\boldsymbol{P})$ *then there is a dual function* $p_s^* \in \partial G \circ \Lambda(u_s - v_{g_s})) \subset Y^*(\Omega)$ *such that* (u_s, p_s^*) *is a solution of mixed problem* $(\boldsymbol{M})$*.*

Therefore, from the variational equivalence of problems $(\boldsymbol{M}) \Leftrightarrow (\boldsymbol{MH})$-$(\boldsymbol{T})$, we can conclude the primal stationary duality principle of the theory.

Corollary 5.2 *Let compatibility conditions* $(\boldsymbol{C}_{[\boldsymbol{\pi}_{\boldsymbol{\Gamma_e}}]})$ *and* $(\boldsymbol{C}_{\boldsymbol{G},\boldsymbol{\Lambda}})$ *be fulfilled. Then stationary macro-hybrid problem* $(\boldsymbol{MH})$-$(\boldsymbol{T})$ *has a solution in a primal mixed sense if, and only if, primal stationary problem* $(\boldsymbol{P})$ *has a solution. That is, if* $(\{u_{e_s}\}, \{p_{e_s}^*\}, \{\lambda_{e_s}^*\}) \in \boldsymbol{V}_{\{\Omega_e\}} \times \boldsymbol{Y^*}_{\{\Omega_e\}} \times \boldsymbol{B}^*_{\{\Gamma_e\}}$ *is a solution of macro-hybrid mixed problem* $(\boldsymbol{MH})$-$(\boldsymbol{T})$ *then* $u_s = \{u_{e_s}\}$ *is a solution of primal problem* $(\boldsymbol{P})$ *and, conversely, if* $u_s \in V(\Omega)$ *is a solution of primal problem* $(\boldsymbol{P})$ *then there is a dual function* $p_s^* \in \partial G \circ \Lambda(u_s - v_{g_s}) \subset Y^*(\Omega)$ *and a dual macro-hybrid function* $\{\lambda_{e_s}^*\} \in \boldsymbol{\partial I}_Q(\{\pi_{\Gamma_e} u_{e_s}\}) \subset \boldsymbol{B}^*_{\{\Gamma_e\}}$ *such that* $(\{u_{e_s}\} = u_s, \{p_{e_s}^*\} = p_s^*, \{\lambda_{e_s}^*\})$ *is a solution of macro-hybrid mixed problem* $(\boldsymbol{MH})$*-* $(\boldsymbol{T})$*.*

Remark 5.3 Due to the potentiality of the stationary problems, macro-hybrid problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$ states the optimality conditions of the Lagrangian, for $(\{v_e\}, \{q_e^*\}, \{\mu_e^*\}) \in \boldsymbol{V_{\{\Omega_e\}}} \times \boldsymbol{Y^*_{\{\Omega_e\}}} \times \boldsymbol{B^*_{\{\Gamma_e\}}}$,

$$\begin{aligned} L_{MH-T}(\{v_e\}, \{q_e^*\}, \{\mu_e^*\}) &= \sum_{e=1}^{E} F_e(v_e) - \sum_{e=1}^{E} G_e^*(q_e^*) - I_{Q^*}(\{\mu_e^*\}) \\ &-\langle\{f_{e_s}^*\}, \{v_e\}\rangle_{V_{\{\Omega_e\}}} - \langle\{g_{e_s}\}, \{q_e^*\}\rangle_{Y^*_{\{\Omega_e\}}} + \langle\{q_e^*\}, \{\Lambda_e v_e\}\rangle_{Y_{\{\Omega_e\}}} \\ &+\langle\{\mu_e^*\}, \{\pi_{\Gamma_e} v_e\}\rangle_{B_{\{\Gamma_e\}}}, \end{aligned} \tag{18}$$

which is, in fact, the macro-hybrid version of the two-field Lagrangian, for $(v, q^*) \in V(\Omega) \times Y^*(\Omega)$,

$$L_M(v, q^*) = F(v) - G^*(q^*) - \langle f_s^*, v\rangle_{V(\Omega)} - \langle g_s, q^*\rangle_{Y^*(\Omega)} + \langle q^*, \Lambda v\rangle_{Y(\Omega)} \tag{19}$$

of mixed problem $(\boldsymbol{M})$. Moreover, primal problem $(\boldsymbol{P})$ states the optimality condition of the "potential energy" functional

$$L_P(v) = F(v) + G(\Lambda v - g_s) - \langle f_s^*, v\rangle_{V(\Omega)}, \tag{20}$$

which is superpotential (10), $\varphi = F + \widetilde{G}_g$, of evolution primal subdifferntial problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}})$.

Remark 5.4 From an optimization approach, potential primal stationary problem $(\widetilde{\widetilde{\boldsymbol{P}}})$ corresponds to the minimization problem

$$(\widetilde{\widetilde{\boldsymbol{O_P}}}) \begin{cases} \text{Find } u_s \in \mathcal{D}(\varphi) \in V(\Omega): \\ \varphi(u_s) \leq \varphi(v), \forall v \in V(\Omega). \end{cases}$$

Hence, primal problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{P}}}})$ is solvable if φ is coercive, and its solvability is unique if further φ is strictly convex (see, e.g. [11]).

