A Sequence Space Defined by a Sequence of Modulus Functions

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Abstract. In the present paper we study a sequence space $m(F, \varphi, p)$ defined by a sequence of modulus functions and examine some topological properties of this space.

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1. Introduction and Preliminaries

A function $f : [0, \infty) \to [0, \infty)$ is said to be a modulus function if it satisfies the following:

1. $f(x) = 0$ if and only if $x = 0$;
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0, y \geq 0$;
3. $f$ is increasing;
4. $f$ is continuous from right at 0.

It follows that $f$ must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then $f(x)$ is unbounded. For more details see ([1], [2], [5] etc.).
Let \( w \) be the set of all sequences, real or complex numbers \( x = (x_k) \) normed by \( \|x\| = \sup_{k} |x_k| \), where \( k \in \mathbb{N} \), the set of positive integers.

Let \( \mathcal{C} \) denote the space whose elements are finite sets of distinct positive integers. Given any elements \( \sigma \) of \( \mathcal{C} \), we denote by \( c(\sigma) \) the sequence \( \{c_n(\sigma)\} \) which is such that \( c_n(\sigma) = 1 \) if \( n \in \sigma \), \( c_n(\sigma) = 0 \) otherwise. Further \( \mathcal{C}_s = \{\sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq s\} \).

Throughout the paper \( \varphi_n \) denotes a non-decreasing sequence of positive numbers such that \( n\varphi_{n+1} \leq (n+1)\varphi_n \) for all \( n \in \mathbb{N} \).

If \( x = (x_k) \) is a sequence, then \( S(x) \) denotes the set of all permutation of the elements of \( (x_k) \). A sequence space \( E \) is said to be symmetric if \( S(x) \subset E \) for all \( x \in E \). A sequence space \( E \) is said to be solid if \( (y_n) \in E \) whenever \( (x_n) \in E \) and \( |y_n| \leq |x_n| \) for all \( n \in \mathbb{N} \).

A BK-space is a Banach sequence space \( E \) in which the coordinate maps are continuous, i.e if \( (x_k^{(n)}) \in E \), then

\[
\|(x_k^{(n)}) - (x_k)\| \to 0 \text{ as } n \to \infty
\]

\[
\Rightarrow \|(x_k^{(n)}) - (x_k)\| \to 0 \text{ as } n \to \infty, \text{ for each fixed } k.
\]

The space \( m(\varphi) \) was defined and introduced by Sargent [4] as follows:

\[
m(\varphi) = \left\{ x = \{x_k\} \in w : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \left( \frac{1}{\varphi_s} \sum_{k \in \sigma} |x_k| \right) < \infty \right\}.
\]

The space \( m(\varphi) \) was extended to \( (m, \varphi, p) \) by Tripathy and Sen [6] as follows:

\[
m(\varphi, p) = \left\{ x = \{x_k\} \in w : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \left( \frac{1}{\varphi_s} \sum_{k \in \sigma} |x_k|^p \right)^{\frac{1}{p}} < \infty \right\}.
\]

Let \( F = (f_k) \) be a sequence of modulus functions. In this paper we define the sequence space \( m(F, \varphi, p) \) as:

\[
m(F, \varphi, p) = \left\{ x = (x_k) \in w : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\varphi_s} \left( \sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k|}{\rho} \right) \right]^p \right)^{\frac{1}{p}} < \infty, \text{ for some } \rho > 0 \right\}.
\]
2. Main Results

Theorem 1. The space $m(F, \varphi, p)$ is complete.

