Multiplication Operators and Dynamical Systems on Weighted Spaces of Vector-Valued Holomorphic Functions on Banach Spaces

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Abstract. Let $U_X$ be a balanced open subset of a Banach space $X$. Let $V$ and $W$ be two Nachbin families of weights on $U_X$. Let $B(E, F)$ be the space of all continuous linear operators from a locally convex Hausdorff space $E$ into a locally convex Hausdorff space $F$. Let $HV(U_X, F)$ and $HW(U_X, E)$ be the weighted locally convex spaces of vector-valued holomorphic functions. In this paper, we investigate the operator-valued maps $\psi : U_X \to B(E, F)$ which generate multiplication operators and invertible multiplication operators $M_\psi$ on the spaces $HV(U_X, F)$ and $HW(U_X, E)$ for general Nachbin families of weights $V$ and $W$ and for single continuous weights $v$ and $w$ on the open unit ball $B_X$ of a Banach space $X$. A $C_0$-group of multiplication operators and a (linear) dynamical system is also obtained as an application of these operators.

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1. Introduction

The theory of multiplication operators has extensively been studied during the last three decades or so on different spaces of functions. It is evident from the literature that multiplication operators are appearing in different areas of mathematical sciences like dynamical systems and theory of semigroups (e.g. see [15], [26]) and isometries (e.g. see [12], [13]) besides their role in the theory of operator algebras and operator spaces (e.g. see [2]).

In recent years, many authors are attracted towards the study of multiplication operators on different spaces of analytic functions. On Bergman spaces, these operators have earlier been investigated by (among others) Axler [3], Bercovici [4], Vukotic [28] and Zhu [29] whereas on Hardy spaces, we refer to Arveson [2], Campbell and Leach [9] and Feldman [11]. On Bloch spaces these operators are studied by Arazy [1] and Brown and Shields [8]. Besides these well-known analytic function spaces, these operators on some other spaces of analytic functions has also been investigated by Bonet, Domanski and Lindstrom [7], Ohno and Takagi [19], Shields and Williams [23], Laitila and Tylli [16] and Manhas [17,18]. Also, since the theory of composition operators has extensively been studied during the last three decades or so on these spaces of analytic functions (e.g. see [10], [22], [25]), the classes of multiplication operators and composition operators are playing important role in the study of weighted composition operators. In the present paper, our efforts are to make a study of multiplication operators on the weighted locally convex spaces of vector-valued holomorphic functions defined on the unit ball of a Banach space which generalizes some of the results of Laitila and Tylli [16] and Manhas [18].

The preliminaries needed for presenting the results are reported in Section 2. In the third section, we characterize operator-valued holomorphic functions which generate multiplication operators whereas invertible multiplication operators are reported in Section 4. Finally, in Sections 5, as an application of these operators, we obtained a (linear) dynamical system associated with these multiplication operators.

2. Preliminaries

Throughout this paper $U_X$ and $U_Y$ will denote balanced open subsets of Banach spaces $X$ and $Y$, respectively. By $B_X$ and $B_Y$, we denote open unit balls of $X$ and $Y$, respectively. Let $E$ and $F$ be the locally convex Hausdorff spaces. By $\text{cs}(E)$ and $\text{cs}(F)$, we denote the sets of all continuous seminorms on $E$ and $F$, respectively. Let $H(U_X, E)$ be the linear space of all holomorphic functions $f : U_X \to E$. A weight is an upper semicontinuous function $v : U_X \to [0, \infty)$. A
set of weights $V$ is called a Nachbin family if for every $v_1, v_2 \in V$ and $\lambda > 0$, there exists $v \in V$ such that $\lambda v_1 \leq v$ and $\lambda v_2 \leq v$ on $U_X$. In what follows $V$ denotes a Nachbin family of weights such that for every $x \in U_X$, there exists $v \in V$ for which $v(x) > 0$. A subset $B \subseteq U_X$ is $U_X$-bounded if it is bounded and its distance to $X \setminus U_X$ is greater than zero. A function $f : U_X \to E$ is said to vanish at infinity outside $U_X$-bounded sets if for each $p \in cs(E)$ and $\epsilon > 0$, there exists a $U_X$-bounded set $B$ such that $p(f(x)) < \epsilon$, for every $x \in U_X \setminus B$.

