On a Class of Banach Sequence Spaces Analogous to the Space of Popov

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Abstract

Azimi and Ledari have introduced a class of hereditarily $\ell^p$ Banach spaces ($Z_p$) analogous to the space of Popov. We show that (i). The Banach space $Z_1$ contains asymptotically isometric copies of $\ell^1$ and $Z_p$ is not weakly sequentially complete. (ii). if $p \in \{0\} \cup [1, +\infty)$, then $Z_p$ does not possess the Schur property. (iii) Let $p \in [1, +\infty)$. (a) If $\inf_n p_n < p$, then the natural operator from $X_{\alpha,p}$ to $Z_p$ is unbounded. (b) If $\inf_n p_n \geq p$, then the natural operator from $Z_p$ to $X_{\alpha,p}$ is unbounded.

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1 Introduction

The existence of hereditarily $\ell^1$ Banach spaces failing the Schur property was shown first by Bourgain [7]. In 1986 a class of hereditarily $\ell^1$ Banach sequence spaces was introduced by Hagler and Azimi which it does not possess the Schur property [4].

In [1], [3], [4] and [5] further geometric investigation of this spaces is carried out. In [1] A.D. Andrew investigated the question of whether those spaces are prime. (An infinite-dimensional Banach Space $X$ is said to be prime if every infinite-dimensional subspace of $X$ is isomorphic to $X$.) Specially, he showed that each complemented non weakly sequentially complete subspace of $X_{\alpha,1}$ contains a complemented isomorph of itself. In [3] P. Azimi studied further properties of the $X_{\alpha,1}$ spaces. In particular, he proved that the predual of any member $X$ of this class contains asymptotically isometric copies of $c_0$ and consequently $X$ contains asymptotically isometric copies of $\ell^1$. 
Then, in 2002 these spaces were extended to a new class of hereditarily \( \ell^p \) Banach spaces, \( X_{\alpha,p} \) [2]. In [2] Azimi showed that if \( X \) denote a specific \( X_{\alpha,p} \) space, then \( X \) contains \( \ell^p \) hereditarily complemented, \( X \) is a dual space and the predual of \( X \) contains complemented subspace isomorphic to \( \ell^q \) where \( \frac{1}{p} + \frac{1}{q} = 1 \). Excellent sources of information on the \( X_{\alpha,p} \) Banach spaces are [2, 3].

In 2005, Popov constructed a new class of hereditarily \( \ell^1 \) subspaces of \( L_1 \) without the Schur property ([10]) and generalized his result to a class of hereditarily \( \ell^p \) Banach spaces in [11]. Azimi and Ledari in 2006 have used of the spaces \( X_{\alpha,p} \) to introduce and study a new class of hereditarily \( \ell^p \) spaces, analogous to the space of Popov(\( Z_p \)) [6]. In particular, they have shown that for \( p = 1 \) the spaces are further examples of hereditarily \( \ell^1 \) Banach spaces which fail the Schur property and for the case \( p = 0 \) the spaces are hereditarily \( c_0 \).

In this article, we show that for \( p \in \{0\} \cup [1, +\infty) \), \( Z_p \) does not possess the Schur property and under some conditions for \( p \in [1, +\infty) \) the natural operator from \( X_{\alpha,p} \) to \( Z_p \) is unbounded, also the natural operator from \( Z_p \) to \( X_{\alpha,p} \) is unbounded.

\section{Preliminary Notes}

In this section the definition of the \( X_{\alpha,p} \) spaces is given. First, by a block we mean an interval \( F \) (finite or infinite) of integers. For a block \( F \) and \( x = (t_1, t_2, \ldots) \) a sequence of scalars such that \( \sum_j t_j \) converges, define \( <x, F> = \sum_{j \in F} t_j \). To define the norm, we consider special sequences of blocks and special sequences of nonnegative reals. Specifically, we call a sequence (finite or infinite) \( F_1, F_2, \ldots, F_n, \ldots \) (where each \( F_i \) is a finite block) admissible if \( \max F_i < \min F_{i+1} \) for \( i = 1, 2, 3, \ldots \).

Let us now consider a sequence \( (\alpha_i) \) of nonnegative reals (whose terms are used as weighting factors in the definition of the norm) which satisfies the following properties:

1. \( \alpha_1 = 1 \) and \( \alpha_{i+1} \leq \alpha_i \) for \( i = 1, 2, 3, \ldots \),
2. \( \lim_{i \to \infty} \alpha_i = 0 \),
3. \( \sum_{i=1}^{\infty} \alpha_i = \infty \).

