A Nonlocal Problem for a Multi-Term Fractional-Order Differential Equation

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Abstract
In this paper we study the existence of solution for a nonlocal problems of a multi-term arbitrary (fractional) orders differential equation. The corresponding integral condition problem will be considered.

Keywords: Fractional calculus, nonlocal condition, integral condition, multi-term differential equation, multi-term fractional-orders functional integral equation

1 Introduction
Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to ([1]-[6] and [9]-[12]), and references therein. Recently, the authors (see [10]) studied the the existence of solution for the nonlocal problem

\[
\frac{dx}{dt} = f(t, D^\alpha x(t)), \quad t \in (0, 1] \quad \text{and} \quad \alpha \in (0, 1]
\]

\[
x(0) + \sum_{k=1}^{m} a_k x(t_k) = x_o, \quad t_k \in (0, 1]
\]
when \(f\) is \(L^1\)–Caratheodory.
In this work we study the existence of at least one solution for the nonlocal
problem of the arbitrary (fractional) order differential equation
\[ x'(t) = f(t, D^{\alpha_1}x(t), D^{\alpha_2}x(t), \ldots, D^{\alpha_n}x(t)), \alpha_i \in (0, 1), \text{ a.e. } t \in (0, 1) \]  
with the nonlocal condition
\[ \sum_{k=1}^{m} a_k x(\tau_k) = x_0, \quad \tau_k \in (a, b) \subset (0, 1). \]  
As an application, we deduce the existence of solution for the nonlocal problem of the differential equation (3) with the nonlocal integral condition
\[ \int_a^b x(s) \, ds = x_0, \quad (a, b) \subset (0, 1). \]  

2 preliminaries

Let \( L^1(I) \) denotes the class of Lebesgue integrable functions on the interval \( I = [0,1], \) where \( 0 \leq a < b < \infty \) and let \( \Gamma(.) \) denotes the gamma function. 

**Definition 2.1** The fractional-order integral of the function \( f \in L_1[a, b] \) of order \( \beta > 0 \) is defined by (see [14])
\[ I^\beta_a f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f(s) \, ds, \]

**Definition 2.2** The Caputo fractional-order derivative of \( f(t) \) of order \( \alpha \in (0, 1] \) is defined as (see [13] and [14])
\[ D^\alpha_a f(t) = I^{1-\alpha}_a \frac{d}{dt} f(t) = \int_a^t \frac{(t - s)^{-\alpha}}{\Gamma(1 - \alpha)} \frac{d}{ds} f(s) \, ds. \]

The following theorems will be needed

**Theorem 2.1** (Schauder fixed point theorem [7])
Let \( E \) be a Banach space and \( Q \) be a convex subset of \( E \), and \( T : Q \rightarrow Q \) is compact, continuous map, Then \( T \) has at least one fixed point in \( Q \).

**Theorem 2.2** (Kolmogorov compactness criterion [8])
Let \( \Omega \subseteq L^p(0, 1), 1 \leq p < \infty \).

(i) \( \Omega \) is bounded in \( L^p(0, 1) \), and

(ii) \( u_h \rightarrow u \) as \( h \rightarrow 0 \) uniformly with respect to \( u \in \Omega \), then \( \Omega \) is relatively compact in \( L^p(0, 1) \), where
\[ u_h(t) = \frac{1}{h} \int_t^{t+h} u(s) \, ds. \]
3 Main results

Consider firstly the fractional-order functional integral equation

\[ y(t) = f(t, I^{1-\alpha_1}y(t), \ldots, I^{1-\alpha_n}y(t)), \]  
(6)

**Definition 3.1** The function \( y \) is called a solution of the fractional-order functional integral equation (6), if \( y \in L_1[0,1] \) and satisfies (6).

