Directionlets and Some Generalized Nonlinear Selfsimilarities

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Abstract

In this paper we review the notion of directional regularity and we introduce quite equivalent definitions using polar systems. Next, we prove a wavelet characterization of such regularities with directional wavelets known as directionlets. A class of nonlinear anisotropic self similar functions is then introduced and proved to be compatible with directionlets for the detection of the singularities. At the end, we propose some prospects to be adapted for the study of spherical detected signals.

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1 Introduction

Wavelets have nowadays a reputable interest and success in different domains. The particularity in a wavelet basis is the fact that all the elements of a basis are deduced from one wavelet called mother wavelet by translation, dilatation and rotation. The last parameter is firstly introduced by Antoine et al (See [1] and [2]) to obtain some directional selectivity of the wavelet transform in higher dimensions. These wavelets are called directional wavelets and they are examples of multidimensional wavelets which are useful when dealing with higher dimensional problems. One class is based on tensor products of one dimensional wavelets. However, it has been observed in the literature that tensor product wavelets are not sufficient to study a variety of applications especially for those where a cartesian representation is not adequate. Consequently, multidimensional wavelets with general dilation matrices have been
introduced. The present work lies within the scope of applications of multidimensional wavelets. One motivation of the present work is related to [3] where the author has brought a significant study of the phenomenon of seismology based on wavelets. In [3] some directional wavelets are applied based on a modification of Morlet one to study such a phenomenon. It was a straightforward step in the understanding of such a problem but it remains incomplete as it is noticed also in the conclusion of the cited reference. One cause is the fact that directional wavelets considered there have the ability to detect or to characterize a function in a fixed direction while they remain blind in others. The method was able to adapt a fixed angle of wavelet transforms. It allowed the detections of faults in such directions but it did not induce a fast numerical algorithm. Moreover, the computation of dips is extremely delicate near the faults. So it necessitates other tools to be adapted to this computation. We thought that the essential extension of [3] is to use a wavelet well adopted to the geometric structure of the surface of detection. Secondly, it is due to the anisotropic behavior in the propagation of the seismic waves caused by their anisotropic directional aspects. So a formulation taking into account this cause may induce good results.

The present paper is organized as follows. In section 2, we introduced the notion of multi-regularity or directional regularity and we showed the natural effect of directional wavelets in characterizing it. We noticed from numerical studies of some examples in [3] that some among them present a scaling law that may be estimated with self similarity. This leads us to a comparison with the existing self-similar models. However, compared to these models, the few examples cited in [3] present some higher regularities than the regularity expected for the classical self-similar functions. This allows us to define a new class of functions including the existing ones taking into account the increase of regularity and the directional behavior. This is the objet of section 3 where we introduced the non isotropic generalized self-similar functions and give some properties of such a class. We remark indeed that these functions are not multifractal as it is the case of the existing models. A natural idea that came in mind is the fact that seismic waves propagates across spherical shaped surfaces. So, it may be suitable to use wavelets adapted to manifolds such as spheres to study such a phenomenon. The most studies in this direction are based on cartesian wavelets composed with rotations. In order to avoid the disadvantages of such a construction, we perform in the last section some method based on harmonic structures defined on spheres to define may be a multi-resolution analysis on $L^2(S^n)$. 
2 Main Results

The starting point was the fact that non isotropic self-similar objects can present irregular deformations at each iteration and direction. This was due to the use of contractions having different ratios in each direction. So that, any global estimation of the wavelet coefficients near the singularity does not lead to a full description of the behavior of the function. This fact allows us to think about adapting some relation taking into account regularity, the direction of propagation and the deformation in each iteration.

Let \((r, \theta)\) be the polar coordinates in \(\mathbb{R}^2\). For \(\alpha \in [0, \pi]\) let \(\Delta : \theta = \alpha\) the line of direction \(\alpha\) at the origin and for \(\beta < \gamma \in [0, \pi]\), \(C(\beta, \gamma)\) designates the cone centered at \(o\) and limited by the lines \(\theta = \beta\) and \(\theta = \gamma\).