For \( x = (t_1, t_2, t_3, \ldots) \) a finitely nonzero sequence of scalars, define

\[
||x|| = \max (\sum_{i=1}^{n} \alpha_i <x, F_i> |^p)^{\frac{1}{p}}
\]

where the max is taken over all \( n \), and all admissible sequences \( F_1, F_2, \ldots, F_n \).

Let \( X_{\alpha,p} \) be the completion of the finitely of non zero sequences of scalars \( x = (t_1, t_2, \ldots) \) in this norm.
Now we go through the construction of the $Z_p$ spaces.

Before we define the new spaces, let $(\alpha_i)$ be a fixed sequence, and $(X_{\alpha,p_n})$ be a sequence of Banach spaces as above with $\infty > p_1 > p_2 > ... > 1$. The direct sum of these spaces in the sense of $\ell^p$ is defined as the linear space

$$X_p = (\sum_{n=1}^\infty \bigoplus X_{\alpha,p_n})_p$$

with $p \in [1, \infty)$, which is the space of all sequences $x = (x^1, x^2, ...)$, where $x^n \in X_{\alpha,p_n}$ for $n = 1, 2, ...$ and $\|x\|_p = (\sum_{n=1}^\infty \|x^n\|_{\alpha,p_n}^p)^{\frac{1}{p}} < \infty$.

The direct sum of these spaces in the sense of $c_0$ is defined as the linear space

$$X_0 = (\sum_{n=1}^\infty \bigoplus X_{\alpha,p_n})_0$$

which is the space of all sequences $x = (x^1, x^2, ...)$, where $x^n \in X_{\alpha,p_n}$ for $n = 1, 2, ...$ for which $\lim_n \|x^n\|_{\alpha,p_n} = 0$ and the norm $\|x\|_0 = \max_n \|x^n\|_{\alpha,p_n} < \infty$.

Now, fix a sequence $(\alpha_i)$ of reals which satisfies the above conditions and a sequence $(p_n)$ of reals with $\infty > p_1 > p_2 > ... > 1$. Consider the sequence space $X_{\alpha,p_n}$ as above. For each $n \geq 1$, denote by $(\overline{e}_{i,n})$ the unit vector basis of $X_{\alpha,p_n}$ and by $(e_{i,n})$ its natural copy in $X_{\alpha,p_n}$, i.e. $e_{i,n} = (0, ..., 0, e_{i,n}, 0, ...) \in X_{\alpha,p_n}$.

Let $\delta_n > 0$ and $\Delta = (\delta_n)$ such that $\sum_{n=1}^\infty \delta_n^p = 1$, if $p \geq 1$ and $\lim_n \delta_n = 0$ and $\max_n \delta_n = 1$, if $p = 0$. For each $i \geq 1$ put $z_i = \sum_{n=1}^\infty \delta_n e_{i,n}$. Let $Z_p$ be the closed linear span of $(z_i)_{i=1}^\infty$.

We recall the main properties of $Z_p$ spaces (theorem 1 of [6]).

**Theorem 2.1** (i) the Banach space $Z_p$ is hereditarily $\ell^p$ for $p > 1$.

(ii) for $p = 1$ the space $Z_1$ is hereditarily $\ell^1$ and fails the Schur property.

(iii) The space $Z_0$ is hereditarily $c_0$.

**Definition 2.2** Let $X$ be an arbitrary Banach space. Then

a) $X$ has the nowhere Schur property if $X$ contains no infinite dimensional closed subspace with the Schur property.

b) $X$ has the nowhere dual Schur property if $X$ contains no infinite dimensional closed subspace such that its dual has the Schur property.

**Definition 2.3** A Banach space $X$ has the Schur property if every weak convergent sequence is norm convergent.

The above theorem and theorem 1.3 of [9] have the following consequence.

**Theorem 2.4** $Z_1$ possesses the nowhere dual Schur property.

### 3 Main Results

**Remark 3.1** For each integer $n$,

$$\| \sum_{n=1}^k (e_{2n} - e_{2n-1}) \|_{\alpha,p_n} = (\sum_{n=1}^{2k} \alpha_n)^{\frac{1}{p_n}}.$$
This follows from the obvious selection of the admissible sequence \( F_i = \{ i \} \) for \( i = 1, 2, ..., 2n \) and the definition of the norm on \( X_{\alpha, p} \) or more precisely see Corollary 2.3. of [5].