Now, we define the weighted spaces of holomorphic functions associated with $V$ as follows:

$$HV(U_X, E) = \left\{ f \in H(U_X, E) : \|f\|_{v,p} = \sup \{v(x)p(f(x)) : x \in U_X\} < \infty, \right.\left. \text{for every } v \in V \text{ and } p \in cs(E) \right\},$$

and

$$HV_0(U_X, E) = \left\{ f \in HV(U_X, E) : uf \text{ vanishes at infinity outside } U_X - \text{bounded sets for every } v \in V \right\}.$$

Both spaces are endowed with weighted locally convex topology generated by the family $\{\|\cdot\|_{v,p} : v \in V \text{ and } p \in cs(E)\}$ of seminorms. With this topology the spaces $HV(U_X, E)$ and $HV_0(U_X, E)$ are called the weighted locally convex spaces of vector-valued holomorphic functions. These spaces have a basis of closed absolutely convex neighbourhoods of the form

$$B_{v,p} = \left\{ f \in HV(U_X, E) : \|f\|_{v,p} \leq 1 \right\}.$$ If $E = \mathbb{C}$, then we write $HV(U_X, E) = HV(U_X), \ HV_0(U_X, E) = HV_0(U_X)$ and $B_v = \{f \in HV(U_X) : \|f\|_v \leq 1\}$. If the Nachbin family $V$ consists of a single continuous weight $v$ such that $v(x) > 0$ for all $x \in U_X$, then $HV(U_X, E)$ and $HV_0(U_X, E)$ are denoted by $H_v(U_X, E)$ and $H_{v0}(U_X, E)$, respectively. The space of bounded holomorphic functions $f : U_X \to E$ is denoted by $H^\infty(U_X, E)$.

A weight $v$ is said to be radial if $v(\lambda x) = v(x)$ for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and every $x \in B_X$ [5]. Following [6], for given any continuous weight $v$, we consider an associated growth condition $u : B_X \to (0, \infty)$ defined by $u(x) = \frac{1}{v(x)}$. With this new function we can rewrite $B_v = \{f \in H_v(B_X) : \|f\| \leq u\}$. From this, $\tilde{u} : B_X \to (0, \infty)$ is defined by $\tilde{u}(x) = \sup \{|f(x)| : f \in B_v\}$ and a new associated weight $\tilde{v} = \frac{1}{\tilde{u}}$. All these functions are defined by Bierstedt, Bonet and Taskinen for open subsets of $\mathbb{C}^N$ in [6]. The following relations between weights for open sets in $\mathbb{C}^N$ are proved in [6, Proposition 1.2]. The same arguments work for the unit ball of a Banach Space.

**Proposition 2.1.** Let $X$ be a Banach Space and let $v$ be a continuous weight defined on $B_X$. Then the following hold:
(i) $0 < v \leq \tilde{v}$ and $\tilde{v}$ bounded and continuous whenever $v$ is bounded and continuous.

(ii) $\tilde{u}$ (respectively $\tilde{v}$) is radial and decreasing or increasing whenever $u$ (respectively $v$) is so.

(iii) $\|f\|_v \leq 1$ if and only if $\|f\|_{\tilde{v}} \leq 1$.

(iv) For each $x \in B_X$, there exists $f_x \in B_v$ such that $f_x(x) = \frac{1}{\tilde{v}(x)}$.

A weight $v$ is said to be essential if there exists $c > 0$ such that $v(x) \leq \tilde{v}(x) \leq cv(x)$, for all $x \in B_X$ \cite{27}. For more information on weighted spaces of holomorphic functions defined on open subsets of $\mathbb{C}^N$, we refer to \cite{5, 6} and for more details on weighted spaces of holomorphic functions defined on the unit ball of a Banach space, we refer to \cite{14, 21}.

By $B(E, F)$, we denote the vector space of all continuous linear operators from $E$ into $F$. We denote by $B$, the family of all bounded subsets of $E$. For each $B \in B$ and $p \in cs(F)$, we define the seminorm $\|T\|_{p,B} = \sup \{p(T(y)) : y \in B\}, T \in B(E, F)$. Then clearly the family $\{\|\cdot\|_{p,B} : p \in cs(F), B \in B\}$ defines the locally convex topology on $B(E, F)$.

**Remark 2.1.** Let $f \in H(B_X, E)$. Let $v$ a continuous weight on $B_X$ and let $p \in cs(E)$. We denote $N_p = \{y \in E : p(y) \leq 1\}$. Then from Proposition 2.1(iii), it follows that

$$\|f\|_{v,p} = \sup \{v(x)p(f(x)) : x \in B_X\}$$

$$= \sup_{y^* \in N_p} \sup_{x \in B_X} v(x) |(y^*of)(x)|$$

$$= \sup_{y^* \in N_p} \sup_{x \in B_X} \tilde{v}(x) |(y^*of)(x)|$$

$$= \sup \{\tilde{v}(x)p(f(x)) : x \in B_X\} = \|f\|_{\tilde{v},p}.