As a complementary approach to the classical result of Remark 5.4, we shall apply the primal duality principle of Corollary 5.2 for the solvability analysis of problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$. Here we shall follow the fixed-point subdifferential strategy proposed in [9], for variational nonlinear constrained boundary value problems. Let $M : V(\Omega) \to V^*(\Omega)$ be a given m-linearly strongly monotone and a-Lipschitz continuous operator. Then the variational equation of primal problem $(\boldsymbol{P})$ is equivalently expressed by the M-preconditioned augmented equation, for $u_s^* \in \partial F(u_s)$,

$$\begin{aligned} & M(u_s) - r(u_s^* - f_s^*) \in (M + r\partial\widetilde{G_{g_s}})(u_s) \\ \iff & u_s = G^r_{u_s^*}(u_s) \equiv J^r_{M,\partial\widetilde{G_{g_s}}}(M(u_s) - r(u_s^* - f_s^*)), \end{aligned} \tag{21}$$

with exact penalization parameter $r > 0$. Here, $\widetilde{G_{g_s}} \equiv G \circ \Lambda((\cdot) - v_{g_s})$ with $\mathcal{D}(\widetilde{G_{g_s}}) = \mathcal{D}(G \circ \Lambda) + v_{g_s}$ (see (8) and (9)), and $J^r_{M,\partial\widetilde{G_{g_s}}} = (M + r\partial\widetilde{G_{g_s}})^{-1} : V^*(\Omega) \to V(\Omega)$ is the M-resolvent operator of the maximal monotone subdifferential $\partial\widetilde{G_{g_s}} = \partial(G \circ \Lambda)((\cdot) - v_{g_s})$, a well defined $1/m$-firm contraction [13, 4, 3]. Hence, primal problem $(\boldsymbol{P})$ is characterized by the M-resolvent fixed-point problem

$$(\widetilde{\boldsymbol{P}}) \begin{cases} \text{Find } u_s \in \mathcal{D}(\partial\widetilde{G_{g_s}}) \subset V(\Omega): \text{ for } u_s^* \in \partial F(u_s), \\ u_s = G^r_{u_s^*}(u_s), \end{cases}$$

in terms of which mixed existence analysis can be performed, as well as proximal-point resolution algorithms may be derived [3]. Hence, adopting as sufficient condition for existence the monotonicity of the composition primal operator,

$$(\boldsymbol{C}_{\boldsymbol{\partial}\widetilde{\boldsymbol{G}_{\boldsymbol{g_s}}}}) \begin{cases} \partial\widetilde{G_{g_s}} : V(\Omega) \to 2^{V^*(\Omega)} \text{ is linearly strongly monotone; i.e. } \exists\ \alpha > 0 : \\ \langle u^* - v^*, u - v\rangle_{V(\Omega)} \geq \alpha \|u - v\|^2_{V(\Omega)},\ \forall u, v \in V(\Omega), \\ u^* \in \partial\widetilde{G_{g_s}}(u), v^* \in \partial\widetilde{G_{g_s}}(v), \end{cases}$$

the following existence result is established.

Theorem 5.5 *Let condition* $(\boldsymbol{C}_{\boldsymbol{\partial}\widetilde{\boldsymbol{G}_{\boldsymbol{g_s}}}})$ *be satisfied. Then stationary primal problem* $(\boldsymbol{P})$ *possesses a unique solution.*

Proof The unique primal solvability is a consequence of the Banach Fixed-Point Theorem. Indeed, the α-monotonicity $\boldsymbol{C}_{\boldsymbol{\partial}\widetilde{\boldsymbol{G}_{\boldsymbol{g_s}}}}$, in conjunction with the $1/m$-firm contraction of M-resolvent operator $J^r_{M,\partial\widetilde{G_{g_s}}}$, guarantees the contraction property of the fixed-point operator $G^r_{u^*_s} : \mathcal{D}(\partial\widetilde{G_{g_s}}) \to \mathcal{D}(\partial\widetilde{G_{g_s}})$, with a sufficiently big exact penalization parameter $r > 0$ (see [3]).
Therefore, from Theorems 5.1, 5.4 and Corollary 5.2, the solvability of stationary macro-hybrid mixed problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$ is achieved.

Corollary 5.6 *Let conditions* $(\boldsymbol{C}_{[\boldsymbol{\pi}_{\boldsymbol{\Gamma_e}}]})$, $(\boldsymbol{C}_{\boldsymbol{G},\boldsymbol{\Lambda}})$ *and* $(\boldsymbol{C}_{\boldsymbol{\partial}\widetilde{\boldsymbol{G}_{\boldsymbol{g_s}}}})$ *be fulfilled. Then problem* $(\boldsymbol{MH})$-$(\boldsymbol{T})$*, equivalent to stationary mixed problem* $(\boldsymbol{M})$*, has a unique solution in a primal mixed sense.*

5.2 Dual stationary analysis

Continuing with the dual analysis of stationary macro-hybrid mixed problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$, but now in a dual sense as associated to dual evolution problem $(\boldsymbol{\mathcal{MH}^*})$, we consider the dual compatibility condition $(\boldsymbol{C}_{\boldsymbol{F^*},-\boldsymbol{\Lambda^T}})$ under which compositional operator equality (7*) holds; that is $\partial(F^* \circ (-\Lambda^T)) = \Lambda\partial F^* \circ (-\Lambda^T)$. Then the dual stationary composition duality principle can be concluded.