**Definition 2.1** A bounded function \(F : \mathbb{R}^2 \to \mathbb{R}\) is said to be regular of order \(\omega(\alpha)\) according to the direction \(\alpha\) at \(X_0\), if for all \(\varepsilon > 0\) there exists a constant \(C > 0\), a polynomial \(P\) and a neighborhood \(W\) of \(X_0\) such that

\[
|F(X) - P(X - X_0)| \leq C|X - X_0|^{\omega(\alpha)} \quad (1)
\]

for all \(X \in (X_0 + C(\alpha - \varepsilon, \alpha + \varepsilon)) \cap W\) and \(\varepsilon \to 0\).

The following theorem gives a wavelet characterization of the directional regularity.

**Theorem 2.2** Let \(u = (\cos \alpha, \sin \alpha)\) be the unitary directional arrow of \(\Delta\) and denote

\[
\psi(X) = D^n(\exp (-||X||^2 + i < X, u >)) \quad X \in \mathbb{R}^2.
\]

For \(a > 0\) and \(b = (b_1, b_2)\) denote \(\psi_{a,b}(X) = \frac{1}{a} \psi \left( \frac{X - b}{a} \right)\). We have

1. If \(F\) is \(\omega(\alpha)\) regular at \(x_0\) then

\[
|<F, \psi_{a,b}>| \leq C a^{1+\omega(\alpha)} \left( 1 + \frac{|X_0 - b|}{a} \right)^{\omega(\alpha)}.
\]

2. Conversely, suppose that \(F \in C^\eta(\mathbb{R}^2)\) for some \(\eta > 0\) and that

\[
|<F, \psi_{a,b}>| \leq C a^{1+\omega(\alpha)} \left( 1 + \frac{|X_0 - b|}{a} \right)^{\omega^\prime}.
\]

for some \(\omega^\prime < \omega(\alpha)\), then \(F\) is \(\omega(\alpha)\) regular at \(X_0\) according to the direction \(\alpha\).
Let us examine this result and give its equivalence especially for cartesian directions. Denote \( F_{y_0} \) the function defined by \( F_{y_0}(x) = F(x, y_0) \) and \( F_{x_0} \) the one defined by \( F_{x_0}(y) = F(x_0, y) \). For \( \alpha = 0 \), the cone \( x_0 + C(-\varepsilon, \varepsilon) \), for \( \varepsilon \to 0 \), can be assimilated to the \( x \)-axis and for \( \alpha = \frac{\pi}{2} \), the cone \( x_0 + C(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon) \), for \( \varepsilon \to 0 \), can be assimilated to the \( y \)-axis. So, we guess that Theorem 2.2 yields the following assertions.

- \( F_{y_0} \) is \( \omega_1 \) regular at \( x_0 \) iff
  \[
  | < F_{y_0}, \psi_{a,b} > | \leq C a^{1+\omega_1} \left( 1 + \frac{|x_0 - b|}{a} \right)^{\omega_1}.
  \]

Similarly, for the \( y \)-direction, we expect the following to hold.

- \( F_{x_0} \) is \( \omega_2 \) regular at \( y_0 \) iff
  \[
  | < F_{x_0}, \psi_{a,b} > | \leq C a^{1+\omega_2} \left( 1 + \frac{|y_0 - b|}{a} \right)^{\omega_2}.
  \]

However, such assertions need in fact some uniformity in all directions to be able to let \( \varepsilon \to 0 \). This means that the formulations above in \( x \) and \( y \) axes take into account the values of the function \( F \) on a neighborhood of the considered axe. For this reasons, we introduced the following definition which we expect to overcome the problem.

