Fixed Point Theorem for Fuzzy Contractive Mappings in Fuzzy Metric Spaces

Arben Isufati
Department of Mathematics and Computer Sciences
Faculty of Natural Sciences
University of Gjirokastra
Gjirokastra, Albania
benisufati@yahoo.com

Elida Hoxha
Department of Mathematics
Faculty of Natural Sciences
University of Tirana
Tirana, Albania
hoxhaelida@yahoo.com

Abstract
In this paper we give a fixed-point theorem for fuzzy contractive mappings in the George and Veeramani $p$-complete fuzzy metric spaces.

Keywords: Fuzzy contractive mapping; $p$-complete fuzzy metric spaces.

1 Introduction
After the definition of the concept of fuzzy metric space by some authors[1, 6, 7] the fixed point theory on these spaces has been developing[5, 8, 9,10, etc ]. Gregori and Sapena introduced the notion of fuzzy contractive mapping and proved fixed point theorems for these mappings in $M$-complete GV(George and Veeramani) fuzzy metric spaces. Mihet, in [8, 9,10] respectively, extend and improve the results in [4] using different types of completeness Recently Gregori et.al [2] introduced the notion of $p$-complete GV fuzzy metric spaces.
The purpose of this work is to extend the results of Gregori and Sapena in the case of $p$-complete GV fuzzy metric spaces.

2 Preliminaries

Definition 2.1 [1] A fuzzy metric space is an ordered triple $(X, M, T)$ such that $X$ is a (nonempty) set, $T$ is continuous $t$-norm and $M$ is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$:

\begin{align*}
(GV1) & \quad M(x, y, t) > 0, \\
(GV2) & \quad M(x, y, t) = 1 \text{ if and only if } x = y, \\
(GV3) & \quad M(x, y, t) = M(y, x, t), \\
(GV4) & \quad T(M(x, y, t), M(y, z, s)) \leq M(x, z, t+s) \\
(GV5) & \quad M(x, y, \cdot):(0, \infty) \rightarrow [0,1] \text{ is continuous.}
\end{align*}

If $(X, M, T)$ is a fuzzy metric space we say that $(M, T)$ or $M$ is a fuzzy metric on $X$. Also, we say that $(X, M)$ or, simply, $X$ is a fuzzy metric space.

George and Veeramani proved in [1] that every fuzzy metric $M$ on $X$ generates a topology $\tau_M$ on $X$ which has as a base the family of open sets of the form \( \{B_M(x, \varepsilon, t) : x \in X, 0 < \varepsilon < 1, t > 0\} \), where

\[ B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\} \text{ for all } x \in X, \varepsilon \in (0,1) \text{ and } t > 0. \]

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, T)$ is said to be convergent to a point $x \in X$ (denoted by $\lim_{n \to \infty} x_n = x$), if and only if $\lim_{n \to \infty} M(x_n, x, t) = 1$, for all $t > 0$.

Definition 2.2[1] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, T)$ is said to be $M$-Cauchy sequence if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

A fuzzy metric space in which every $M$-Cauchy sequence is convergent with respect to $\tau_M$, is said to be $M$-complete.

Definition 2.3[8] Let $(X, M, T)$ be a GV fuzzy metric space. A sequence $(x_n)_{n \in N}$ in $X$ is said to be point convergent to $x_0 \in X$ (shown as $x_n \xrightarrow{p} x_0$), or $p$-convergent, if there exists $t_0 > 0$ such that $\lim_{n \to \infty} M(x_n, x_0, t_0) = 1$.

In such case we say that $(x_n)_{n \in N}$ is $p$-convergent to $x_0$ for $t_0 > 0$, or simply, $(x_n)_{n \in N}$ is $p$-convergent.
Equivalently, \((x_n)_{n\in\mathbb{N}}\) is \(p\)-convergent if there exists \(x_0 \in X\) and \(t_0 > 0\) such that \(\{x_n\}\) is eventually in \(B(x_0,r,t_0)\) for each \(r \in (0,1)\).