**Theorem 3.2** \( Z_p \) is not weakly sequentially complete.

**Proof 3.3** We prove that the sequence \( (z_i) \) is a weak Cauchy sequence in \( Z_p \) with no weak limit in \( Z_p \). If the sequence \( (z_i) \) were not weak Cauchy, we could find \( n_1 < m_1 < n_2 < m_2 < ..., \) a \( \delta > 0 \) and an \( f \in Z_p^* \) with \( \| f \| = 1 \) and \( f(z_{n_i} - z_{m_i}) \geq \delta \) for all \( i \). Thus,

\[
\left\| \frac{1}{N} \sum_{i=1}^{N} (z_{n_i} - z_{m_i}) \right\|_p^p > \delta,
\]

for all \( N \), but

\[
\left\| \frac{1}{N} \sum_{i=1}^{N} (z_{n_i} - z_{m_i}) \right\|_p^p = \frac{1}{N} \sum_{n=1}^{\infty} \delta_n^p (\sum_{i=1}^{2N} \alpha_i)^\frac{p}{\sigma_m} \\
\leq \sum_{n=1}^{\infty} \delta_n^p \frac{1}{N} (\sum_{i=1}^{2N} \alpha_i)^\frac{p}{\sigma_m} \to 0,
\]

which is a contradiction. Thus, the sequence \( (z_i) \) is weak Cauchy.

Suppose that this sequence has a weak limit \( x \in Z_1 \). If \( x = (x^1, x^2, x^3, ...) \), then \( x_j^i = < x^i, \{ j \} > = \lim_{i \to \infty} < \delta_n \varphi_{i,n}, \{ j \} > = 0 \), so \( x = 0 \).

On the other hand, \( < x^i, N > = \lim_{i \to \infty} < \delta_n \varphi_{i,n}, N > = \delta_n \), which is a contradiction.

The following lemma is due to Parviz Azimi (Lemma 3.14. of [3]).

**Lemma 3.4** Let \( (x_n) \) be a sequence of vectors in a Banach space \( X \) such that for every increasing sequence \( (n_k) \) of integers,

\[
\lim_{k \to \infty} \left\| \frac{x_{n_1} + x_{n_2} + ... + x_{n_k}}{k} \right\| = 0
\]

then \( x_n \to 0 \) weakly.

**Theorem 3.5** Let \( p \in \{ 0 \} \cup [1, +\infty) \), then \( Z_p \) does not possess the Schur property.

**Proof 3.6** The case \( p = 1 \) is lemma 2.8 of [6]. For each \( i \geq 1 \), put \( z_i = \sum_{n=1}^{\infty} \delta_n e_{i,n} \). Let \( u_i = z_{2i} - z_{2i-1} \). Therefore, if we assume that \( p \in [1, +\infty) \),

\[
\left\| \sum_{i=1}^{k} u_{n_i} \right\|_p^p = \left\| \sum_{i=1}^{k} (z_{2n_i} - z_{2n_i-1}) \right\|_p^p = \left\| \sum_{m=1}^{\infty} \delta_m (e_{2m-n_i} - e_{2m-n_i}) \right\|_p^p \\
= \sum_{m=1}^{\infty} \delta_m \left( \sum_{i=1}^{k} \left( e_{2m-n_i} - e_{2m-n_i-1} \right) \right)^\frac{p}{\sigma_m} \\
= \sum_{m=1}^{\infty} \delta_m \left( \sum_{i=1}^{2k} \alpha_i \right)^\frac{p}{\sigma_m}.
\]
since $\alpha_i \to 0$, we have

$$\lim_{k \to \infty} \frac{||u_1 + u_2 + \ldots + u_k||_0}{k} = \lim_{k \to \infty} \frac{(\sum_{m=1}^{\infty} \delta_m ((\sum_{i=1}^{2k} \alpha_i) \frac{1}{p^m})^\frac{1}{p})}{k} = 0.$$ 

On the other hand,

$$||u_i||_p^p = ||z_{2i} - z_{2i-1}||_p^p = \left( \sum_{m=1}^{\infty} \delta_m (1 + \alpha_2)^\frac{1}{p^m} \right)^p \leq \sum_{m=1}^{\infty} \delta_m^p = 1.$$ 

When $p = 0$, we have

$$\frac{\sum_{i=1}^{k} u_n_i}{k}||_0 = \frac{\sum_{i=1}^{k} (z_{2n_i} - z_{2n_i-1})}{k}||_0 = \frac{\sum_{i=1}^{k} (\sum_{m=1}^{\infty} \delta_m (e_{2n_i,m} - e_{2n_i-1,m}))}{k}||_0 = \max_m \delta_m ||(\sum_{i=1}^{\infty} (e_{2n_i,m} - e_{2n_i-1,m})||_\alpha,p^m.$$ 