Proof. Let $\{x^{(n)}\}$ be a Cauchy sequence in $m(F, \varphi, p)$. Then

$$\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\varphi_s} \left\{ \sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k|}{\rho} \right) \right]^p \right\}^{1/p} < \infty,$$

for some $\rho > 0$ and for all $n \in \mathbb{N}$. For each $\epsilon > 0$, there exists a positive integer $n_0$ such that

$$\|x^{(m)} - x^{(n)}\|_{m(F, \varphi, p)} < \epsilon,$$

for all $m, n \geq n_0$. This implies that

$$\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\varphi_s} \left\{ \sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k - x_n|}{\rho} \right) \right]^p \right\}^{1/p} < \epsilon$$

(1)

for some $\rho > 0$ and for all $m, n \geq n_0$. Hence $|x_k^{(m)} - x_k^{(n)}| < \epsilon \varphi_1$ for all $m, n \geq n_0$ and for all $k \in \mathbb{N}$, showing that for each fixed $k(1 \leq k < \infty)$, the sequence $\{x_k^{(n)}\}$ is a Cauchy sequence in $\mathbb{C}$.

Let $x_k^{(n)} \rightarrow x_k$ as $n \rightarrow \infty$. We define $x = (x_1, x_2, \cdots)$. We need to show that $x \in m(F, \varphi, p)$ and $x^{(n)} \rightarrow x$. From (1) we get, for each fixed $s$.

$$\sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k^{(m)} - x_k^{(n)}|}{\rho} \right) \right]^p < \epsilon^p \varphi_s^p, \text{ for some } \rho > 0, \text{ for all } m, n \geq n_0 \text{ and } \sigma \in \mathcal{C}_s.$$

Taking $n \rightarrow \infty$, we get

$$\sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k|}{\rho} \right) \right]^p < \epsilon^p \varphi_s^p, \text{ for some } \rho > 0, \text{ for all } m, n \geq n_0 \text{ and } \sigma \in \mathcal{C}_s.$$

This implies that

$$\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\varphi_s} \left\{ \sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k - x_k|}{\rho} \right) \right]^p \right\}^{1/p} < \epsilon$$

(2)

for some $\rho > 0$, for all $m, n \geq n_0$.

$$\Rightarrow x^{(n)} - x \in m(F, \varphi, p), \text{ for all } n \geq n_0.$$

Hence $x = x^{(n_0)} + x - x^{(n_0)} \in m(F, \varphi, p)$ as $m(F, \varphi, p)$ is a linear space. From (2), $\|x^{(n)} - x\|_{m(F, \varphi, p)} < \epsilon$, for all $m, n \geq n_0$, which implies that $\|x^{(n)} - x\|_{m(F, \varphi, p)} \rightarrow 0$ as $n \rightarrow \infty$. Hence $m(F, \varphi, p)$ ($1 \leq p < \infty$) is a Banach space.
Theorem 2. The space \( m(F,\varphi,p) \) is a BK-space.

Proof. Suppose that \( \|x^{(n)} - x\|_{m(F,\varphi,p)} \to 0 \) as \( n \to \infty \). For each \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that
\[
\|x^{(n)} - x\| < \epsilon \quad \text{for all} \quad n \geq n_0.
\]
This implies that
\[
\sup_{s \geq 1} \sup_{\sigma \in C_s} \varphi_s \{ \sum_{k \in \sigma} \left[ f_k \left( \frac{|x^{(n)}_k - x_k|}{\rho} \right) \right]^p \}^{1/p} < \epsilon \quad \text{for some} \quad \rho > 0, \quad \text{for all} \quad n \geq n_0.
\]
Consequently
\[
|x^{(n)}_k - x_k| < \epsilon \varphi_1, \quad \text{for all} \quad n \geq n_0 \quad \text{and for all} \quad k.
\]
So \( |x^{(n)}_k - x_k| \to 0 \) as \( n \to \infty \). This completes the proof.

Corollary 3. (i) The space \( m(F,\varphi,p) \) is a symmetric space. If \( x \in m(F,\varphi,p) \) and \( v \in S(x) \), then
\[
\|v\|_{m(F,\varphi,p)} = \|x\|_{m(F,\varphi,p)}.
\]
(ii) The space \( m(F,\varphi,p) \) is a normal space.

Proof. It is obvious.

Theorem 4. \( m(\varphi) \subseteq m(F,\varphi,p) \).

