Existence of Positive Periodic Solution
of an Impulsive Delay SIS Model

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Abstract

In this paper, the impulsive delay epidemic SIS model is considered. By using the continuation theory for k-set contractions, the sufficient conditions of the existence of positive \( \omega \)-periodic solutions of the impulsive delay SIS model are obtained.

Keywords: SIS model; k-set contractions; Existence; Periodic solution

1. Introduction

Many real-world evolution processes which depend on their prehistory and are subject to short time disturbances can be modeled by impulsive delay differential equations. Such processes occur in the theory of optimal control, population dynamics, biology, economics, etc. For details, see [1,2] and references therein. In the last few years, the existence problems of positive periodic solutions of differential equations with impulsive effects and/or delay have been studied by many researchers [3-5,7,8].

In [4], the author considered the impulsive Logistic model. The sufficient conditions of the existence and asymptotic stability of T-periodic solution were obtained. In [5], the author studied an impulsive delay differential equation and sufficient conditions are obtained for the existence and global attractivity of periodic positive solutions. It is shown that under appropriate linear periodic impulsive perturbations, the impulsive delay differential equation preserves the original periodicity and global attractive properties of the non-impulsive delay differential equation. In [6], an epidemic SIS model with periodic contact
rate was first introduced by H.W. Hethcote and asymptotic behavior in the epidemic model is considered.

The main purpose of this paper is to study the impulsive delay epidemic SIS model. By using the continuation theory for k-set contractions [10], the existence of positive periodic solution of this model is considered and sufficient conditions are obtained for the existence of periodic positive solutions.

This paper is organized in three sections including the introduction. Section 2 formulates the problem and presents the preliminary results. The sufficient conditions for the existence of positive $\omega$–periodic solution of the model are established in section 3.

2. Preliminaries

Consider the following impulsive delay epidemic SIS model

\[
\begin{align*}
\begin{cases}
x'(t) = & \lambda(t)x(t)[(1-x(t-\tau)] - \mu(t)x(t), \quad t \neq t_k, \\
x(t_k^+) - x(t_k) = & b_kx(t_k), \quad k = 1, 2, \ldots
\end{cases}
\end{align*}
\]

(1)

where $x(t)$ is the infected fraction of the population at time $t$. $\lambda(t)$ is the contact rate of the population at time $t$. $\mu(t)$ is the rate of recovery. $\tau$ is the gestation period. $b_k$ is impulsive perturbation at the moments of time $t_k, k = 1, 2, \ldots$.

The following assumption will be needed throughout the paper.

(A1) $0 < t_1 < t_2 < \ldots$ are fixed impulsive points with $\lim_{k \to \infty} t_k = \infty$.

(A2) $\lambda(t), \mu(t) \in (0, 1), \ [0, 1]$ are locally integrable functions.

(A3) $\{b_k\}$ is a real sequence and $1 + b_k > 0, k = 1, 2, \ldots$.

(A4) $\lambda(t), \mu(t), \prod_{0< t_k < t}(1+b_k)$ are positive continuous $\omega$-periodic functions and in the sequence the product equals to unity if the number of factors is zero.

We shall consider (1) with the initial condition

\[x(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad \phi(t) \in L([-\tau, 0], (0, \infty)),\]

(2)

where $L([-\tau, 0], (0, \infty))$ denotes the set of Lebesgue measurable functions on $[-\tau, 0]$.

By a solution $x(t)$ of (1) satisfying initial condition (2) we mean an absolutely continuous function $x(t)$ on $[-\tau, \infty)$, and satisfies conditions: $x(t_k^+)$ and $x(t_k)$ exist for any $t_k, k = 1, 2, \ldots$, and $x(t)$ satisfies (1) for almost everywhere in $[0, \infty)$ and at impulsive points $t_k$ may have discontinuity of the first kind.