\section{Characterizations of Multiplication Operators}

In this section we characterize operator-valued holomorphic functions which generate multiplication operators on the weighted spaces $HV(U_X, F)$ and $HW(U_X, E)$ and the spaces $H_v(B_X, F)$ and $H_w(B_X, E)$.

**Proposition 3.1.** Let $V$ and $W$ be Nachbin families of weights on $U_X$ and let $\psi : U_X \to B(E, F)$ be a holomorphic mapping. Then $M_\psi : HW(U_X, E) \to HV(U_X, F)$ is a multiplication operator if for every $v \in V$ and $p \in cs(F)$, there exist $w \in W$ and $q \in cs(E)$ such that $v(x)p(\psi_x(y)) \leq w(x)q(y)$, for every $x \in U_X$ and $y \in E$.

**Proof.** To show that $M_\psi$ is a multiplication operator, it is enough to show that $M_\psi$ is continuous at the origin. For, let $v \in V$ and $p \in cs(F)$. Then by the
given condition there exist \( w \in W \) and \( q \in cs(E) \) such that \( v(x) p(\psi_x(y)) \leq w(x) q(y) \), for every \( x \in U_X \) and \( y \in E \). We claim that \( M_\psi(B_{w,q}) \subseteq B_{v,p} \). Let \( f \in B_{w,q} \). Then \( \|f\|_{w,q} \leq 1 \). Now, consider
\[
\|M_\psi f\|_{v,p} = \sup \left\{ v(x) p(\psi_x(f(x)) : x \in U_X \right\}
\leq \sup \left\{ w(x) q(f(x)) : x \in U_X \right\} = \|f\|_{w,q} \leq 1.
\]
Thus \( M_\psi f \in B_{v,p} \). This completes the proof.

**Remark 3.1.** From Proposition 3.1, it follows that every bounded holomorphic function \( \psi : U_X \to B(E,F) \) induces the multiplication operator \( M_\psi : HV(U_X,E) \to HV(U_X,F) \) for any Nachbin family of weights \( V \) on \( U_X \). Also, if \( V = \{\lambda \chi_K : \lambda \geq 0, K \subseteq U_X, K \text{ compact set} \} \), then every operator-valued holomorphic map \( \psi : U_X \to B(E,F) \) induces the multiplication operator \( M_\psi : HV(U_X,E) \to HV(U_X,F) \). This makes it clear that even unbounded holomorphic operator-valued mappings induce multiplication operators on some of the weighted spaces whereas it is not true for other spaces of analytic functions. For instance, Arveson [2] and Axler [3] have shown that only bounded analytic functions give rise to multiplication operators on Hardy spaces and Bergman spaces, respectively. Also, the same behaviour has shown by Bonet, Domanski and Lindstrom [7] on the weighted Banach spaces of analytic functions \( H^\infty_v(D) \) defined by a single continuous weight \( v \). Thus the behaviour of the multiplication operators on the weighted locally convex spaces of holomorphic functions is very much influenced by different Nachbin families of weights \( V \) on \( U_X \).

**Theorem 3.1.** Let \( v \) and \( w \) be continuous weights on \( B_X \) and let \( \psi : B_X \to B(E,F) \) be a holomorphic mapping. Then \( M_\psi : H_v(B_X,E) \to H_v(B_X,F) \) is a multiplication operator if and only if for every \( p \in cs(F) \), there exists \( q \in cs(E) \) such that \( v(x) p(\psi_x(y)) \leq \tilde{w}(x) q(y) \), for every \( x \in B_X \) and \( y \in E \).

**Proof.** The sufficient part follows from Proposition 3.1 and Remark 2.1. Conversely, suppose that \( M_\psi \) is a multiplication operator. Let \( p \in cs(F) \). Then by the continuity of \( M_\psi \) at the origin, there exists \( q \in cs(E) \) such that \( M_\psi(B_{w,q}) \subseteq B_{v,p} \). We claim that \( v(x) p(\psi_x(y)) \leq \tilde{w}(x) q(y) \), for every \( x \in B_X \) and \( y \in E \). Fix \( x_0 \in B_X \) and \( y_0 \in E \). We shall prove that \( v(x_0) p(\psi_{x_0}(y_0)) \leq \tilde{w}(x_0) q(y_0) \).