**Definition 2.3** A bounded function \( F : \mathbb{R}^2 \to \mathbb{R} \) is multi-regular of order \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}_+^2 \) at a point \( X_0 = (x_0, y_0) \) if there exists a polynomial \( P \) of degree less than \( \alpha \) and a constant \( C > 0 \) such that

\[
|F(x, y) - P(x - x_0, y - y_0)| \leq C (|x - x_0|^{\alpha_1} + |y - y_0|^{\alpha_2}).
\]

A main remark in this definition is the fact that the polynomial \( P \) may not be unique. To overcome this problem we assume the following assertions to hold.

- \( \mathcal{A}_1 \). The degree of the polynomial \( P(x, y) = \sum_{r_1, r_2} a_{r_1, r_2} x^{r_1} y^{r_2} \) is less that the number \( \alpha = (\alpha_1, \alpha_2) \) iff
  \[
  \frac{r_1}{\alpha_1} + \frac{r_2}{\alpha_2} < 1.
  \]

- \( \mathcal{A}_2 \). The multi-regularity degree \( \alpha = (\alpha_1, \alpha_2) \) in Definition 2.3 satisfies the order
  \[
  \alpha_1 \geq \alpha_2.
  \]
We now establish a wavelet characterization of the multi-regularity. For this aim, we adopt the following notations. Let $\psi$ be an analyzing mother wavelet in $\mathbb{R}^2$ supported on the cone $u + C(-\varepsilon, +\varepsilon)$ where $u$ is a vector in $\mathbb{R}^2$ chosen as a direction. Next, for $(j_1, j_2) \in \mathbb{N}^2$, $(k_1, k_2) \in \mathbb{Z}^2$, denote $\lambda = (j_1, k_1; j_2, k_2)$ and 

$$
\psi_{\lambda, u}(x) = e^{i(xu_1 + yu_2)}2^{(j_1 + j_2)}\psi(2^{j_1}x - k_1, 2^{j_2}y - k_2).
$$

We obtained the following result.

**Theorem 2.4** Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a bounded function and $C_{\lambda, u}(F)$ its wavelet transform relatively to $\psi_{\lambda, u}$.

- If $F$ is $\alpha$ regular at $(x_0, y_0)$ then 
  
  $$
  |C_{\lambda, u}(F)| \leq C2^{-(j_1 + j_2)} \left[ 2^{-\alpha_1 j_1} + |x_0 - k_1|^{\alpha_1} + 2^{-\alpha_2 j_2} + |y_0 - k_2|^{\alpha_2} \right].
  $$

- Conversely, suppose that $F \in C^\eta(\mathbb{R})$ for some $\eta > 0$ and that 
  $$
  |C_{\lambda, u}(F)| \leq C2^{-(j_1 + j_2)} \left[ 2^{-\alpha_1 j_1} + |x_0 - k_1|^{\alpha'_1} + 2^{-\alpha_2 j_2} + |y_0 - k_2|^{\alpha'_2} \right].
  $$

for some $\alpha'_i < \alpha_i$, then $F$ is $\alpha$ regular at $(x_0, y_0)$ at the direction $u$.

Next, we introduce the definition of the directional Hölder exponent which we aim to be compatible with Definition 2.3 with polar coordinates.

**Definition 2.5** The Hölder exponent of the function $F$ at the point $(x, y)$ relatively to the direction $u$ is 

$$
\mathcal{AH}_{F, u}(x, y) = \sup\{\alpha; F \in C^{\alpha, \beta}_u(x, y) \text{ for some } \beta\}
$$

Next, a second aim of the present paper is to introduce some self-similar classes of functions characterized by higher order regularity. The idea starts by considering some self-similar functional equations involving convolution products. Recall that self-similar functions are firstly introduced by Jaffard in [8] as follows.

