Exterior Set in Biminimal Structure Spaces

Supunnee Sompong

Department of Mathematics and Statistics
Faculty of Science and Technology
Sakon Nakhon Rajabhat University
Sakon Nakhon 47000, Thailand
s_sanpinij@yahoo.com

Abstract

The purpose of this paper is to introduce the concept and some fundamental properties of exterior set in biminimal structure space.

Keywords: minimal structure space, biminimal structure space, exterior

1 Introduction

The concept of minimal structure space was introduced by V. Popa and T. Noiri [5]. They also introduced the concepts of $m_X$-open set and $m_X$-closed set and characterize those sets using $m_X$-closure and $m_X$-interior operators respectively. J.C. Kelly [2] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. The notion of biminimal structure space was introduced by C. Boonpok [1] in 2010. Also he studied $m_1^Xm_2^X$-closed sets and $m_1^Xm_2^X$-open sets in biminimal structure spaces. In this paper, we introduced the concept of exterior set in biminimal structure space and studied some fundamental of their properties.

2 Preliminaries

In this section we recall the notions, notations and some results in [1] and [3].

Definition 2.1. [3] Let $X$ be a nonempty set and $P(X)$ the power set of $X$. A subfamily $m_X$ of $P(X)$ is called a minimal structure (briefly m-structure) on $X$ if $\emptyset \in m_X$ and $X \in m_X$

By $(X, m_X)$, we denote a nonempty set $X$ with an m-structure $m_X$ on $X$ and it is called an m-space.
Definition 2.2. [1] Let $X$ be a nonempty set and $m_X^1$, $m_X^2$ be minimal structures on $X$. A triple $(X, m_X^1, m_X^2)$ is called a biminimal structure space (briefly bim-space).

We defined all of elements in $m_X^1$ and $m_X^2$ are open sets.

Definition 2.3. [1] A subset of a biminimal structure space $(X, m_X^1, m_X^2)$ is called $m_X^1m_X^2$-closed if $A = m^1\text{Cl}(m^2\text{Cl}(A))$. The complement of $m_X^1m_X^2$-closed set is called $m_X^1m_X^2$-open.

Proposition 2.4. [1] Let $(X, m_X^1, m_X^2)$ be a biminimal structure space. Then $A$ is $m_X^1m_X^2$-open subset of $(X, m_X^1, m_X^2)$ if and only if $A = m^1\text{Int}(m^2\text{Int}(A))$.

Proposition 2.5. [1] Let $(X, m_X^1, m_X^2)$ be a biminimal structure space. If $A$ and $B$ are $m_X^1m_X^2$-closed subsets of $(X, m_X^1, m_X^2)$, then $A \cap B$ is $m_X^1m_X^2$-closed.

Proposition 2.6. [1] Let $(X, m_X^1, m_X^2)$ be a biminimal structure space. If $A$ and $B$ are $m_X^1m_X^2$-open subsets of $(X, m_X^1, m_X^2)$, then $A \cup B$ is $m_X^1m_X^2$-open.

3 Exterior

In this section, we introduce the concept and study some fundamental properties of exterior set in biminimal structure space.

Definition 3.1. Let $(X, m_X^1, m_X^2)$ be a biminimal structure space, $A$ be a subset of $X$ and $x \in X$. We called $x$ is $m_X^1m_X^2$-exterior point of $A$ if $x \in m^1\text{Int}(m^2\text{Int}(X \setminus A))$. We denote the set of all $m_X^1m_X^2$-exterior point of $A$ by $m\text{Ext}_{ij}(A)$ where $i, j = 1, 2$ and $i \neq j$.

From definition we have $m\text{Ext}_{ij}(A) = X \setminus m^1\text{Cl}(m^2\text{Cl}(A))$.

Example 3.2. Let $X = \{1, 2, 3\}$. Define $m$-structures $m_X^1$ and $m_X^2$ on $X$ as follows: $m_X^1 = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $m_X^2 = \{\emptyset, \{2\}, \{1, 3\}, X\}$. Then $m\text{Ext}_{12} (\{2\}) = X \setminus m^1\text{Cl}(m^2\text{Cl}(\{2\})) = \{1\}$ and $m\text{Ext}_{21} (\{2\}) = X \setminus m^2\text{Cl}(m^1\text{Cl}(\{2\})) = \emptyset$.

Lemma 3.3. Let $(X, m_X^1, m_X^2)$ be a biminimal structure space and $A$ be a subset of $X$. Then for any $i, j = 1, 2$ and $i \neq j$, we have:

1. $m\text{Ext}_{ij}(A) \cap A = \emptyset$.
2. $m\text{Ext}_{ij}(\emptyset) = X$.
3. $m\text{Ext}_{ij}(X) = \emptyset$. 
Proof. 1. Since $m\text{Ext}_{ij}(A) = X \setminus m^i\text{Cl}(m^j\text{Cl}(A))$ and $A \subset m^i\text{Cl}(m^j\text{Cl}(A))$, 
\[
(X \setminus m^i\text{Cl}(m^j\text{Cl}(A))) \cap A \subseteq (X \setminus A) \cap A = \emptyset.
\]
Therefore $(X \setminus m^i\text{Cl}(m^j\text{Cl}(A))) \cap A = \emptyset$. Hence $m\text{Ext}_{ij}(A) \cap A = \emptyset$.