Consider the following assumption

(i) \( f : [0,1] \times \mathbb{R}^n \to \mathbb{R} \) be a function with the following properties:

(a) for each \( t \in [0,1] \), \( f(t,.) \) is continuous,
(b) for each \( x \in \mathbb{R}^n \), \( f(.,x) \) is measurable,

(ii) there exists an integrable function \( a, a \in L_1[0,1] \) and constants \( b_i > 0, i = 1, 2 \), such that

\[ |f(t,x)| \leq a(t) + \sum_{i=1}^n b_i |x_i|, \text{ for each } t \in [0,1], x \in \mathbb{R}^n, \]

**Theorem 3.1** Let the assumptions (i) and (ii) are satisfied. If \( \sum_{i=1}^n \frac{b_i}{\Gamma(2-\alpha_i)} < 1 \), then the fractional-order functional integral equation (6) has at least one solution \( y \in L_1[0,1] \).

**Proof.** Define the operator \( T \) associated with equation (6) by

\[ Ty(t) = f(t, I^{1-\alpha_1}y(t), \ldots, I^{1-\alpha_n}y(t)). \]

Let \( B_r = \{ y \in L_1(I) : \|y\| < r, r > 0 \} \) and let \( y \) be an arbitrary element in \( B_r \). Then from the assumptions (i) and (ii), we obtain

\[ \|Ty\|_{L_1} = \int_0^1 |Ty(t)| \, dt \leq \int_0^1 |f(t, I^{1-\alpha_1}y(t), \ldots, I^{1-\alpha_n}y(t))| \, dt \]
\[ \leq \int_0^1 |a(t)| \, dt + \sum_{i=1}^n b_i \int_0^1 \int_s^t \frac{(t-s)^{1-\alpha_i}}{\Gamma(1-\alpha_i)} |y(s)| \, ds \, dt \]
\[ \leq \|a\| + \sum_{i=1}^n b_i \int_0^1 \frac{(t-s)^{1-\alpha_i}}{(1-\alpha_i)\Gamma(1-\alpha_i)} |y(s)| \, ds \]
\[ \leq \|a\| + \sum_{i=1}^n b_i \int_0^1 \frac{1}{\Gamma(2-\alpha_i)} |y(s)| \, ds \]
\[ \leq \|a\| + \sum_{i=1}^n \frac{b_i}{\Gamma(2-\alpha_i)} \|y\|_{L_1} \leq r, \]
which implies that the operator $T$ maps $B_r$ into itself. Assumption (i) implies that $T$ is continuous. Now, we will show that $T$ is compact, to apply Theorem 2.2. So, let $\Omega$ be a bounded subset of $B_r$. Then $T(\Omega)$ is bounded in $L_1[0, 1]$, i.e. condition (i) of Theorem 2.2 is satisfied. It remains to show that $(Ty)_h \to Ty$ in $L_1[0, 1]$ as $h \to 0$, uniformly with respect to $Ty \in T \Omega$. Now

$$|| (Ty)_h - Ty || = \int_0^1 |(Ty)_h(t) - (Ty)(t)| \, dt$$

$$= \int_0^1 \frac{1}{h} \int_t^{t+h} (Ty)(s) \, ds - (Ty)(t) \, dt$$

$$\leq \int_0^1 \left( \frac{1}{h} \int_t^{t+h} |(Ty)(s) - (Ty)(t)| \, ds \right) \, dt$$

$$\leq \int_0^1 \frac{1}{h} \int_t^{t+h} |f(s, I^{1-\alpha_1}y(t), \ldots, I^{1-\alpha_n}y(t)) - f(t, I^{1-\alpha_1}y(t), \ldots, I^{1-\alpha_n}y(t))| \, ds \, dt.$$

And $y \in \Omega$ implies (by assumption (ii)) that $f \in L_1(0, 1)$, then

$$\frac{1}{h} - \int_t^{t+h} |f(s, I^{1-\alpha_1}y(t), \ldots, I^{1-\alpha_n}y(t)) - f(t, I^{1-\alpha_1}y(t), \ldots, I^{1-\alpha_n}y(t))| \, ds \to 0$$

Therefore, by Theorem 2.2, we have that $T(\Omega)$ is relatively compact, that is, $T$ is a compact operator, then the operator $T$ has a fixed point in $B_r$, which proves the existence of solution $y \in L_1[0, 1]$ equation (6).

For the existence of solution of the nonlocal problem (3)- (4) we have the following theorem

**Theorem 3.2** Let the assumptions of Theorem 3.1 are satisfied. Then nonlocal problem (3)- (4) has at least one solution $x \in AC[0, 1]$. 