**Definition 2.6** A bounded function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be self-similar of order $k \geq 0$ if 

- there exists a bounded domain $\Omega$ and contractive similitudes $S_1, ..., S_d$ satisfying 
  $$
  S_i(\Omega) \subset \Omega
  $$
  $$
  S_i(\Omega) \cap S_j(\Omega) = \emptyset
  $$
there exists $\lambda_1, \ldots, \lambda_d$ such that $0 < \lambda_i < 1$ and a function $g, C^k$ with all derivatives of order less than $k$ having fast decay.

$$F(x) = \sum_{i=0}^{d} \lambda_i F(S_i^{-1}(x)) + g(x).$$

there exists $x_0 \in \Omega$ such that $F \notin C^k(x_0)$.

These functions are multifractal as well as their singularities sets. In the present paper, we seek candidates related to self similarity but with more regularity properties. One idea to increase the regularity of functions is to apply convolution products. This was the object of the following definition of generalized non isotropic self-similar functions.

**Definition 2.7** A bounded function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be a generalized non isotropic $p$-self-similar functions of order $r \geq 0$ if it satisfies

1. there exists a bounded domain $\Omega$ and anisotropic $C^r$ ($r \in \mathbb{N}$) contractions $S_j$ s.t.
   $$S_j(\Omega) \subset \Omega \quad \text{and} \quad S_i(\Omega) \cap S_j(\Omega) = \emptyset, \; \forall \; i \neq j.$$

2. there exist $(p_{j,k}, 1 \leq k \leq p, 1 \leq j \leq N_k)$ in $]0,1[$ such that
   $$F(x) = \sum_{k=1}^{p} \sum_{j=1}^{N_k} p_{j,k} F^{*k}(S^{-1}_j(x))$$
   where $F^{*k}$ is the $k$ convolutions of the function $F$ with itself.

3. $F^{*k}(S^{-1}_j(x)) = 0$ whenever $x \notin \Omega_i = S_i(\Omega)$.

4. there exists $x_0 \in \Omega$ such that $F \notin C^r(x_0)$.

Let us comment on these functions. For the sake of simplicity we will restrict to the case $p = 2$ and denote $p_{j,1} = p_j$ and

$$F_{NL}(x) = \sum_{j=1}^{N_2} p_{j,2}(F \ast F)(S^{-1}_j(x)).$$

The identity $F = \sum_{j=1}^{N_1} p_j F \circ S_j^{-1} + F_{NL}$ is nonlinear in $F$. 

• The classical self-similar functions can be already obtained with \( p = 2 \). In this case, the function \( F \) is multifractal and is supported by the invariant compact self-similar set
\[
K = \bigcup_j S_j(K).
\]
This set is usually a fractal which forces \( F \) to be singular.

• The situation in Definition 2.7 is different. The non-linear analog of the self-similar set is
\[
K = \bigcup_k \bigcup_j S_j(K + K + \cdots + K).
\]
Such a set is usually not fractal.

• It may exist some bad cases where \( F \) is supported by an affine hyperplane which will be known as the degenerate case.

• For \( p \geq 1 \), the function \( F \) has a fractal component defined by
\[
F_1(x) = \sum_{j=1}^{N_1} p_{j,k} F(S_{j}^{-1}(x)).
\]

We will show next that these functions have some regularity properties that get stronger as the linear terms in the identity get smaller. For \( j, n \in \mathbb{N} \), \( x \in \Omega \) and \( L \) large enough, denote
\[
\Omega_{j,n} = \{ i ; 2^{-n} \leq |\Pi_j(S_i(\Omega))| < 22^{-n} \}
\]
and
\[
\Omega_{j,n}(x) = \{ i \in \Omega_{j,n} ; |\Pi_j(x - x_i)| \leq L2^{-n} \}.
\]
\( \Pi_j \) denotes the projection onto the \( j^{th} \)-direction. Denote also
\[
\Lambda_{j,n}(x) = \{ \lambda = (k_1, k_2, ..., k_d) \in \mathbb{Z}^d ; |x_j - k_j 2^{-n}| \leq L2^{-n} \}.
\]