Theorem 5.1* *Let compatibility condition* $(\boldsymbol{C}_{\boldsymbol{F^*},-\boldsymbol{\Lambda^T}})$ *be fulfilled. Then stationary problem* $(\boldsymbol{M})$ *is solvable in a dual mixed sense if, and only if, dual stationary problem*

$$(\boldsymbol{D}) \begin{cases} \textit{Find } p^*_s \in Y^*(\Omega) : \\ 0 \in \partial G^*(p^*_s) + \partial(F^* \circ (-\Lambda^T))(p^*_s + q^*_{f^*_s}) + g_s, \quad \textit{in } Y(\Omega), \end{cases}$$

equivalently expressed by

$$(\widetilde{\widetilde{\boldsymbol{D}}})\begin{cases} \textit{Find } p_s^* \in Y^*(\Omega) : \\ \\ 0 \in \partial\varphi^*(p_s^*) + g_s, \quad \textit{in } Y(\Omega), \end{cases}$$

is solvable, where $q^*_{f^*_s}$ is a $-\Lambda^T$-preimage of function f^*_s: $-\Lambda^T q^*_{f^*_s} = f^*_s$. That is, if $(u_s, p^*_s) \in V(\Omega) \times Y^*(\Omega)$ is a solution of mixed problem $(\boldsymbol{M})$ then p^*_s is a solution of dual problem $(\boldsymbol{D})$ and, conversely, if $p^*_s \in Y^*(\Omega)$ is a solution of dual problem $(\boldsymbol{D})$ then there is a primal function $u_s \in \partial F^* \circ (-\Lambda^T)(p^*_s + q^*_{f^*_s}) \subset V(\Omega)$ such that (u_s, p^*_s) is a solution of mixed problem $(\boldsymbol{M})$.

Therefore, from the variational equivalence of problems $(\boldsymbol{M})$ and $(\boldsymbol{MH})$-$(\boldsymbol{T})$, the dual stationary duality principle of the theory is achieved.

Corollary 5.2* *Let compatibility conditions $(\boldsymbol{C}_{[\boldsymbol{\pi_{\Gamma_e}}]})$ and $(\boldsymbol{C}_{\boldsymbol{F^*},-\boldsymbol{\Lambda^T}})$ be satisfied. Then stationary macro-hybrid problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$ is solvable in a dual mixed sense if, and only if, dual stationary problem $(\boldsymbol{D})$ is solvable. That is, if $(\{u_{e_s}\}, \{p^*_{e_s}\}, \{\lambda^*_{e_s}\}) \in \boldsymbol{V}_{\{\Omega_e\}} \times \boldsymbol{Y^*}_{\{\Omega_e\}} \times \boldsymbol{B}^*_{\{\Gamma_e\}}$ is a solution of macro-hybrid mixed problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$ then $p^*_s = \{p^*_{e_s}\}$ is a solution of dual problem $(\boldsymbol{D})$ and, conversely, if $p^*_s \in Y^*(\Omega)$ is a solution of dual problem $(\boldsymbol{D})$ then there is a primal function $u_s \in \partial F^* \circ (-\Lambda^T)(p^*_s + q^*_{f^*_s}) \subset V(\Omega)$ and a dual macro-hybrid function $\{\lambda^*_{e_s}\} \in \boldsymbol{\partial I}_Q(\{\pi_{\Gamma_e} u_{e_s}\}) \subset \boldsymbol{B}^*_{\{\Gamma_e\}}$ such that $(\{u_{e_s}\} = u_s, \{p^*_s\} = p^*_s, \{\lambda^*_{e_s}\})$ is a solution of macro-hybrid mixed problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$.*

Remark 5.3* As in the previous analysis, from the potentiality of the stationary problems, but now in a dual mixed sense, macro-hybrid problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$ states as before the optimality conditions of the Lagrangian L_{MH-T}, given by (18), that is the macro-hybrid version of the two-field Lagrangian L_M of mixed problem $(\boldsymbol{M})$, defined by (19). Moreover, now, dual problem $(\boldsymbol{D})$ states the optimality condition of the "complementary energy" functional

$$L_D(q^*) = G^*(q^*) + F^*(-\Lambda^T q^* + f^*_s) + \langle g_s, q^* \rangle_{Y^*(\Omega)}, \tag{20*}$$

which is superpotential (10*), $\varphi^* = G^* + F^*_{f^*}$, of evolution dual subdifferential problem $(\widetilde{\widetilde{\mathcal{D}}})$.

Remark 5.4* Potential dual stationary problem $(\widetilde{\widetilde{\boldsymbol{D}}})$ corresponds to the minimization problem

$$(\widetilde{\widetilde{\boldsymbol{O_D}}})\begin{cases}\text{Find } p_s^* \in \mathcal{D}(\varphi^*) \in Y^*(\Omega) : \\ \\ \varphi^*(p_s^*) \leq \varphi^*(y^*), \forall y^* \in Y^*(\Omega).\end{cases}$$

Hence, dual problem $(\widetilde{\widetilde{\boldsymbol{\mathcal{D}}}})$ is solvable if φ^* is coercive, and its solvability is unique if φ^* is strictly convex.

Next, proceeding as in the previous subsection, and as a complementary approach to Remark 4.5*, we apply the dual duality principle of Corollary 5.2*, for the solvability analysis of problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$, but now in a dual mixed sense, we introduce an m^*-linearly strongly monotone and a^*-Lipschitz continuous operator $M^* : Y^*(\Omega) \to Y(\Omega)$, and express the variational equation of dual problem $(\boldsymbol{D})$ equivalently, for $p_s \in \partial G^*(p_s^*)$, by

$$\begin{aligned} &M^*(p_s^*) - r^*(p_s + g_s) \in (M^* + r^*\partial \widetilde{F_{f_s^*}^*})(p_s^*) \\ &\iff p_s^* = F_{p_s}^{r^*}(p_s^*) \equiv J^{r^*}_{M^*,\partial \widetilde{F_{f_s^*}^*}}(M^*(p_s^*) - r^*(p_s + g_s)), \end{aligned} \tag{21*}$$

