Analytical Solution for an Arithmetic Asian Option Using Mellin Transforms

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Abstract

The analytical solution of the Black-Scholes PDE for Asian options is not known as an explicit formula, this is due to the fact that, the arithmetic average of a set of lognormal random variables is not lognormally distributed. In this paper, we derive a closed form solution for a continuous arithmetic Asian option by means of partial differential equations (PDEs). We provide a new method for solving arithmetic Asian options PDEs using Mellin transform in a stock price to reduce the second order PDE of the arithmetic Asian options to the first order. In addition, we use partial differential equations to have a final solution for the continuous arithmetic Asian options PDE.

Keywords: Partial differential equations, arithmetic Asian option, Mellin transform

1 Introduction

Asian options are securities with payoffs that depend on the average value of an underlying stock-price over its lifespan. Asian options are less sensitive to market fluctuations near the expiry date. However these options have proved to be much more difficult to price than other options. as a result, there are many technique developed in the literature to address the problem, some of them; Geman and Yor [4] obtain an analytical formula for pricing Asian option using Laplace transform in time of the Asian option. However, this transform exists for some cases. Rogers and Shi [6] reduce the PDE for Asian option to the PDE in two variables instead three, and then, they use numerical procedure to solve it. Also they derive lower-bound formulas for Asian options by computing the expectation based on some zero-mean Gaussian variable. Zhang [8] presents a theory of continuously-sampled Asian option pricing; he solves
the PDE with perturbation approach. Yang et. al. [7] derive quasi analytical expressions for price and hedge arithmetic Asian call option. Elshegmani et. al. [3] derive a modified arithmetic Asian options PDE, together with its analytical solution. Jodar et. al. [5] obtain a direct solution for the classical Black-Scholes equation using Mellin transform. In this paper we offer an analytical solution for arithmetic Asian option by applying Mellin transform to the partial differential equation for arithmetic Asian option. We have a direct solution without using any program Mathematica or any numerical procedure.

From Dewynine [2], and Rogers [6] the value of the arithmetic Asian options $V(t, S, A)$ satisfies:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial A} - rV = 0$$

(1)

$$V(T, S, A) = \varphi(S, A)$$

With boundary condition, $S$ is the stock price, $r$ is the interest rate, $\sigma$ is the asset volatility, $T$ the expiration date, and $A = \frac{1}{t} \int_0^t S(t) dt$ is the average of the stock price at time $t$.

There are four different types of the arithmetic Asian options according to the payoff function $\varphi(S, A)$:

1. Fixed strike call: $\varphi(S, A) = (A - K)^+$
2. Fixed strike put: $\varphi(S, A) = (K - A)^+$
3. Floating strike call: $\varphi(S, A) = (S - A)^+$
4. Floating strike put: $\varphi(S, A) = (A - S)^+$

Where $K$ is the strike price.

To solve equation (1) we will apply Mellin transform in $S$:

Throughout of this paper the set of all absolute Lebesgue integrable functions in a set $J$ of the real line will be denoted by $L^1(J)$. A function $f(x)$ is Mellin transformable if the function $f(x)x^{z-1}$ is in $L^1([0, \infty])$ for some $z > 0$. Then the Mellin transforms of the function $f(x)$, denoted by $\mathcal{M}[f(x)](Z)$ is defined by

$$f^*(Z) = \mathcal{M}[f(x)](Z) = \int_0^\infty f(x)x^{Z-1}dx,$$
The inverse Mellin transform of \( f^*(Z) \) is given by

\[
\mathcal{M}^{-1} \left[ f^*(Z) \right] (x) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} f^*(Z) x^{-Z} dZ, \beta > Z
\]

Some properties of Mellin transforms [1]:

\[
\mathcal{M} \left[ x \frac{\partial f(x)}{\partial x} \right] (Z) = -Z f^*(Z)
\]

\[
\mathcal{M} \left[ x^2 \frac{\partial^2 f}{\partial x^2} \right] (Z) = (Z^2 + Z) f^*(Z)
\]

\[
\mathcal{M} \left[ x f(x) \right] (Z) = f^*(Z + 1)
\]

Since \( S > 0 \) so we will also use this property

\[
\mathcal{M} \left[ x f(x - x_0) \right] (z) = -\frac{x_0^{1+z}}{1+z}
\]