**Theorem 2.8** Suppose that \( F \) is not degenerate and that
\[
p_{j,1} = 0 \ \forall \ j \quad \text{and} \quad S_j = r_j R_j x + b_j,
\]
with \( 0 < r_j < 2^{-p} \) and \( R_j \) orthogonal. Then \( F \) is regular with fast decay Fourier transform. More precisely, there exists constants \( C, a > 0 \) such that
\[
|\hat{F}(x)| \leq e^{-C|x|^a}, \quad \forall \ |x| \geq 1.
\]
Next, consider for \((x, y) \in \mathbb{R}^2\) and \(\delta_1, \ldots, \delta_p\) positive
\[
C_\delta(x, y) = \sum_k \delta_k X_k(x, y) - 1
\]
with
\[
X_k(x, y) = \sum_j p_{j,k}^2 r_j^{-y}. 
\]

It consists of convex functions increasing in \(y\) and decreasing in \(x\). Define finally the associated separative function
\[
\varphi_\delta(x) = \sup \{y; \quad C_\delta(x, y) < 0\}. 
\]

For \(p_j^1 \neq 0\), we have proved the following result.

**Theorem 2.9** Suppose that \(F\) is not degenerate and that
\[
p_{j,k} > 0 \quad \forall \ j, k \quad \text{and} \quad S_j = r_j R_j x + b_j, 
\]
with \(0 < r_j < 2^{-p}\) and \(R_j\) orthogonal. Then

1. The Fourier transform \(\hat{F}\) of \(F\) vanishes at infinity and there exists constants \(C, \delta > 0\) satisfying for all \(0 < a < \varphi_\delta(1)\),
\[
|\hat{F}(x)| \leq C|x|^{-a}, \quad \forall \ |x| \geq 1.
\]

2. For all \(j = 1, 2\) and \(x \in K\),
\[
\alpha_{F_j}(x) = \liminf_{n \to \infty} \inf_{\lambda \in \Lambda_{j,n}(x)} \frac{\log |C_{n,\lambda}(F)|}{-n \log 2} \geq \liminf_{n \to \infty} \inf_{i \in \Omega_{j,n}(x)} \frac{\log p_i}{\log \Pi(S_i(\Omega))},
\]
where \(C_{n,\lambda}(F)\) is the wavelet transform of \(F\) at the scale \(n\) and the position \(\lambda\) relatively to an analyzing wavelet.

### 3 Proof of main results

The purpose in this section is to develop the proofs of our main results.

**Proof of Theorem 2.2** For the sake of simplicity we suppose that \(x_0 = 0\), \(\omega < 1\) and \(n = 1\).

1) Denote firstly
\[
\tilde{\psi}_{a,\delta}^\alpha(\zeta) = \left(\frac{4(\zeta_1 - b_1 + z_2 - b_2)^2}{a^2} + (\cos \alpha + \sin \alpha)^2\right)^{1/2} \exp \left(-|\zeta - b/a|^2\right). 
\]
We have
\[ |<F, \psi_{a,b}>| = |\int F(\zeta) \psi_{a,b}(\zeta) d\zeta| \]
\[ \leq C \int r^{\omega(\alpha)} |\psi_{a,b}(\zeta)| d\zeta \]
\[ \leq C \int r^{\omega(\alpha)} \tilde{\psi}_{a,b}^{\omega(\alpha)}(\zeta) d\zeta \]
\[ \leq Ca \int |aX + b|^{\omega(\alpha)} 1 + r^2 \exp (-r^2) dX. \]

Consequently,
\[ |<F, \psi_{a,b}>| \leq Ca \int_{0}^{+\infty} (a^{\omega(\alpha)} r^{\omega(\alpha)} + |b|^{\omega(\alpha)})(1 + r^2)^{1/2} e^{-r^2} dr \]
\[ \leq Ca(a^{\omega(\alpha)} + |b|^{\omega(\alpha)}) \]
\[ \leq Ca^{1+\omega(\alpha)} (1 + \frac{b}{a})^{\omega(\alpha)}. \]