an M^*-preconditioned augmented equation, with exact penalization parameter $r^* > 0$. Here, $\widetilde{F_{f_s^*}^*} \equiv F^* \circ (-\Lambda^T)((\cdot) + q_{f_s^*}^*)$ with $\mathcal{D}(\widetilde{F_{f_s^*}^*}) = \mathcal{D}(F^* \circ (-\Lambda^T)) - q_{f_s^*}^*$ (see (8) and (9)) and $J^{r^*}_{M^*,\partial \widetilde{F_{f_s^*}^*}} = (M^* + r^*\partial \widetilde{F_{f_s^*}^*})^{-1} : Y(\Omega) \to Y^*(\Omega)$ is the M^*-resolvent operator of the maximal monotone subdifferential $\partial \widetilde{F_{f_s^*}^*} = \partial(F^* \circ (-\Lambda^T))((\cdot) + q_{f_s^*}^*)$, a well defined $1/m^*$-firm contraction [13, 4]. Hence, dual problem $(\boldsymbol{D})$ is characterized by the M^*-resolvent fixed-point problem

$$(\widetilde{\boldsymbol{D}})\begin{cases}\text{Find } p_s^* \in \mathcal{D}(\partial \widetilde{F_{f_s^*}^*}) \subset Y^*(\Omega) : \text{ for } p_s \in \partial G^*(p_s^*), \\ \\ p_s^* = F_{p_s}^{r^*}(p_s^*),\end{cases}$$

from which we can establish mixed existence as well as proximal-point resolution algorithms [3]. Hence, adopting the monotonicity of the composition dual operator,

$$(\boldsymbol{C}_{\boldsymbol{\partial \widetilde{F_{f_s^*}^*}}})\begin{cases}\partial \widetilde{F_{f_s^*}^*} : Y^*(\Omega) \to 2^{Y(\Omega)} \text{ is linearly strongly monotone; i.e. } \exists\ \alpha^* > 0 : \\ \\ \langle p - q, p^* - q^* \rangle_{Y^*(\Omega)} \geq \alpha^* \|p^* - q^*\|^2_{Y^*(\Omega)},\ \forall p^*, q^* \in Y^*(\Omega), \\ \\ p \in \partial \widetilde{F_{f_s^*}^*}(p^*), q \in \partial \widetilde{F_{f_s^*}^*}(q^*),\end{cases}$$

as in the previous subsection the following existence result is concluded.

Theorem 5.5* *Let condition* $(\boldsymbol{C}_{\partial \widetilde{\boldsymbol{F}^*_{f^*_s}}})$ *be fulfilled. Then stationary dual problem* $(\boldsymbol{D})$ *has a unique solution.*

Consequently, from Theorems 5.1*, 5.4* and Corollary 5.2*, the solvability of stationary macro-hybrid mixed problem $(\boldsymbol{MH})$-$(\boldsymbol{T})$ is obtained.

Corollary 5.6* *Let conditions* $(\boldsymbol{C}_{[\boldsymbol{\pi}_{\boldsymbol{\Gamma}_{\mathbf{e}}}]})$, $(\boldsymbol{C}_{\boldsymbol{F}^*,-\boldsymbol{\Lambda}^{\boldsymbol{T}}})$ *and* $(\boldsymbol{C}_{\partial \widetilde{\boldsymbol{F}^*_{f^*_s}}})$ *be satisfied. Then problem* $(\boldsymbol{MH})$-$(\boldsymbol{T})$*, equivalent to stationary mixed problem* $(\boldsymbol{M})$*, possesses a unique solution in a dual mixed sense.*

6 An Application: Control Mixed Nonlinear Diffusion Problem

In order to exemplify the present evolution variational theory, we elaborate in this final section on a distributed control macro-hybrid mixed nonlinear diffusion problem. The controlled diffusion physical process is locally modeled by the mixed system of conservation, constitutive and distributed control equations

$$
\left.\begin{aligned}
&\frac{du_e}{dt} + div\ \boldsymbol{w}_e = -b^*_e - s^*_e, \\
&\boldsymbol{w}_e = -\boldsymbol{grad}\ u_e, \\
&u_e \in \partial\psi^*_e(b^*_e), \\
&u_e \in \partial\phi^*_e(s^*_e),
\end{aligned}\right\} \text{ in } \Omega_e \times (0,T),
\qquad (22)
$$

$$
u_e(0) = u_{e_0}, \text{ in } \Omega_e,
$$

where $\{\Omega_e\}$, $e = 1, 2, ..., E$, corresponds to the disjoint and connected subdomains of a non-overlapping decomposition of the domain $\Omega \subset \Re^d$, $d \in \{1, 2, 3\}$, of the physical system that evolves along the time interval $(0, T)$, $T > 0$. Also, u_e denotes the diffusive local field of the process (a mass concentration or the temperature in some practical cases), with linear flux vector field $\boldsymbol{w}_e$, b^*_e corresponds to the divergence of the nonlinear flux vector field to be denoted by $\widetilde{\boldsymbol{w}}_e$, and s^*_e is the local control distributed source field. Further, $\partial\phi^*_e$ is the monotone subdifferential of a dual proper convex and lower semicontinuous local control mechanism [14]. In addition, for the local nonlinear diffusion constitutive equation, we have its potential and subdifferential

$$\psi_e(u_e) = \frac{1}{p}\int_\Omega \|\boldsymbol{grad}\ u_e\|^p d\Omega,\ 2 < p < +\infty,$$
$$\partial\psi_e(u_e) = -div\ (\|\boldsymbol{grad}\ u_e\|^{p-2}\ \boldsymbol{grad}\ u_e), \tag{23}$$

where the differential is in the Gâteaux sense [21].