Clearly \(\{x_n\}\) is convergent to \(x_0\) if and only if \(\{x_n\}\) is \(p\)-convergent to \(x_0\) for all \(t > 0\).

The following properties hold [2,8]:

1. If \(\lim_{n\to\infty} M(x_n,x,t_1) = 1\) and \(\lim_{n\to\infty} M(x_n,y,t_2) = 1\) then \(x = y\).
2. If \(\lim_{n\to\infty} M(x_n,x_0,t_0) = 1\) then \(\lim_{n\to\infty} M(x_n,x_0,t_0) = 1\) for each subsequence \(\langle x_n \rangle\) of \(\{x_n\}\).
3. (Corollary 6, [1]) If \(\{x_n\}\) is \(p\)-convergent to \(x_0\) and it is convergent, then \(\{x_n\}\) converges to \(x_0\).
4. (Examples 8,13 [2]) There are \(p\)-convergent sequences which are not convergent.

**Definition 2.4** [2] Let \((X,M)\) be a fuzzy metric space. A sequence \(\{x_n\}\) in \(X\) is said to be \(p\)-Cauchy if for each \(\varepsilon \in (0,1)\) there are \(n_0 \in \mathbb{N}\) and \(t_0 > 0\) such that \(M(x_n,x_m,t_0) > 1 - \varepsilon\).

In such case we say that \(\{x_n\}\) is \(p\)-Cauchy for \(t_0 > 0\), or simply, \(\{x_n\}\) is \(p\)-Cauchy.

Clearly \(\{x_n\}\) is Cauchy sequence in the sense of Definition 2.2 if and only if \(\{x_n\}\) is \(p\)-Cauchy for all \(t > 0\). Obviously, \(p\)-convergent sequences are \(p\)-Cauchy.

**Definition 2.5** [2] The fuzzy metric space \((X,M)\) is called \(p\)-complete if every \(p\)-Cauchy sequence in \(X\) is \(p\)-convergent to some point of \(X\). In such case \((X,M)\) is called \(p\)-complete.

**Definition 2.6** [4] A fuzzy contractive mapping on a fuzzy metric space \((X,M)\) with the property \(M(x,y,t) \neq 0\ \forall x,y \in X, \forall t > 0\) is a self-mapping \(A\) of \(X\) satisfying

\[
\frac{1}{M(A(x),A(y),t)} - 1 \leq k \left( \frac{1}{M(x,y,t)} - 1 \right)
\]

where \(k\) is a fixed number in \((0,1)\).

As Mihet remarked in [9] we rewrite the above contractivity condition in the following equivalent form:
Definition 2.7[9] A self-mapping \( f \) of \( X \) is called fuzzy \( \psi \)-contractive mapping if satisfying the following condition

\[
M(Ax, Ay, t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))}.
\]

where \( \psi \) is a continuous mapping such that \( \psi(\lambda) > \lambda \) for all \( \lambda \in (0, 1) \). It is easy to see that a fuzzy contractive mapping is a fuzzy \( \psi \)-contractive mapping with \( \psi(\lambda) = \frac{\lambda}{\lambda + k(1 - \lambda)} \), \( k \in (0, 1) \).

3 Main results

Theorem 3.1 Let \((X, M)\) be a \( p \)-complete GV fuzzy metric space and \( A: X \to X \) be a fuzzy \( \psi \)-contractive mapping. Then \( A \) has a unique fixed point.

Proof. Let \( x_n = A^n(x), n \in N \) \((A(x) = x)\). Since \( A \) fuzzy \( \psi \)-contractive mapping the inequality \( M(A(x), A(y), t) \geq \psi(M(x, y, t)) \) holds \( \forall x, y \in X \) and \( \forall t > 0 \) we have \( M(A(x_1), A(x_2), t) \geq \psi(M(x_1, x_2, t)) > M(x_1, x_2, t), \forall t > 0 \) and then, by induction \( M(A(x_{n+1}), A(x_{n+2}), t) \geq M(x_n, x_{n+1}, t), \forall t > 0 \).