Again since $\alpha_i \to 0$, we have

$$\lim_{k \to \infty} \frac{||u_1 + u_2 + \ldots + u_k||_0}{k} = \lim_{k \to \infty} \frac{\max_m \delta_m ((\sum_{i=1}^{2k} \alpha_i) \frac{1}{p^m})}{k} = 0.$$ 

But,

$$||u_i||_0 = ||z_{2i} - z_{2i-1}||_0 = \max_m \delta_m (1 + \alpha_2)^\frac{1}{p^m} \geq \max_m \delta_m = 1.$$ 

Hence, if $p \in \{0\} \cup [1, +\infty)$, the sequence $(u_i)$ is a weakly null sequence in $Z_p$ but not in norm.

**Definition 3.7** Let $X$ and $Y$ be any of the spaces $X_{\alpha,p}(1 \leq p < \infty)$, $Z_p(1 \leq p < \infty)$ with their natural bases $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ respectively. The formal (maybe, unbounded) operator $T : X \to Y$ which extends by linearity and continuity the equality $Tx_n = y_n$ is called the natural operator from $X$ to $Y$.

**Theorem 3.8** Let $p \in [1, +\infty)$.

(i) If $\inf_n p_n < p$, then the natural operator from $X_{\alpha,p}$ to $Z_p$ is unbounded.

(ii) If $\inf_n p_n \geq p$, then the natural operator from $Z_p$ to $X_{\alpha,p}$ is unbounded.
Proof 3.9 (i) We have
\[ \sum_{n=1}^{\infty} \delta_n^p k_{\frac{\mu}{p_n}} = \sum_{n=1}^{\infty} \delta_n^p \| \sum_{i=1}^{k} e_{i,n} \|_{p_n,\alpha}^p = \| \sum_{n=1}^{\infty} \delta_n^p (\sum_{i=1}^{k} e_{i,n}) \|_{p_n,\alpha}^p \]
\[ = \| \sum_{i=1}^{k} \sum_{n=1}^{\infty} \delta_n^p (e_{i,n}) \|_{p_n,\alpha}^p = \| \sum_{i=1}^{k} z_i \|_{p_n}^p = \| \sum_{i=1}^{k} T e_i \|_{p_n}^p \]
\[ = \| T (\sum_{i=1}^{k} e_i) \|_{p_n}^p \leq \| T \|_{p_n}^p \sum_{i=1}^{k} e_i \|_{p_n,\alpha}^p = \| T \|_{p_n}^p k. \]

Therefore,
\[ \| T \|_{p_n}^p \geq \sum_{n=1}^{\infty} \frac{\delta_n^p k_{\frac{\mu}{p_n}}}{k} = \sum_{n=1}^{\infty} \delta_n^p k_{\frac{\mu}{p_n}}^{-1} \]

If \( \inf_n p_n < p \), then there exists an \( n_0 \) such that \( p_{n_0} < p \) and hence
\[ \| T \|_{p_n}^p \geq \sum_{n=1}^{\infty} \delta_n^p k_{\frac{\mu}{p_{n_0}}}^{-1} \geq \delta_n^p k_{\frac{\mu}{p_{n_0}}}^{-1} \rightarrow \infty, \]
as \( k \rightarrow \infty \).

(ii) Suppose now \( \inf_n p_n \geq p \). In this case \( \frac{\mu}{p_n} - 1 < 0 \) for each \( n \). Let \( \varepsilon > 0 \) is arbitrary, choose \( n_0 \) so that \( \sum_{n=1}^{n_0} \delta_n^p < \frac{\varepsilon}{2} \). Then choose \( k_0 \) such that
\[ \max_{1 \leq n \leq n_0} \delta_n^p k_{\frac{\mu}{p_{n_0}}}^{-1} < \frac{\varepsilon}{2n_0} \]
for \( m > m_0 \). Then for such \( m \) we have
\[ \| T \|_{p_n}^p \geq \frac{1}{\sum_{n=1}^{\infty} \delta_n^p k_{\frac{\mu}{p_n}}^{-1}} = \frac{1}{\sum_{n=1}^{n_0} \delta_n^p k_{\frac{\mu}{p_{n_0}}}^{-1} + \sum_{n=n_0+1}^{\infty} \delta_n^p k_{\frac{\mu}{p_n}}^{-1}} \]
\[ \geq \frac{1}{\sum_{n=1}^{n_0} (\max_{1 \leq n \leq n_0} \delta_n^p k_{\frac{\mu}{p_{n_0}}}^{-1}) + \sum_{n=n_0+1}^{\infty} \delta_n^p k_{\frac{\mu}{p_n}}^{-1}} \]
\[ \geq \frac{1}{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}} \rightarrow \infty, \]
as \( k \rightarrow \infty \).