Proof. Suppose that \( x \in m(\varphi) \). Then
\[
\|x\|_{m(\varphi)} = \sup_{s \geq 1} \sup_{\sigma \in C_s} \varphi_s \{ \sum_{k \in \sigma} |x_k| \} = K < \infty.
\]
Hence for each fixed \( s \),
\[
\sum_{k \in \sigma} |x_k| \leq K \varphi_s, \quad \sigma \in C_s, \quad \text{for some} \quad \rho > 0
\]
so that
\[
\sup_{s \geq 1} \sup_{\sigma \in C_s} \varphi_s \{ \sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k|}{\rho} \right) \right]^p \}^{1/p} \leq K, \quad \text{for some} \quad \rho > 0.
\]
Thus \( x \in m(F,\varphi,p) \). Hence \( m(\varphi) \subseteq m(F,\varphi,p) \).

Theorem 5. \( m(F,\varphi,p) \subseteq m(F,\psi,p) \) if and only if \( \sup_{s \geq 1} \frac{\varphi_s}{\psi_s} < \infty \).
Proof. Let $\sup_{s \geq 1} \left( \frac{\varphi_s}{\psi_s} \right) = K < \infty$. Then $\varphi_s \leq K \psi_s$. Now if $(x_k) \in m(F, \varphi, p)$, then
\[
\sup_{s \geq 1} \left( \frac{1}{\varphi_s} \left\{ \sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k|}{\rho} \right) \right]^p \right\}^{1/p} \right) < \infty, \text{ for some } \rho > 0.
\]
This implies that
\[
\sup_{s \geq 1} \left( \frac{1}{K \psi_s} \left\{ \sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k|}{\rho} \right) \right]^p \right\}^{1/p} \right) < \infty, \text{ for some } \rho > 0,
\]
so that $\| x \|_{m(F, \psi, p)} < \infty$. Hence $m(F, \varphi, p) \subseteq m(F, \psi, p)$. Conversely, suppose that $m(F, \varphi, p) \subseteq m(F, \psi, p)$. We need to show that
\[
\sup_{s \geq 1} \left( \frac{\varphi_s}{\psi_s} \right) = \sup_{s \geq 1} (\eta_s) < \infty.
\]
Let $\sup(\eta_s) = \infty$. Then there exists a subsequence $(\eta_{s_i})$ of $(\eta_s)$ such that $\lim_{i \to \infty} (\eta_{s_i}) = \infty$. Then for $(x_k) \in m(F, \varphi, p)$ we have
\[
\sup_{s \geq 1} \left( \frac{1}{\psi_s} \left\{ \sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k|}{\rho} \right) \right]^p \right\}^{1/p} \right) \geq \sup_{s \geq 1} \left( \frac{\psi_s}{\varphi_{s_i}} \left\{ \sum_{k \in \sigma} \left[ f_k \left( \frac{|x_k|}{\rho} \right) \right]^p \right\}^{1/p} \right)
\]
\[
= \infty.
\]
for some $\rho > 0$. This implies that $(x_k) \not\in m(F, \psi, p)$, a contradiction which completes the proof.

Theorem 6. $\ell^p \subseteq m(F, \varphi, p) \subset \ell^\infty$.

Proof. Since $m(F, \varphi, p) = \ell^p$ for $f_k(x) = x$ and $\varphi_n = 1$, for all $n \in \mathbb{N}$, it follows that $\ell^p \subseteq m(F, \varphi, p)$. Next, let $x \in m(F, \varphi, p)$. Then
\[
\sup_{s \geq 1} \left( \frac{1}{\varphi_s} \left\{ \sum_{n \in \sigma} \left[ f_k \left( \frac{|x_k|}{\rho} \right) \right]^p \right\}^{1/p} \right) = K < \infty, \text{ for some } \rho > 0.
\]
This implies that $|x_k| \leq K \varphi_1$, for all $k \in \mathbb{N}$, so that $x \in \ell^\infty$. Thus $m(F, \varphi, p) \subset \ell^\infty$.

References


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