Under assumption (A1)–(A4), obviously, all solution of (1) and (2) are positive $[0, \infty]$. We make a transformation $y(t) = \ln x(t)$, then $y'(t) = \frac{x'(t)}{x(t)}, x(t) =
$e^{y(t)}$ and by substituting them into (1) and (2) we can obtain
\[
\begin{aligned}
\{ \ y'(t) &= \lambda(t) - \mu(t) - \lambda(t)e^{y(t-\tau)}, \ t \neq t_k, \\
 y(t_k) - y(t_k^-) &= \ln(1 + b_k), \ k = 1, 2, \ldots, 
\end{aligned}
\] (3)
and the initial condition
\[
y(t) = \ln \phi(t), \ -\tau \leq t \leq 0, \ \phi(t) \in L([-\tau, 0], (0, \infty)).
\] (4)

For investigating Eqs. (3) and (4), we introduce following non-impulsive delay differential equation
\[
z'(t) = \lambda(t) - \mu(t) - \lambda(t)e^{\sum_{0<t_j<t} \ln(1+b_j)}\ e^{z(t-\tau)}
\] (5)
with the initial condition
\[
z(t) = \ln \phi(t), \ -\tau \leq t \leq 0, \ \phi(t) \in L([-\tau, 0], (0, \infty)).
\] (6)

**Theorem 2.1.** If $z(t)$ is a solution of (5) and (6) on $[-\tau, \infty)$, then $y(t) = z(t) + \sum_{0<t_j<t} \ln(1+b_j)$ is a solution of (3) and (4). And if $y(t)$ is a solution of (3) and (4) on $[-\tau, \infty)$, then $z(t) = y(t) - \sum_{0<t_j<t} \ln(1+b_j)$ is a solution of (5) and (6).

The proof of the Theorem 2.1 is similar to that of Theorem 1 [10] and is omitted.

We give a brief explanation of the abstract continuation theory for k-set contractions that will be used in proof of the main results of the paper.

Let $Z$ be a Banach space. For a bounded subset $A \subset Z$, let $\Gamma_Z(A)$ denote the (Kuratovski) measure of non-compactness defined by
\[
\Gamma_Z(A) = \inf \{ \ \delta > 0 : \exists \text{ a finite number of subsets } A_i \subset A, \\
A = \cup_i A_i, \text{diam}(A_i) \leq \delta \}. 
\]
Here, diam$(A_i)$ denotes the maximum distance between the points in the set $A_i$. Let $X$ and $Y$ be Banach spaces and $\Omega$ a bounded open subset of $X$. A continuous and bounded map $N : \bar{\Omega} \rightarrow Y$ is called k-set-contractive if for any bounded $A \subset \bar{\Omega}$ we have $\Gamma_Y(N(A)) \leq k \Gamma_X(A)$. Also, for a continuous and bounded map $T : X \rightarrow Y$ we define
\[
l(T) = \sup \{ r \geq 0 : \forall \text{ bounded subset } A \subset X, r \Gamma_X(A) \leq \Gamma_Y(T(A)) \}. 
\]

**Theorem 2.2[11].** Let $L : X \rightarrow Y$ be a Fredholm operator of index zero, and $q(t) \in Y$ be a fixed point. Suppose that $N : \bar{\Omega} \rightarrow Y$ is k-set contractive.
with \( k < l(L) \), where \( \Omega \subset X \) is bounded, open, and symmetric about \( 0 \in \Omega \). Suppose further that:

(i) \( Lx \neq \lambda Nx + \lambda q(t) \) for \( x \in \partial \Omega, \, \lambda \in (0, 1) \) and

(ii) \([QN(x) + Qq(t), x] \cdot [QN(-x) + Qq(t), x] < 0 \), for \( x \in \text{Ker}(L) \cap \partial \Omega \).

Where \([\cdot, \cdot]\) is a bilinear form on \( Y \times X \) and \( Q \) is the projection of \( Y \) onto \( \text{coker}(L) \).

Then there exists \( x \in \bar{\Omega} \) such that \( Lx - Nx = q(t) \).

### 3. Main results

For the convenience of investigation, we still denote \( z(t) \) by \( x(t) \) in Eqs. (5) and (6), then the new form is obtained

\[
x'(t) = q(t) - p(t)e^{x(t-\tau)},
\]

with the initial condition

\[
x(t) = \ln \phi(t), \, -\tau \leq t \leq 0, \, \phi(t) \in L([-\tau, 0], (0, \infty)),
\]

where \( q(t) = \lambda(t) - \mu(t), \, p(t) = \lambda(t) \prod_{0 < j, \epsilon_{t_0} \in \mathbb{R}} (1 + b_j) \) are positive continuous \( \omega \)-periodic functions.