2. $m\text{Ext}_{ij}(\emptyset) = X \setminus m^i\text{Cl}(m^j\text{Cl}(\emptyset)) = X \setminus \emptyset = X$.

3. $m\text{Ext}_{ij}(X) = X \setminus m^i\text{Cl}(m^j\text{Cl}(X)) = X \setminus X = \emptyset$.

\[\square\]

**Theorem 3.4.** Let $(X, m^1_X, m^2_X)$ be a biminimal structure space and $A, B$ be a subset of $X$. If $A \subseteq B$, then $m\text{Ext}_{ij}(B) \subseteq m\text{Ext}_{ij}(A)$ where $i, j = 1, 2$ and $i \neq j$.

**Proof.** Assume that $(X, m^1_X, m^2_X)$ is a biminimal structure space, $A, B$ are subset of $X$ and $A \subseteq B$. Thus $m^i\text{Cl}(m^j\text{Cl}(A)) \subseteq m^i\text{Cl}(m^j\text{Cl}(B))$ and $X \setminus m^i\text{Cl}(m^j\text{Cl}(B)) \subseteq X \setminus m^i\text{Cl}(m^j\text{Cl}(A))$. Hence $m\text{Ext}_{ij}(B) \subseteq m\text{Ext}_{ij}(A)$ for any $i, j = 1, 2$ and $i \neq j$.

\[\square\]

**Theorem 3.5.** Let $(X, m^1_X, m^2_X)$ be a biminimal structure space and $A$ be a subset of $X$. Then for any $i, j = 1, 2$ and $i \neq j$, $A$ is $m^1_X m^2_X$-closed if and only if $m\text{Ext}_{ij}(A) = X \setminus A$.

**Proof.** Let $A$ be a subset of $X$.

$(\Longrightarrow)$ Assume that $A$ is $m^1_X m^2_X$-closed. Thus $A = m^i\text{Cl}(m^j\text{Cl}(A))$.

Therefore $m\text{Ext}_{ij}(A) = X \setminus m^i\text{Cl}(m^j\text{Cl}(A)) = X \setminus A$.

$(\Longleftarrow)$ Assume that $m\text{Ext}_{ij}(A) = X \setminus A$. Thus $X \setminus m^i\text{Cl}(m^j\text{Cl}(A)) = X \setminus A$. Consequently $m^i\text{Cl}(m^j\text{Cl}(A)) = A$ and $A$ is $m^1_X m^2_X$-closed.

\[\square\]

**Corollary 3.6.** Let $(X, m^1_X, m^2_X)$ be a biminimal structure space and $A$ be a subset of $X$. Then for any $i, j = 1, 2$ and $i \neq j$, $A$ is $m^1_X m^2_X$-open if and only if $m\text{Ext}_{ij}(X \setminus A) = A$.

**Proof.** Let $A$ be a subset of $X$.

$(\Longrightarrow)$ Assume that $A$ is $m^1_X m^2_X$-open. Thus $X \setminus A$ is $m^1_X m^2_X$-closed. Therefore $m\text{Ext}_{ij}(X \setminus A) = X \setminus (X \setminus A) = A$.

$(\Longleftarrow)$ Assume that $m\text{Ext}_{ij}(X \setminus A) = A$.

Thus $A = m\text{Ext}_{ij}(X \setminus A) = X \setminus m^i\text{Cl}(m^j\text{Cl}(X \setminus A)) = m^i\text{Int}(m^j\text{Int}(A))$. Hence $A$ is $m^1_X m^2_X$-open.

\[\square\]
Theorem 3.7. Let \((X, m_X^1, m_X^2)\) be a biminimal structure space and \(A\) be a subset of \(X\). If \(A\) is \(m_X^1 m_X^2\)-closed, then \(m_{Ext} (X \setminus m_{Ext} (A)) = m_{Ext} (A)\), where \(i, j = 1, 2\) and \(i \neq j\).

Proof. Assume that \(A\) is \(m_X^1 m_X^2\)-closed. Thus \(m_{Ext} (A) = X \setminus A\). Hence \(m_{Ext} (X \setminus m_{Ext} (A)) = m_{Ext} (X \setminus (X \setminus A)) = m_{Ext} (A)\).

Example 3.8. Let \(X = \{1, 2, 3\}\). Define \(m\)-structures \(m_X^1\) and \(m_X^2\) on \(X\) as follows: \(m_X^1 = \{\emptyset, \{1\}, \{2, 3\}, X\}\) and \(m_X^2 = \{\emptyset, \{2\}, \{1, 3\}, \{2, 3\}, X\}\). Then \(m_{Ext} (\{2\}) = X \setminus m^1 \text{Cl}(m^1 \text{Cl}(\{2\}))\) and \(m_{Ext} (\{3\}) = X \setminus m^2 \text{Cl}(m^2 \text{Cl}(\{3\}))\).