Using Eq. (2), (3) and (4) applying Mellin transform on Eq. (1).

\[
\frac{\partial \hat{V}(t, Z, A)}{\partial t} + \frac{1}{2} \sigma^2 (Z^2 + Z) \hat{V}(t, Z, A) - rZ \hat{V} + \frac{\partial \hat{V}(t, Z + 1, A)}{\partial A}
\]

\[-r \hat{V}(t, Z, A) = 0
\]

\[
\frac{\partial \hat{V}(t, Z, A)}{\partial t} + \frac{\partial \hat{V}(t, Z + 1, A)}{\partial A} + \left[ \frac{1}{2} \sigma^2 (Z^2 + Z) - rZ - r \right] \hat{V}(t, Z, A) = 0
\]

\[
\hat{V}(T, Z, A) = \varphi(Z, A)
\]

The payoff function \( \varphi(S, A) \) for the case of fixed strike call and put its still the same, but for floating strike call and put will be transformed as followings:

For the case of floating strike call: \( \varphi(Z, A) = (-\frac{S_0^{1+z}}{1+z} - A)^+ \),

And for floating strike put: \( \varphi(Z, A) = (A + \frac{S_0^{1+z}}{1+z})^+ \)
Now make the following change of the variables on Eq. (7)

\[ \hat{V}(t, Z, A) = f(t, Z, A) \exp \left[ \frac{1}{2} \sigma^2 (Z^2 + Z) - rZ - r \right] \tau, \tau = T - t \]

Eq. (1.7) is transformed to

\[ \frac{\partial f(t, Z, A)}{\partial \tau} = \frac{\partial f(t, Z + 1, A)}{\partial A} \] (8)

In order to get rid from the variable \( Z + 1 \) of the function in the left side we shall applying Fourier transform in \( Z \)

\[ \frac{\partial \hat{f}(t, \hat{Z}, A)}{\partial \tau} = e^{i\hat{Z}} \frac{\partial \hat{f}(t, \hat{Z}, A)}{\partial A} \] (9)

The solution of Eq. (9) is

\[ \hat{f}(\tau, \hat{Z}, A) = c \left( \tau + Ae^{-i\hat{Z}} \right) \] (10)

\[ \hat{f}(0, \hat{Z}, A) = \varphi(\hat{Z}, A) = cAe^{i\hat{Z}} \]

\[ c = \frac{\varphi(\hat{Z}, A)}{Ae^{i\hat{Z}}} = \varphi(\hat{Z}, A) \]

\[ \hat{f}(\tau, \hat{Z}, A) = \varphi(\hat{Z}, A) \left( \tau + Ae^{-i\hat{Z}} \right) \] (11)

Applying inverse Fourier transform in \( \hat{Z} \)

\[ f(\tau, Z, A) = \varphi(Z, A) \left( \tau \delta(\hat{Z}) + A\delta(\hat{Z} - 1) \right) \] (12)

\[ \hat{V}(\tau, Z, A) = \varphi(Z, A) \left( \tau \delta(\hat{Z}) + A\delta(\hat{Z} - 1) \right) \]

\[ \exp \left[ \frac{1}{2} \sigma^2 (Z^2 + Z) - rZ - r \right] \tau \] (13)

\[ V(t, S, A) = \mathcal{M}^{-1} \left[ \hat{V}(t, Z, A) \right] \]
Analytical solution for an arithmetic Asian option

\[ V(t, S, A) = \frac{1}{2\pi} \int_{-\infty}^{\beta+i\infty} S^{-z} \varphi(Z, A) \left( (T-t) \delta(\hat{Z}) + A\delta(\hat{Z} - 1) \right) \]

\[ \exp \left[ \frac{1}{2} \sigma^2 \left( Z^2 + Z \right) - rZ - r \right] (T-t) dZ \]  

(14)

To prove that, expression (14) is a direct solution for Eq. (1) we will differentiate the function \( V(t, S, A) \) respect to all variable and then, substituting in Eq. (1)

\[ \frac{\partial V}{\partial t} = -\left[ \frac{1}{2} \sigma^2 \left( Z^2 + Z \right) - rZ - r \right] V(t, S, A) + \]

\[ \frac{-1}{2\pi} \int_{-\infty}^{\beta+i\infty} \left\{ S^{-z} \varphi(Z, A) \delta(\hat{Z}) \exp \left[ \frac{1}{2} \sigma^2 \left( Z^2 + Z \right) - rZ - r \right] (T-t) \right\} dZ \]  