Denote \( Y = C^0_\omega \) is a linear Banach space of real-valued \( \omega \)-periodic functions on \( \mathbb{R} \). In \( C^0_\omega \), for \( x \in C^0_\omega \), the norm is defined by \( |x|_0 = \sup_{t \in \mathbb{R}} |x(t)| \). Let \( X = C^1_\omega \) denote the linear space of \( \omega \)-periodic functions with the first-order continuous derivative. \( C^1_\omega \) is a Banach space with norm \( |x|_1 = \max \{|x|_0, |x'|_0\} \).

Let \( L : X \rightarrow Y \) be given by \( Lx = \frac{dx}{dt} = x' \).

Since \( |Lx|_0 = |x'|_0 \leq |x|_1 \), we see that \( L \) is a bounded linear map. Next define a nonlinear map \( N : X \rightarrow Y \) by \( Nx(t) = -p(t)e^{x(t-\tau)} \). Now, Eq. (7) has a solution \( x(t) \) if and only if \( Lx = Nx + q(t) \) for some \( x \in X \).

**Theorem 3.1.** Let

\[
M = \max \left\{ \left| \ln \frac{\bar{q}}{\bar{p}} \right|, R_1, M_1, M_2 \right\}, \quad R_1 = \ln \frac{\bar{q}}{\bar{p}^m} + 2\omega \bar{q}, \quad M_1 = |q|_0 + |p|_0 e^{R_1},
\]

\[
M_2 = M_1 \omega + \max \left\{ \left| \ln \frac{\bar{q}}{|p|_0} \right|, \left| \ln \frac{\bar{q}}{p^m} \right| \right\},
\]

where

\[
\bar{q} = \frac{1}{\omega} \int_0^\omega q(t)dt, \quad \bar{p} = \frac{1}{\omega} \int_0^\omega p(t)dt, \quad p^m = \min_{t \in [0, \omega]} p(t),
\]

\[
|q|_0 = \max_{t \in [0, \omega]} q(t), \quad |p|_0 = \max_{t \in [0, \omega]} p(t).
\]
Suppose that the condition \( p_0 e^M < 1 \) holds, then Eq. (6) has at least one \( \omega \)-periodic solution. Therefore, the system (1) has at least one positive \( \omega \)-periodic solution.

**Proof.** Let \( A \subset \bar{\Omega} \) be a bounded subset and let \( \eta = \Gamma_X(A) \), then for any \( \varepsilon > 0 \), there is a finite family of subset \( A_i \) with \( A = \cup_i A_i \) and \( \text{diam}_1(A_i) \leq \eta + \varepsilon \). Now it follows from the fact that \( g(t, x_1) = p(t)e^{x_1} \) is uniformly continuous on any compact subset of \( R \times R \) that, for any \( x, u \in A_i \), exists a \( \sigma \in (x, u) \),

\[
|N_x - Nu|_0 = \sup_{0 \leq t \leq \omega} |g(t, x(t - \tau)) - g(t, u(t - \tau))| \\
\leq |p_0| \sup_{0 \leq t \leq \omega} |e^{x(t-\tau)} - e^{u(t-\tau)}| \\
= |p_0| \sup_{0 \leq t \leq \omega} |e^{\sigma(t-\tau)}||x(t - \tau) - u(t - \tau)|.
\]

In this case, \( \sigma |_1 < r \), therefore, \( |N_x - Nu|_0 \leq |p_0| e^r |x - u|_0 \leq k_0 \eta + k_0 \varepsilon \). i.e.

\[
\Gamma_Y(N(A)) \leq k_0 \Gamma_X(A).
\]

Therefore, the map \( N \) is \( k_0 \)-set contractive.

If \( Lx = \lambda Nx + \lambda q(t) \) for any \( x(t) \in X, \lambda \in (0, 1) \), i.e.

\[
x'(t) = \lambda [q(t) - p(t)e^{x(t-\tau)}].
\]

Integrating (9) from 0 to \( \omega \), we have

\[
\int_0^\omega p(t)e^{x(t-\tau)}dt = \int_0^\omega q(t)dt.
\]

We have

\[
\int_0^\omega |x'(t)|dt \leq \lambda \left[ \int_0^\omega q(t)dt + \int_0^\omega p(t)e^{x(t-\tau)}dt \right] < 2 \int_0^\omega q(t)dt = 2\omega \bar{q}.
\]