Remark 6.1 For a classical interpretation of local nonlinear diffusion problem (22), we observe that by convex dualization, the nonlinear constitutive equation $u_e \in \partial\psi_e^*(b_e^*)$ is equivalently expressed by $b_e^* \in \partial\psi_e(u_e)$, ψ^* standing for the conjugate of ψ. Hence, taking into account definition $b_e^* = div\ \widetilde{\boldsymbol{w}}_e$, the nonlinear diffusion constitutive equation of the model example, is characterized by

$$\widetilde{\boldsymbol{w}}_e = -\|\boldsymbol{grad}\ u_e\|^{p-2}\boldsymbol{grad}\ u_e, \tag{24}$$

and, consequently, the total local flux vector field of diffusion is given by the sum $\boldsymbol{w}_e + \widetilde{\boldsymbol{w}}_e$.

For the sake of simplicity, we will consider the homogeneous Dirichlet boundary condition: for $e = 1, 2, ..., E$,

$$u_e = 0,\ \text{on}\ \partial\Omega_{D_e} \times (0, T), \tag{25}$$

where $\partial\Omega_{D_e} = \partial\Omega \cap \partial\Omega_e$, assuming the domain boundary, $\partial\Omega$, is Lipschitz continuous.

6.1 Primal evolution variational problem

As a primal Ω-field space $V(\Omega)$ of u-diffusive functions, and dual Ω-field space $Y^*(\Omega)$ of $\boldsymbol{w}$-linear flux vector, b^*-nonlinear flux divergence and s^*-distributed source functions, we shall consider the spatial functional framework of reflexive Banach Sobolev spaces [1]

$$V(\Omega) = W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega):\ \pi v = 0 \text{ a.e. on } \partial\Omega\},\ 2 < p < +\infty,$$
$$Y^*(\Omega) = \boldsymbol{L}^{p^*}(\Omega) \times W^{-1,p^*}(\Omega) \times W^{-1,p^*}(\Omega),\ p^* = p/(p-1), \tag{26}$$

which are decomposable in the sense of formulae (1). Hence, $Y^*(\Omega)$ is the topological dual of the space $Y(\Omega) = \boldsymbol{L}^p(\Omega) \times W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, and the continuous linear internal boundary trace operator satisfies macro-hybrid compatibility condition $(\boldsymbol{C}_{[\boldsymbol{\pi}_{\boldsymbol{\Gamma}_e}]})$; that is

$$(\boldsymbol{C}_{[\boldsymbol{\pi}_{\boldsymbol{\Gamma}_e}]_{cd}})\ [\pi_{\Gamma_e}] \in \mathcal{L}(\boldsymbol{V}_{\{\Omega_e\}}, \boldsymbol{B}_{\{\Gamma_e\}} \text{ is surjective,}$$

where $\boldsymbol{V}_{\{\Omega_e\}} \equiv \prod_{e=1}^{E} W_0^{1,p}(\Omega_e)$ and $\boldsymbol{B}_{\{\Gamma_e\}} \equiv \prod_{e=1}^{E} W^{1/p^*,p}(\Gamma_e)$. Moreover, corresponding primal transmission admissibility subspace $\boldsymbol{Q} \subset \boldsymbol{B}_{\{\Gamma_e\}}$ of Dirichlet interface traces, continuous in the $W^{1/p^*,p}$-sense, has as polar subspace the dual transmission admissibility subspace (2), $\boldsymbol{Q}^* \subset \boldsymbol{B}^*_{\{\Gamma_e\}} = \prod_{e=1}^{E} W^{-1/p^*,p^*}(\Gamma_e)$, of Neumann interface traces $[\delta_{\Gamma_e}] \in \mathcal{L}(\boldsymbol{L}^{p^*}_{\{\Omega_e\}} = \prod_{e=1}^{E} \boldsymbol{L}^{p^*}(\Omega_e), \boldsymbol{B}^*_{\{\Gamma_e\}})$, continuous in the $W^{-1/p^*,p^*}$-weak sense.

Hence, proceeding as usual, by integration and application of the primal variational Green's formula, for $\boldsymbol{w} \in \boldsymbol{L}^{p^*}_{\Omega}$,

$$div\ \boldsymbol{w} + grad^T \boldsymbol{w} = \pi^T \delta \boldsymbol{w},\ \text{in } W^{-1,p^*}(\Omega), \tag{27}$$

to the conservation divergence equation $(22)_1$, the primal evolution macro-hybrid mixed variational formulation of control nonlinear diffusion problem (22)-(25), with spatial functional framework (26), turns out to be

$$(\boldsymbol{\mathcal{MH}}_{cd}) \begin{cases} \text{Find } \{u_e\} \in \boldsymbol{\mathcal{W}}_{\{\Omega_e\}} \text{ and } (\{\boldsymbol{w}_e\}, \{b^*_e\}, \{s^*_e\}) \in \boldsymbol{\mathcal{Y}^*}_{\{\Omega_e\}} : \\ -\Big(\{grad^T \boldsymbol{w}_e\} + \{b^*_e\} + \{s^*_e\}\Big) - \{\pi^T_{\Gamma_e} \lambda^*_e\} \\ \qquad \in \left\{\dfrac{du_e}{dt}\right\} + \{\partial 0(u_e)\}, \qquad \text{in } \boldsymbol{\mathcal{V}^*}_{\{\Omega_e\}}, \\ \{(\boldsymbol{grad}\ u_e, u_e, u_e)\} \in \{(\boldsymbol{w}_e, \partial\psi^*(b^*_e), \partial\phi^*(s^*_e))\}, \quad \text{in } \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}}, \\ \{u_e(0)\} = \{u_{0_e}\}, \end{cases}$$