Therefore for every \( t > 0 \), the sequence \((M(x_n, x_{n+1}, t))_{n \in N}\) of numbers in \((0,1]\) is non-decreasing. Since for a \( t > 0 \), \( M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_n, x_{n+1}, t)) \) and \( \psi \) is continuous, we have \( l \geq \psi(l) \). This implies \( l = 1 \) and therefore for all \( t > 0 \),

\[
\lim_{n \to \infty} M(x_n, x_{n+1}, t) = 1
\]

Now we show that \( \{x_n\} \) is a \( p \)-Cauchy sequence.
Supposing this is not true, then there exists \( \epsilon \in (0, 1) \), such that for each \( k \in N \) there exist \( m(k), n(k) \geq k \) and

\[
M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \epsilon \quad \text{for all} \quad t > 0.
\]

Let, for each \( k \in N \), \( m(k), n(k) \) be the last integers satisfying the above inequality, such that \( M(x_{m(k)-1}, x_{n(k)}, t) > 1 - \epsilon \).
Then for each \( k \in N \)
\[
1 - \varepsilon \geq M(x_{m(k)}, x_{n(k)}, t)
\]
\[
\geq M(x_{m(k)-1}, x_{n(k)}, t) * M(x_{m(k)-1}, x_{m(k)}, t)
\]
\[
\geq (1 - \varepsilon) * M(x_{m(k)-1}, x_{m(k)}, t)
\]
Thus \( \lim_{k \to \infty} M(x_{m(k)}, x_{n(k)}, t) = 1 - \varepsilon \), for all \( t > 0 \).

Then
\[
M(x_{m(k)}, x_{n(k)}, t) \geq M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)+1}, x_{n(k)+1}, t) * M(x_{n(k)+1}, x_{n(k)}, t)
\]
\[
\geq M(x_{m(k)}, x_{m(k)+1}, t) * \psi(M(x_{m(k)}, x_{n(k)}, t)) * M(x_{n(k)+1}, x_{n(k)}, t)
\]
Letting \( k \to \infty \), we obtain
\[
1 - \varepsilon \geq 1 * \psi(1 - \varepsilon) * 1 > 1 - \varepsilon
\]
which is a contradiction.

Thus \( \{x_n\} \) is a \( p \)-Cauchy sequence, that is for each \( \varepsilon \in (0,1) \) there are \( n_0 \in N \)
and \( t_0 > 0 \) such that \( M(x_n, x_{n_0}, t_0) > 1 - \varepsilon \). Since \((X, M)\) is \( p \)-complete then there exists \( x_0 \in X \) such that \( x_n \to x_0 \).

By
\[
M(f(x_n), f(x_0), t_0) \geq \psi(M(x_{n-1}, x_0, t_0)) > M(x_{n-1}, x_0, t_0),
\]
we have \( f(x_{n-1}) = x_n \to f(x_0) \), and since the \( p \)-convergence is Frechet in a George and Veeramani fuzzy metric space, therefore \( f(x_0) = x_0 \) which means \( x_0 \)
is a fixed point and the proof is complete.

**Corollary 3.2** Let \((X, M)\) be a \( p \)-complete GV fuzzy metric space and \( A: X \to X \) be a fuzzy contractive mapping in the sense of Gregori and Sapena. Then \( A \) has a fixed point.

**Proof.** The proof following by Theorem 3.1 for \( \psi(\lambda) = \frac{\lambda}{\lambda + k(1 - \lambda)} \), \( k \in (0,1) \).

Also for \( \psi(\lambda) = 1 - k + k\lambda \) we have the following:

**Corollary 3.3.** Let \((X, M)\) be a \( p \)-complete GV fuzzy metric space and \( A: X \to X \) be a fuzzy contractive mapping such that:
\[
M(fx, fy, t) \geq 1 - k + kM(x, y, t), \ \forall x, y \in X \ \forall t > 0
\]
Then \( A \) has a fixed point.
References


Received: December, 2011