Definition 3.10 Let \( X \) be a Banach space. We say that \( X \) contains asymptotically isometric copy of \( \ell^1 \) if for any \( \varepsilon_n \downarrow 0 \) \((0 < \varepsilon_n \leq 1)\), \( X \) contains a norm-one sequence \( (x_n) \) such that for all \( m \) and scalars \( \{a_n : 1 \leq n \leq m\} \)
\[ \sum_{n=1}^{m} (1 - \varepsilon_n) |a_n| \leq || \sum_{n=1}^{m} a_n x_n || \leq \sum_{n=1}^{m} (1 + \varepsilon_n) |a_n|. \]

Theorem 3.11 The Banach space \( Z_1 \) contains asymptotically isometric copies of \( \ell^1 \).
Proof 3.12 Let $E_0$ be an infinite dimensional subspace of $Z_1$ and choose a sequence of positive numbers $\varepsilon_s$ such that $\frac{\delta_s}{4} \leq \varepsilon_s$, for all $s \in \mathbb{N}$ and satisfy the conditions of Lemma 2.4 of [11], where $(\delta_s)$ is as in definition of $Z_1$.

Using Lemma 2.3 of [11], we construct inductively sequences $(x_s)_{s=1}^{\infty} \subseteq E_0$ and $(z_s)_{s=1}^{\infty} \subseteq Z_1 - \{0\}$ of the form $u_s = \sum_{i=j_s+1}^{j_{s+1}} a_i z_i$ where $j_1 < j_2 < ...$ and $||u_s|| = 1$ so that $||u_s - x_s|| \leq \varepsilon_s$.

To see that this can be done, let $j_1 = 1$. Choose by Lemma 2.3 of [11] an $x_1 \in Z_1 - \{0\}$ and $u_1 = \sum_{i=j_1+1}^{j_1+1} a_i z_i$ such that $||u_1|| = 1$ and $||u_1 - x_1|| \leq \frac{\delta_1}{4}$. Continuing the procedure in the obvious manner, we construct the desired sequences.

Therefore, for each scalars $(a_s)_{s=1}^{m}$ one has

$$
\sum_{n=1}^{m} (1 - \varepsilon_n)|a_n| \leq \sum_{n=1}^{m} (|a_n||u_n|| - ||u_n - x_n||) \leq \sum_{n=1}^{m} |a_n||u_n|| \\
\leq \sum_{n=1}^{m} |a_n||(||u_n|| + ||u_n - x_n||) \leq \sum_{n=1}^{m} (1 + \varepsilon_n)|a_n|.
$$

therefore, $Z_1$ contains asymptotically isometric copies of $\ell^1$

The following theorem is an immediate consequence of theorem 2 of [8] and above theorem.

Theorem 3.13 (i) The dual $Z_1^*$ of $Z_1$ contains subspaces isometrically isomorphic to $C[0,1]^*$.

(ii) $C(\Delta)$ is isometric to a quotient space of $Z_1$ where $\Delta$ is the Cantor set.

(iii) $L_1$ is linearly isometric to a subspace of $Z_1$.

Remark 3.14 The identity operator from $X_{\alpha,p}$ to $X_{\alpha,q}$ is unbounded, where $p > q$ (Theorem 2.4 of [5]). Obviously, the identity operator from $Z_1$ to $Z_0$ is bounded. Since we have

$$
||I(\sum_{i=1}^{k} t_i z_i)||_0 = ||\sum_{i=1}^{k} t_i z_i||_0 = \max_{n} \delta_n ||\sum_{i=1}^{k} t_i e_{i,n}||_{\alpha,p_n} \\
\leq \sum_{n=1}^{\infty} \delta_n ||\sum_{i=1}^{k} t_i e_{i,n}||_{\alpha,p_n} = ||\sum_{i=1}^{k} t_i z_i||_1.
$$

So, $||I|| \leq 1$.

There is still a further question concerning the structure of $Z_p$.

Problem 3.15 (i) Is the identity operator from $Z_0$ to $Z_p (1 \leq p < \infty)$ bounded?

(ii) Is the identity operator from $Z_p (1 < p < \infty)$ to $Z_0$ bounded?
Problem 3.16 Is the Banach space $Z_p(1 < p < \infty)$ contains asymptotically isometric copies of $\ell^p$?

Problem 3.17 Let $p \in \{0\} \cup [1, +\infty)$. Is then $Z_p$ possess the Dunford-Pettis property (DPP)?

References


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