synchronized by the internal boundary dual transmission problem

$$(\boldsymbol{\mathcal{T}}_{cd}) \begin{cases} \text{Find } \{\lambda^*_e\} \in \boldsymbol{\mathcal{B}}^*_{\{\Gamma_e\}} : \\ \{\pi_{\Gamma_e} u_e\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda^*_e\}), \quad \text{in } \boldsymbol{\mathcal{B}}_{\{\Gamma_e\}}, \end{cases}$$

where transmission field $\{\lambda^*_e\}$ corresponds to the internal boundary normal fluxes $[\delta_e \boldsymbol{w}_e] = \{\boldsymbol{w}_e \cdot \boldsymbol{n}_e\} \in \boldsymbol{B}^*_{\{\Gamma_e\}}$. Further, in virtue of Lemma 2.1, a global interpretation of this problem, $(\boldsymbol{\mathcal{MH}}_{cd})$-$(\boldsymbol{\mathcal{T}}_{cd})$, is given by the primal evolution mixed problem

$$(\boldsymbol{\mathcal{M}}_{cd}) \begin{cases} \text{Find } u \in \mathcal{W} \text{ and } (\boldsymbol{w}, b^*, s^*) \in \boldsymbol{\mathcal{Y}^*} : \\ -\Big(grad^T \boldsymbol{w} + b^* + s^*\Big) \in \dfrac{du}{dt} + \partial 0(u), \qquad \text{in } \mathcal{V}^*, \\ (\boldsymbol{grad}\ u, u, u) \in (\boldsymbol{w}, \partial\psi^*(b^*), \partial\phi^*(s^*)), \quad \text{in } \boldsymbol{\mathcal{Y}}, \\ u(0) = u_0, \end{cases}$$

which is equivalent in a solvability sense. Here, the presence of the zero subdifferential operator $\partial 0$ (whose inverse graph is not trivial) is just to emphasize the subdifferential duality approach of the theory.

Next, for the application of the composition duality principles established in Section 3, we have as primal evolution compatibility conditions $(\boldsymbol{C}_{\boldsymbol{\Lambda}^{\boldsymbol{T}}})$ and $(\boldsymbol{C}_{\boldsymbol{G},\boldsymbol{\Lambda}})$ the following:

$$(\boldsymbol{C}_{\boldsymbol{\Lambda}^{\boldsymbol{T}}{}_{cd}})\ grad^T \in \mathcal{L}(\boldsymbol{L}^{p^*}(\Omega), W^{-1,p^*}(\Omega)) \text{ has a closed range,}$$

$$(\boldsymbol{C}_{\boldsymbol{G},\boldsymbol{\Lambda}_{cd}})\ \mathrm{int}\mathcal{D}(\phi)\Big(\subset W_0^{1,p}(\Omega)\Big) \neq \emptyset,$$

the first one being classically satisfied [17], while the second one is a natural condition for control mechanisms [14]. Then, under condition $(\boldsymbol{C}_{\boldsymbol{G},\boldsymbol{\Lambda}_{cd}})$, from Theorem 3.7 macro-hybrid mixed problem $(\boldsymbol{\mathcal{MH}}_{cd})$-$(\boldsymbol{\mathcal{T}}_{cd})$ is solvable if, and only if, the primal evolution variational problem

$$(\widetilde{\boldsymbol{\mathcal{P}}}_{cd}) \begin{cases} \text{Find } u \in \mathcal{W}: \\ \\ 0 \in \dfrac{du}{dt} + grad^T \boldsymbol{grad}(u) + \partial\psi(u) + \partial\phi(u), \quad \text{in } \mathcal{V}^*, \\ \\ u(0) = u_0, \end{cases}$$

is solvable. Moreover, due to property $(\boldsymbol{C}_{\boldsymbol{\Lambda}^{\boldsymbol{T}}{}_{cd}})$, compositional primal problem $\widetilde{\boldsymbol{\mathcal{P}}}_{cd}$ may be interpreted by corresponding primal problem $\boldsymbol{\mathcal{P}}_{cd}$ of Theorem 3.2, with primal admissibility set, (4), characterized by its indicator subdifferential, (5).

Remark 6.2 We note that the primal component $\partial\psi(u)$ as a functional in $\mathcal{V}^*$, according to definitions (23), Green's formula (27) and boundary condition (25), has the explicit variational form

$$\partial\psi(u) = grad^T(\|\boldsymbol{grad}\ u\|^{p-2}\boldsymbol{grad})(u), \tag{28}$$

the nonlinear flux term of the primal evolution variational formulation.

In order to apply primal existence Theorem 4.1 to control nonlinear diffusion problem $(\widetilde{\boldsymbol{\mathcal{P}}}_{cd})$, we identify its superpotential and subdifferential, for $v \in W_0^{1,p}(\Omega)$, as

$$\varphi(v) = (\psi + \phi)(v) = \frac{1}{2}\int_\Omega \|\boldsymbol{grad}\ v\|^2 d\Omega + \frac{1}{p}\int_\Omega \|\boldsymbol{grad}\ v\|^p d\Omega + \phi(v), \tag{29}$$

$$\partial\varphi(v) = \partial(\psi+\phi)(v) = grad^T(1+\|\boldsymbol{grad}\, v\|^{p-2})\boldsymbol{grad}(v) + \partial\phi(v). \qquad (30)$$