**Proof.** Consider the nonlocal fractional differential equation

$$x'(t) = f(t, D^{\alpha_1}x(t), D^{\alpha_2}x(t), \ldots, D^{\alpha_n}x(t)) , \alpha_i \in (0, 1), \text{ a.e. } t \in (0, 1),$$

Let $y(t) = x'(t)$, then

$$x(t) = x(0) + Iy(t)$$

(7)

and $y$ is the solution of the fractional-order functional integral equation (6).

Let $t = \tau_k \in (a, b)$ in equation (7), we get

$$x(\tau_k) = \int_0^{\tau_k} y(s) \, ds + x(0)$$

and

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s) \, ds + x(0) \sum_{k=1}^{m} a_k.$$
From equation (4), we get
\[ x_o = \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} y(s) \, ds + x(0) \sum_{k=1}^{m} a_k. \]

Then we obtain
\[ x(0) = A \left( x_o - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} y(s) \, ds \right), \quad \text{where} \quad A = \left( \sum_{k=1}^{m} a_k \right)^{-1}. \]

and
\[ x(t) = A x_o - A \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} y(s) \, ds + \int_{0}^{t} y(s) \, ds \] (8)
which, by Theorem 3.1, has at least one solution \( x \in AC(0, 1) \).

Now, from equation (8), we have
\[ x(0) = \lim_{t \to 0^+} x(t) = A x_o - A \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} y(s) \, ds \]
\[ x(1) = \lim_{t \to 1^-} x(t) = A x_o - A \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} y(s) \, ds + \int_{0}^{1} y(s) \, ds \]
from which we deduce that equation (8) has at least one solution \( x \in AC[0, 1] \).

To complete the proof we prove that equation (8) satisfies the nonlocal problem (3)- (4). Differentiating (8), we obtain
\[ D^{\alpha_i} x(t) = I^{1-\alpha_i} \frac{d}{dt} x(t) = I^{1-\alpha_i} y(t), \quad i = 1, 2 \ldots \]
and
\[ x'(t) = \frac{dx}{dt} = y(t) = f(t, D^{\alpha_1} x(t), D^{\alpha_2} x(t), \ldots, D^{\alpha_n} x(t)). \]

Also from (8), we have
\[ \sum_{k=1}^{m} a_k x(\tau_k) = x_o - \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} y(s) \, ds + \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} y(s) \, ds = x_o, \quad \tau_k \in (a, b) \]

4 Nonlocal integral condition

Let \( x \in AC[0, 1] \) be the solution of the nonlocal problem (1)-(2).
Let \( a_k = t_k - t_{k-1}, \quad \tau_k \in (t_{k-1}, t_k), \quad a = t_0 < t_1 < t_2, \ldots < t_n = b, \)

then the nonlocal condition (2) will be
\[ \sum_{k=1}^{m} (t_k - t_{k-1}) x(\tau_k) = x_o. \]
From the continuity of the solution $x$ of the nonlocal problem (1)-(2) we can obtain
\[ \lim_{m \to \infty} \sum_{k=1}^{m} (t_k - t_{k-1}) x(\tau_k) = \int_{a}^{b} x(s) \, ds. \]
and the nonlocal condition (2) transformed to the integral one
\[ \int_{a}^{b} x(s) \, ds = x_o. \]
Also from the continuity of the function $Iy(t)$, where $y$ is the solution of the functional integral equation (6), we deduce that the solution (8) will be
\[ x(t) = (b - a)^{-1} (x_o - \int_{a}^{b} \int_{0}^{s} y(\theta) \, d\theta \, ds) + \int_{0}^{t} y(s) \, ds. \]
Now, we have the following Theorem

**Theorem 4.1** Let the assumptions of Theorem 3.2 are satisfied. Then there exist at least one solution $x \in AC[0, 1]$ of the nonlocal problem with integral condition,

\[ x'(t) = f(t, D^{\alpha_1}x(t), D^{\alpha_2}x(t), \ldots, D^{\alpha_n}x(t)), \alpha_i \in (0, 1), \text{ a.e. } t \in (0, 1), \int_{a}^{b} x(s) \, ds = x_o, (a,b) \in (0,1). \]

**References**


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