Let \( s = t - \tau \), then

\[
\int_0^\omega p(t)e^{x(t-\tau)}dt = \int_{-\tau}^{\omega-\tau} p(s + \tau)e^{x(s)}ds \geq p_m \int_{-\tau}^{\omega-\tau} e^{x(s)}ds = p_m \int_0^{\omega} e^{x(s)}ds.
\]

It follows from (10) that

\[
\int_0^\omega q(t)dt \geq p_m \int_0^{\omega} e^{x(s)}ds = \omega p_m e^{x(\xi)}
\]

for some \( \xi_1 \in [0, \omega] \). Therefore, we have

\[
x(\xi_1) \leq \ln \frac{\bar{q}}{p_m}.
\]
From (11) and (12), we can see that
\[ x(t) \leq x(\xi_1) + \int_0^\omega |x'(t)| \leq \ln \frac{\bar{q}}{p^m} + 2\omega \bar{q} = R_1. \]

From (9), we have
\[ x'(t) \leq \lambda \left[ q(t) + p(t)e^{x(t-\tau)} \right] < |q|_0 + |p|_0 e^{R_1}. \]

So that
\[ |x'|_0 < |q|_0 + |p|_0 e^{R_1} \equiv M_1. \quad (13) \]

On the other hand, there exists a \( \xi_2 \in [0, \omega] \), such that
\[
\int_0^\omega p(t)e^{x(t-\tau)}dt = \int_{-\tau}^{\omega-\tau} p(s + \tau)e^{x(s)}ds = p(\xi_2) \int_{-\tau}^{\omega-\tau} e^{x(s)}ds = p(\xi_2) \int_0^\omega e^{x(s)}ds.
\]

Hence, from (10), we have
\[
\int_0^\omega e^{x(t)}dt = \frac{\int_0^\omega q(t)dt}{p(\xi_2)}.
\]

So, there exists a \( \xi_3 \in [0, \omega] \), such that
\[ e^{x(\xi_3)} = \frac{\bar{q}}{p(\xi_2)}, \]

i.e.
\[ x(\xi_3) = \ln \frac{\bar{q}}{p(\xi_2)} \]

It is clear that
\[ \ln \frac{\bar{q}}{|p|_0} \leq x(\xi_3) \leq \ln \frac{\bar{q}}{p^m} \]

Therefore,
\[ |x(\xi_3)| \leq max \left\{ \ln \frac{\bar{q}}{|p|_0}, \ln \frac{\bar{q}}{p^m} \right\} \equiv M_0 \quad (14) \]

We get
\[ |x|_0 \leq |x(\xi_3)| + \int_0^\omega |x'|_0 dt < M_0 + M_1 \omega \equiv M_2. \]
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This implies that

$$|x|_1 \leq M \equiv \max \left\{ |\ln \frac{\bar{q}}{\bar{p}}|, R_1, M_1, M_2 \right\}.$$  

Therefore, for $x(t) \in \partial \Omega, \lambda \in (0, 1)$

$$Lx \neq \lambda Nx + \lambda q(t).$$

This implies that the condition (i) of Theorem 2.2 holds.

Define a bounded bilinear form $[\cdot, \cdot]$ on $Y \times X$ by $[y, x] = \int_0^\omega y(t)x(t)dt.$ And define $Q : Y \rightarrow \text{coker}(L)$ by $y \rightarrow \int_0^\omega y(t)dt.$ So that for $x \in \ker(L) \cap \partial \Omega,$ we get

$$[QN(x) + Qq(t), x] \cdot [QN(-x) + Qq(t), x]$$

$$= r^2\omega^2 \left[ \int_0^\omega q(t)dt - e^r \int_0^\omega p(t)dt \right] \cdot \left[ \int_0^\omega q(t)dt - e^{-r} \int_0^\omega p(t)dt \right]$$

$$= r^2\omega^4 (\bar{q} - e^r \bar{p})(\bar{q} - e^{-r} \bar{p}).$$

Since $r > M \geq |\ln \frac{\bar{q}}{\bar{p}}|,$ we have $\bar{q} - e^r \bar{p} < 0,$ $\bar{q} - e^{-r} \bar{p} > 0.$ This implies that the condition (ii) of Theorem 2.2 holds.

Hence, it follows from Theorem 2.2 that there is a function $x(t) \in \bar{\Omega} \subset X$ such that $Lx - Nx = q(t).$ Thus, the proof of Theorem 3.1 is completed.

References


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