Note that in this case the functional $v \mapsto \|\boldsymbol{grad}\, v\|^q_{L^q}$, for $q \geq 2$, is of class C^1, which guaranties the validity of the subdifferential sum rule. This potential and nonlinear diffusion operator is well known to satisfy corresponding boundedness and coercivity conditions $(\boldsymbol{C1_\psi})$ and $(\boldsymbol{C2_\psi})$, as well as their operator versions $(\boldsymbol{C1_{\partial\psi}})$ and $(\boldsymbol{C2_{\partial\psi}})$, of Section 4. In fact, the nonlinear variational operator $\partial\psi = grad^T(1+\|\boldsymbol{grad}\, v\|^{p-2})\boldsymbol{grad} : W_0^{1,p}(\Omega) \to W^{-1,p^*}(\Omega)$, is a Lipschitz continuous bounded, strongly monotone and coercive operator [19, 2]. On the other hand, in general lower semicontinuous convex functionals on Banach spaces are affinely bounded form below by Han -Banach theorem and, consequently, condition $(\boldsymbol{C1_\phi})$ for the primal control superpotential holds true. Therefore, assuming control mechanisms for the problem such that their superpotential $\partial\phi$ satisfies the qualifying conditions, the variational results given by Theorems 4.3, 4.4 and 4.5 apply to primal evolution macro-hybrid mixed model example $(\boldsymbol{\mathcal{MH}}_{cd})$-$(\boldsymbol{\mathcal{T}}_{cd})$. Furthermore, relative to associated primal stationary problems, we note that the nonlinear diffusion operator $\partial\psi$ is further linearly strongly monotone and, consequently, with compatible distributed controls for corresponding monotone condition $(\boldsymbol{C}_{\boldsymbol{\partial}\widetilde{\boldsymbol{G}_{\boldsymbol{gs}}}_{cd}})$, from Corollary 5.5 the existence and uniqueness of solutions to the macro-hybrid stationary problem $(\boldsymbol{MH}_{cd})$-$(\boldsymbol{T}_{cd})$ is guaranteed.

6.2 Dual evolution variational problem

Finally, we next consider the dual evolution variational formulation of local control nonlinear diffusion mixed model example (22)-(25). For a dual approach of the problem, we shall regard now a primal Ω-field space $V(\Omega)$ of $\boldsymbol{w}$-linear flux vector functions, and a dual Ω-field space $Y^*(\Omega)$ of u-diffusive functions, given by

$$\begin{aligned} &V(\Omega) = \boldsymbol{W}(div;\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) : div\ \boldsymbol{v} \in L^p(\Omega)\},\ 2 < p < +\infty, \\ &Y^*(\Omega) = L^{p^*}(\Omega),\ p^* = p/(p-1), \end{aligned} \qquad (26^*)$$

decomposable in accordance with formulae (1), with continuous linear internal boundary trace operator, satisfying macro-hybrid compatibility condition $(\boldsymbol{C}_{[\boldsymbol{\pi_{\Gamma_e}}]})$ in a dual sense; that is

$$(\boldsymbol{C}_{[\boldsymbol{\pi_{\Gamma_e}}]_{cd^*}}) \begin{cases} [\pi_{\Gamma_e}] \in \mathcal{L}(\boldsymbol{V}_{\{\Omega_e\}} = \prod_{e=1}^E \boldsymbol{H}(div;\Omega_e), \boldsymbol{B}_{\{\Gamma_e\}} = \prod_{e=1}^E W^{-1/p^*,p^*}(\Gamma_e)) \\ \\ \text{is surjective.} \end{cases}$$

Thus, primal transmission admissibility subspace $\boldsymbol{Q} \subset \boldsymbol{B}_{\{\Gamma_e\}}$ is of Neumann interface traces, continuous in the $W^{-1/p^*,p^*}$-weak sense, while its polar subspace

(2), $\boldsymbol{Q^*} \subset \boldsymbol{B}^*_{\{\Gamma_e\}} = \prod_{e=1}^{E} W^{1/p^*,p}(\Gamma_e)$, of Dirichlet internal boundary traces $[\delta_{\Gamma_e}] \in \mathcal{L}(\boldsymbol{L}^{p^*}_{\{\Omega_e\}} = \prod_{e=1}^{E} \boldsymbol{L}^{p^*}(\Omega_e), \boldsymbol{B}^*_{\{\Gamma_e\}})$, continuous in the $W^{1/p^*,p}$-sense.

Hence, by integration and application of dual variational Green's formula, for $u \in L^{p^*}$,

$$\boldsymbol{grad}\, u + \boldsymbol{div^T} u = -\boldsymbol{\pi^T} \delta u, \text{ in } (\boldsymbol{W}(div;\Omega))^*, \qquad (27^*),$$

to the gradient linear constitutive equation $(22)_2$, the dual evolution variational formulation of the model example, with spatial functional framework (26^*), is given by

$$(\boldsymbol{\mathcal{MH}^*}_{cd}) \begin{cases} \text{Find } \{\boldsymbol{w}_e\} \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}} \text{ and } \{u_e\} \in \boldsymbol{\mathcal{X}^*}_{\{\Omega_e\}} : \\ -\{\boldsymbol{div}^T u_e\} - \{\boldsymbol{\pi}^T_{\Gamma_e} \lambda^*_e\} = \{\boldsymbol{w}_e\}, & \text{in } \boldsymbol{\mathcal{V}^*}_{\{\Omega_e\}}, \\ \{div\ \boldsymbol{w}_e\} \in \left\{\dfrac{du_e}{dt}\right\} + \{\partial\psi(u_e)\} + \{\partial\phi(u_e)\} & \text{in } \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}}, \\ \{u_e(0)\} = \{u_{0_e}\}. \end{cases}$$