2) We have
\[ F(X) = \int_{0}^{+\infty} \int_{\mathbb{R}^2} <F, \psi_{a,b}> \psi_{a,b}(X) db \frac{da}{a^3} \]
\[ = \sum_{j=-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}^2} <F, \psi_{a,b}> \psi_{a,b}(X) db \frac{da}{a^3}. \]

Denote
\[ F_j(X) = \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}^2} <F, \psi_{a,b}> \psi_{a,b}(X) db \frac{da}{a^3}. \]

One can see that
\[ |F_j(X)| \leq C \int_{\mathbb{R}^2} \int_{2^j}^{2^{j+1}} a^{1+\omega(\alpha)}(1 + \frac{|X-b/a|^{\omega'}}{(1 + \frac{|X-b/a|}{a})^N} db \frac{da}{a^3}. \]

Since \(2^j \leq a \leq 2^{j+1}\), then
\[ |F_j(X)| \leq C2^{j(\omega(\alpha)-1)} \int_{\mathbb{R}^2} \frac{1 + |b/2^j|^{\omega'}}{(1 + |X-b/2^j|)^N} db \]
\[ \leq C2^{j(\omega(\alpha)+1)} \int_{\mathbb{R}^2} \frac{1 + |\zeta - X/2^j|^{\omega'}}{(1 + |\zeta|)^N} d\zeta \]
\[ \leq C2^{j(\omega(\alpha)+1)} \left( 1 + \left( \frac{|X|}{2^j} \right)^{\omega'} \right). \]
Denote now
\[ \widetilde{\Theta}_{a,b}(X) = \left(1 + \frac{|x - b_1|}{a}\right) \left(1 + \frac{|y - b_2|}{a}\right) \exp\left(- \frac{|X - b|^2}{a^2}\right) a^{1+\omega(\alpha)}(1 + \frac{|b|}{a})^\omega. \]

\[ F'_j(X) \] is bounded by
\[ C \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}^2} \frac{1}{a^2} \left(1 + \frac{|x - b_1 + y - b_2|}{a}\right) \widetilde{\Theta}_{a,b}(X) \, db \, da \]
\[ \leq C \int_{2^j}^{2^{j+1}} a^{(\omega(\alpha)-2)} \int_{\mathbb{R}^2} (1 + |\zeta|)^3 (1 + |\zeta|') \exp(-|\zeta|^2) d\zeta \, da \]
\[ \leq C2^{(\omega(\alpha)-1)j} \left(1 + \frac{|X|'}{2^j} \right). \]

Let \( J \) be such that \( 2^{J-1} \leq |X| < 2^J \). We have in one side
\[ \sum_{j \geq J} |F_j(X) - F_j(0)| \leq |X| \sum_{j \geq J} \|F'_j\|_{L^\infty[0,X]} \]
\[ \leq C|X| \sum_{j \geq J} 2^{j(\omega(\alpha)-1)} \left| \frac{X}{2^j} \right|' \]
\[ \leq C2^{J\omega(\alpha)} \leq C|X|^\omega(\alpha). \]

In the other side, we have
\[ \sum_{j \leq J-1} |F_j(X) - F_j(0)| \leq C \sum_{j \leq J-1} \left(2^{j(1+\omega(\alpha))} \left(1 + \left| \frac{X}{2^j} \right|' \right) + 2^{j(1+\omega(\alpha))} \right) \]
\[ \leq C \left(2^{J(1+\omega(\alpha))} + 2^{J(1+\omega(\alpha)-\omega')} |X|' + 2^{J(1+\omega(\alpha))} \right) \]
\[ \leq C2^{J(1+\omega(\alpha))} \leq C|X|^\omega(\alpha). \]