(15)

\[ \frac{\partial V}{\partial S} = \frac{1}{2\pi} \int_{-\infty}^{\beta+i\infty} \left\{ -Z S^{-z-1} \varphi(Z, A) \left( (T-t) \delta(\hat{Z}) + A\delta(\hat{Z} - 1) \right) \right\} \]

\[ \exp \left[ \frac{1}{2} \sigma^2 \left( Z^2 + Z \right) - rZ - r \right] (T-t) dZ \]  

(16)

\[ \frac{\partial^2 V}{\partial S^2} = \frac{1}{2\pi} \int_{-\infty}^{\beta+i\infty} \left\{ -Z(-Z-1) S^{-z-2} \varphi(Z, A) \left( (T-t) \delta(\hat{Z}) + A\delta(\hat{Z} - 1) \right) \right\} \]

\[ \exp \left[ \frac{1}{2} \sigma^2 \left( Z^2 + Z \right) - rZ - r \right] (T-t) dZ \]  

(17)

\[ \frac{\partial V}{\partial A} = \frac{1}{2\pi} \int_{-\infty}^{\beta+i\infty} \left\{ S^{-z} \varphi(Z, A) \left( \delta(\hat{Z} - 1) \right) \right\} \]

\[ \exp \left[ \frac{1}{2} \sigma^2 \left( Z^2 + Z \right) - rZ - r \right] (T-t) dZ \]  

(18)
Substituting Eq. (14) - (18) in Eq. (1) yield;

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial A} - rV =
\]

\[
- \left[ \frac{1}{2} \sigma^2 (Z^2 + Z) - rZ - r \right] V - \frac{1}{2\pi} \int_{\beta-i\infty}^{\beta+i\infty} \{S^{-z} \delta(\hat{Z}) \varphi(Z, A)\} dz
\]

\[
\exp \left[ \frac{1}{2} \sigma^2 (Z^2 + Z) - rZ - r \right] (T-t) \right] V + \exp \left[ \frac{1}{2} \sigma^2 [Z^2 + Z] - r - rZ \right] V + \exp \left[ \frac{1}{2} \sigma^2 [Z^2 + Z] - r - rZ \right] (T-t) dZ
\]

Rearranging group yield

\[
= \left( -\frac{1}{2} \sigma^2 (Z^2 + Z) + rZ + r - rZ + \frac{1}{2} \sigma^2 [Z^2 + Z] - r \right) V + \frac{1}{2\pi} \int_{\beta-i\infty}^{\beta+i\infty} \left\{ S^{-z} (S \delta(\hat{Z} - 1) - \delta(\hat{Z})) \varphi(Z, A) \right\} dz
\]

\[
\exp \left[ \frac{1}{2} \sigma^2 (Z^2 + Z) - rZ - r \right] (T-t) dZ = 0
\]

The last equality because

\[
(S \delta(\hat{Z} - 1) - \delta(\hat{Z})) = e^{-iz} \delta (\hat{Z} - 1) - \delta(\hat{Z}) = \int_{-\infty}^{\infty} e^{-iz} e^{-iz(\hat{Z}-1)} d\hat{Z} - \delta(\hat{Z})
\]

\[
= \int_{-\infty}^{\infty} e^{-iz(\hat{Z}-1+1)} - \delta(\hat{Z}) = \int_{-\infty}^{\infty} e^{-iz\hat{Z}} d\hat{Z} - \delta (\hat{Z}) = \delta (\hat{Z}) - \delta (\hat{Z}) = 0
\]

So \( V(t, S, A) \) is an explicit analytical solution for the arithmetic Asian options. The solution for all types of the continuous arithmetic Asian options can be obtained only by changing the payoff function \( \varphi(Z, A) \) according to the type of options we need to price or value as we have explained before.
2 Conclusion

In this work we approach the problem of computing the price of the arithmetic Asian option by using Mellin transform, we reduce the PDE of the arithmetic Asian options from a second order to the first order, and we got the solution without reducing the dimensions of the PDE, as most authors have done. And then, we use partial differential equations to obtain the final solution.

References


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