Hence \(m_{Ext} (\{2\}) \cap \{3\} = X\), \(m_{Ext} (\{2\}) = X \setminus m^1 \text{Cl}(m^1 \text{Cl}(\{2\})) = \{1\}\) and \(m_{Ext} (\{3\}) = X \setminus m^2 \text{Cl}(m^2 \text{Cl}(\{3\})) = \emptyset\).

Therefore \(m_{Ext} (\{2\}) \cup m_{Ext} (\{3\}) \neq m_{Ext} (\{2\} \cap \{3\})\).

Theorem 3.9. Let \((X, m_X^1, m_X^2)\) be a biminimal structure space and \(A, B\) be subset of \(X\). Then for any \(i, j = 1, 2\) and \(i \neq j\), we have:

1. \(m_{Ext} (A) \cup m_{Ext} (B) \subseteq m_{Ext} (A \cap B)\).

2. If \(A\) and \(B\) are \(m_X^1 m_X^2\)-closed, then \(m_{Ext} (A) \cup m_{Ext} (B) = m_{Ext} (A \cap B)\).

Proof. Assume that \((X, m_X^1, m_X^2)\) is a biminimal structure space, \(A\) and \(B\) are subsets of \(X\).

1. Since \(A \cap B \subseteq A\) and \(A \cap B \subseteq B\), we have \(m_{Ext} (A) \subseteq m_{Ext} (A \cap B)\) and \(m_{Ext} (B) \subseteq m_{Ext} (A \cap B)\).

It follows that \(m_{Ext} (A) \cup m_{Ext} (B) \subseteq m_{Ext} (A \cap B)\).

2. Assume that \(A\) and \(B\) are \(m_X^1 m_X^2\)-closed. Then \(A \cap B\) is \(m_X^1 m_X^2\)-closed. Thus \(m_{Ext} (A \cap B) = X \setminus (A \cap B)\)

\[= (X \setminus A) \cup (X \setminus B) = m_{Ext} (A) \cup m_{Ext} (B).\]

Example 3.10. Let \(X = \{1, 2, 3\}\). Define \(m\)-structures \(m_X^1\) and \(m_X^2\) on \(X\) as follows: \(m_X^1 = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, X\}\) and \(m_X^2 = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, X\}\). Then \(m_{Ext} (\{1\}) = X \setminus m^1 \text{Cl}(m^1 \text{Cl}(\{1\}))\) and \(m_{Ext} (\{2\}) = X \setminus m^2 \text{Cl}(m^2 \text{Cl}(\{2\}))\).

Hence \(m_{Ext} (\{1\}) = X \setminus m^1 \text{Cl}(m^1 \text{Cl}(\{1\})) = X \setminus \{1\} = \{2, 3\}\)

\(m_{Ext} (\{2\}) = X \setminus m^1 \text{Cl}(m^1 \text{Cl}(\{2\})) = X \setminus \{2\} = \{1, 3\}\) and

\(m_{Ext} (\{1\} \cup \{2\}) = m_{Ext} (\{1, 2\}) = X \setminus m^1 \text{Cl}(m^1 \text{Cl}(\{1, 2\})) = \emptyset\).

Therefore \(m_{Ext} (\{1\} \cup \{2\}) \neq m_{Ext} (\{1\}) \cap m_{Ext} (\{2\})\).
Theorem 3.11. Let \((X, m^1_X, m^2_X)\) be a biminimal structure space and \(A, B\) be subsets of \(X\). Then for any \(i, j = 1, 2\) and \(i \neq j\), we have:

1. \(m_{ij}(A \cup B) \subseteq m_{ij}(A) \cap m_{ij}(B)\).
2. If \(A\) and \(B\) are \(m^i_X m^j_X\)-open, then
\[m_{ij}(A \cup B) = m_{ij}(A) \cap m_{ij}(B)\].

Proof. Assume that \((X, m^1_X, m^2_X)\) is a biminimal structure space, \(A\) and \(B\) are subsets of \(X\).

1. Since \(A \subseteq A \cup B\) and \(B \subseteq A \cup B\), we have \(m_{ij}(A \cup B) \subseteq m_{ij}(A)\) and \(m_{ij}(A \cup B) \subseteq m_{ij}(B)\).

It follows that \(m_{ij}(A \cup B) \subseteq m_{ij}(A) \cap m_{ij}(B)\).

2. Assume that \(A\) and \(B\) are \(m^1_X m^2_X\)-open. Then \(A \cup B\) is \(m^1_X m^2_X\)-open. It follows that \(X \setminus A\), \(X \setminus B\) and \(X \setminus (A \cup B)\) are \(m^1_X m^2_X\)-closed. Thus by Theorem 3.9 (2), we have
\[
m_{ij}(X \setminus A) \cup m_{ij}(X \setminus B) = m_{ij}((X \setminus A) \cap (X \setminus B))
= m_{ij}(X \setminus (A \cup B))
= A \cup B.
\]

\(\square\)

References


Received: January, 2011