**Proof of theorem 2.8** Remark firstly that \( |\hat{F}(x)| < 1 \) for all \( x \neq 0 \). Define
\[ r = \min r_j, \quad a = \frac{\text{Log}p}{\text{Log}r} \]
and consider the annular region \( 1 \leq |x| \leq A \) with \( A \leq \frac{1}{r} \). It consists of a compact set. So that, there exists \( C > 0 \) satisfying
\[ |\hat{F}(x)| \leq e^{-C|x|^a}, \quad \forall 1 \leq |x| \leq A. \]
It results that the inequality remains satisfied on $1 \leq |x| \leq \frac{1}{L}$. So repeating the arguments for $\frac{1}{L} A$ instead of $A$, and so on, we obtain

$$|\hat{F}(x)| \leq e^{-C|x|^a}, \quad \forall |x| \geq 1.$$ 

**Proof of Theorem 2.9.** The first point can be proved by following similar arguments as in theorem 2.8 with some suitable modifications. The second point is based on the following lemma.

**Lemma 3.1** Assume that $F * F$ is $C^\eta(\mathbb{R}^2)$ for some $\eta > 0$. It holds that

$$AH_{F, u_j}(x) = \liminf_{p \to +\infty} \left( \inf_{i \in \Omega_{j,p}(x)} \log \left| p_i p_{i2} \cdots p_{in} \right| \log \left| \prod_{j}(S_i(\Omega_i)) \right| \right).$$

where $u_j$ is the canonical $j$-direction in the cartesian system of coordinates.

**Lemma 3.2** Let $F \in L^2(\mathbb{R}^m)$ and $C_{j,\lambda}(F)$ its wavelet transform relatively to an orthonormal wavelet basis $\psi_{j,\lambda}$ of $L^2(\mathbb{R}^m)$. The regularity exponent of $F$ at a point $x$ is

$$\alpha_F(x) = \liminf_{j \to \infty} \inf_{\lambda \in \Lambda_j(x)} \frac{\log |C_{j,\lambda}(F)|}{-j \log 2},$$

where

$$\Lambda_j(x) = \{ \lambda \in \mathbb{Z}^d ; \ |x - \lambda 2^{-j}| \leq L 2^{-j}\}.$$

**4 Some prospects**

In our survey on self-similar systems, we noticed that several among them present self-similar variations according to the argument $\theta$. We mention signals captured by spherical surfaces for example. This fact have allowed us to think about wavelet bases adapted to spheres. In the literature, most ideas are based on compositions of cartesian bases by rotations. In other words, on the action of the groups $SO(n)$. We thought to construct a multi-resolution analysis starting from the sphere itself. To do this, let us recall some exploratory notions. Denote $\mathcal{P}_k$ the space of homogeneous polynomials of degree $k$ on $\mathbb{R}^n$ and $\mathcal{PH}_k$ that of harmonic homogeneous polynomials of degree $k$ on $\mathbb{R}^n$. One knows that

$$\text{dim} \mathcal{P}_k = s_k = C_{n+k-1}^k = \frac{(n+k-1)!}{k!(n-1)!},$$

$$\text{dim} \mathcal{PH}_k = d_k = s_k - s_{k-2} = \frac{(n+2k-2)(n+k-3)}{k!(n-2)!}.$$
and that
\[ P_k = P\mathcal{H}_k \oplus |x|^2 P_{k-2}. \]
Denote \( \mathcal{H}_k \) the space of spherical harmonics of degree \( k \) on \( S^{n-1} \). We have the following orthogonal summation.

**Proposition 4.1**
\[
L^2(S^{n-1}) = \bigoplus_{k=0}^{+\infty} \mathcal{H}_k.
\]

One intends to construct a multi-resolution analysis on \( S^{n-1} \) with approximation wavelet spaces \( W_j = \mathcal{H}_j \). It brought us to think about the following problem.

- Find a mother wavelet \( \psi \in L^2(S^{n-1}) \) such that \( (\psi_j,k) \) forms an orthonormal basis of \( W_j \).
- Given a wavelet \( \psi \) on \( \mathbb{R}^n \), Can its radial transform be a wavelet on \( S^{n-1} \)?
- What can the zonal harmonics offer?

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**References**


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