This problem is synchronized as in the primal evolution case, by the internal boundary dual transmission common problem $(\boldsymbol{\mathcal{T}}_{cd})$. Then, as before, by macro-hybrid compositional dualization (3), a global interpretation of macro-hybrid problem $(\boldsymbol{\mathcal{MH}^*}_{cd})$-$(\boldsymbol{\mathcal{T}}_{cd})$ follows as a dual evolution mixed problem.

$$(\boldsymbol{\mathcal{M}^*}_{cd}) \begin{cases} \text{Find } \boldsymbol{w} \in \boldsymbol{\mathcal{V}} \text{ and } u \in \mathcal{X}^* : \\ -\boldsymbol{div}^T u = \boldsymbol{w}, & \text{in } \boldsymbol{\mathcal{V}^*}, \\ div\ \boldsymbol{w} \in \dfrac{du}{dt} + \partial\psi(u) + \partial\phi(u) & \text{in } \mathcal{Y}, \\ u(0) = u_0. \end{cases}$$

For this dual evolution formulation, compatibility conditions $(\boldsymbol{C_{-\Lambda}})$ and $(\boldsymbol{C_{F^*,-\Lambda^T}})$ of the dual composition duality principles of Section 3, result to be

$$(\boldsymbol{C_{-\Lambda}}_{cd}) \ -div \in \mathcal{L}(\boldsymbol{W}(div;\Omega), L^p(\Omega)) \text{ has a closed range},$$

$$(\boldsymbol{C_{F^*,-\Lambda^T}}_{cd})\ \boldsymbol{\mathcal{R}}(-\boldsymbol{div}^T)\Big(\subset (\boldsymbol{W}(div;\Omega))^*\Big) \neq \emptyset,$$

the first one being a classical property [17], and the second one trivially satisfied. Thereby, from Theorem 3.7*, macro-hybrid mixed problem $(\boldsymbol{\mathcal{MH}^*}_{cd})$-$(\boldsymbol{\mathcal{T}}_{cd})$ has a solution if, and only if, the dual evolution variational problem

$$(\widetilde{\boldsymbol{\mathcal{D}}}_{cd})\begin{cases} \text{Find } u \in \mathcal{X}^* : \\ 0 \in \dfrac{du}{dt} + div\ \boldsymbol{div}^T(u) + \partial\psi(u) + \partial\phi(u), \quad \text{in } \mathcal{Y}, \\ u(0) = u_0, \end{cases}$$

has a solution. Moreover, due to property $(\boldsymbol{C}_{-\boldsymbol{\Lambda}_{cd}})$, compositional dual problem $\widetilde{\boldsymbol{\mathcal{D}}}_{cd}$ may be interpreted by corresponding dual problem $\boldsymbol{\mathcal{D}}_{cd}$ of Theorem 3.2*, with dual admissibility set, (4*), characterized by its indicator subdifferential, (5*).

Remark 6.2* For this dual formulation, the component $\partial\psi(u)$, now as a functional in $\mathcal{Y}$, and according to definitions (23), dual Green's formula (27*) and Dritchlet boundary condition (25), has the explicit form

$$\partial\psi(u) = div\ (\|\boldsymbol{grad}\ u\|^{p-2}\ \boldsymbol{div}^T)(u), \tag{28*}$$

the nonlinear flux term of the dual evolution variational formulation.

For the application of dual existence Theorem 4.1* to control nonlinear diffusion problem $(\widetilde{\boldsymbol{\mathcal{D}}}_{cd})$, we identify its superpotential and subdifferential in a dual sense, for $v \in L^{p^*}(\Omega)$, by

$$\varphi(v) = (\psi + \phi)(v) = \frac{1}{2}\int_\Omega \|\boldsymbol{div}^T\ v\|^2 d\Omega + \frac{1}{p^*}\int_\Omega \|\boldsymbol{div}^T\ v\|^p d\Omega + \phi(v), \tag{29*}$$

$$\partial\varphi(v) = \partial(\psi + \phi)(v) = div\ (1 + \|\boldsymbol{grad}\ v\|^{p-2})\ \boldsymbol{div}^T(v) + \partial\phi(v). \tag{30*}$$

For this dual evolution formulation, its potential and nonlinear diffusion operator, now in an L^{p^*}-variational sense, satisfy boundedness and coercivity conditions $(\boldsymbol{C1}_{\boldsymbol{\varphi^*}})$ and $(\boldsymbol{C2}_{\boldsymbol{\varphi^*}})$, as well as their operator versions $(\boldsymbol{C1}_{\boldsymbol{\partial\varphi^*}})$ and $(\boldsymbol{C2}_{\boldsymbol{\partial\varphi^*}})$, of Section 4 [19]. Consequently, assuming that control operator $\partial\phi$ satisfies such conditions in such a sense too, the qualitative results established by Theorems 4.3*, 4.4* and 4.5* are valid for dual evolution macro-hybrid mixed model problem $(\boldsymbol{\mathcal{MH}^*}_{cd})$-$(\boldsymbol{\mathcal{T}}_{cd})$. On the other hand, we note that for the associated dual stationary problems, the nonlinear diffusion operator $\partial\psi$ is in addition linearly strongly monotone and, assuming distributed controls compatible with corresponding monotone condition $(\boldsymbol{C}_{\boldsymbol{\partial}\widetilde{\boldsymbol{F}^*_{f_s}}_{cd}})$, the existence and uniqueness of solutions to the macro-hybrid dual stationary problem $(\boldsymbol{MH^*}_{cd})$-$(\boldsymbol{T}_{cd})$ follows from Corollary 5.5*.

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Received: February, 2011