Case-I: If \( q(y_0) = 0 \), then it is enough to show that \( p(\psi_{x_0}(y_0)) = 0 \) since \( v(x_0) > 0 \) and \( \tilde{w}(x_0) > 0 \). On the contrary, we assume that \( p(\psi_{x_0}(y_0)) \neq 0 \). By Proposition 2.1 (iv), there exists \( f_{x_0} \in B_w \) such that \( f_{x_0}(x_0) = \frac{1}{\tilde{w}(x_0)} \). Let
$r > 1$ and let $\alpha = \frac{r \tilde{w}(x_0)}{v(x_0) p(\psi_{x_0}(y_0))}$. Then define the function $g_0 : B_X \to E$ as $g_0(x) = \alpha f_{x_0}(x) y_0$, for every $x \in B_X$. Clearly $g_0 \in B_{w,q}$ and hence $M_{\psi} g_0 \in B_{v,p}$. Thus we have $v(x) p(\psi_x(g_0(x))) \leq 1$, for every $x \in B_X$. For $x = x_0$, we have $v(x_0) p(\psi_{x_0}(g_0(x_0))) \leq 1$. That is, $\frac{\alpha v(x_0) p(\psi_{x_0}(y_0))}{\tilde{w}(x_0)} \leq 1$ and further it implies that $r \leq 1$, which is a contradiction. Thus $p(\psi_{x_0}(y_0)) = 0$ and hence the inequality $v(x_0) p(\psi_{x_0}(y_0)) \leq \tilde{w}(x_0) q(y_0)$.

Case-II: Let $q(y_0) \neq 0$. Then, again we choose $f_{x_0} \in B_w$ such that $f_{x_0}(x_0) = \frac{1}{\tilde{w}(x_0)}$. Define the function $h_0 : B_X \to E$ as $h_0(x) = \frac{f_{x_0}(x) y_0}{q(y_0)}$, for every $x \in B_X$. Clearly $h_0 \in B_{w,q}$ and hence $M_{\psi} h_0 \in B_{v,p}$. Further, it implies that $v(x) p(\psi_x(h_0(x))) \leq 1$, for every $x \in B_X$. For $x = x_0$, we have $v(x_0) p(\psi_{x_0}(h_0(x_0))) \leq 1$. That is, $v(x_0) p(\psi_{x_0}(y_0)) \leq \tilde{w}(x_0) q(y_0)$, This completes the proof of the theorem. $\square$

**Remark 3.2.** (i) In Theorem 3.1, if $w$ is an essential continuous weight, then $M_{\psi}$ is a multiplication operator if and only if $v(x) p(\psi_x(y)) \leq w(x) q(y)$, for every $x \in B_X$ and $y \in E$.

(ii) All the results proved above are also hold for the spaces $H_{w_0}(U_X, E)$, $H_{v_0}(U_X, F)$, $H_{w_0}(B_X, E)$, and $H_{v_0}(B_X, F)$.

(iii) Proposition 3.1 and Theorem 3.1 generalizes the results of boundedness of $M_{\psi}$ proved by Manhas [18] for the weighted spaces defined on connected subset $G$ of $\mathbb{C}^N$ and $E = F$ as a Banach space.

(iv) In Theorem 3.1, if we take $E$ and $F$ as Banach spaces, then it reduces to [18, Corollary 3.2]. Also, Theorem 3.1 generalizes the result of boundedness of $M_{\psi}$ proved by Laitila and Tylli [16, Corollary 3.5] for the weighted spaces defined on an open unit disk $D$ and Banach spaces $E$ and $F$. Also, if $B_X = D$, $E = F = \mathbb{C}$ and $v = w$ with $v$ essential, then Theorem 3.1 reduces to Proposition 2.1 of [7].

4. **Invertible Multiplication Operators**

We begin this section by stating an invertibility criterion on a Hausdorff topological vector space [24] which we shall use for characterizing invertible multiplication operators.

**Theorem 4.1.** Let $H$ be a complete Hausdorff topological vector space and let $T : H \to H$ be a continuous linear operator. Then $T$ is invertible if and only if $T$ is bounded below and has dense range.

OR
Let $H$ be a as Hausdorff topological vector space and let $T : H \to H$ be a continuous linear operator. Then $T$ is invertible if and only if $T$ is bounded below and onto.

In the above invertible criterion, a generalized definition of bounded below operators on Hausdorff topological vector spaces is used. Now, we give this definition as it is needed for proving some of the results of this section. A continuous linear operator $T$ on a Hausdorff topological vector space $H$ is said to be bounded below if for every neighbourhood $N$ of the origin in $H$, there exists a neighbourhood $M$ of the origin in $H$ such that $T(N^c) \subseteq M^c$, where the symbol ‘c’ stands for the complement of the neighbourhood in $H$. Also, if $H = E$ is a locally convex Hausdorff space, then $T$ is bounded below if for every $p \in cs(E)$, there exists $q \in cs(E)$ such that $p(y) \leq q(T(y))$, for every $y \in E$.

**Proposition 4.1.** Let $v$ and $w$ be continuous weights on $B_X$ and let $\psi : B_X \to B(E)$ be a holomorphic mapping such that $M_\psi : H_w(B_X,E) \to H_v(B_X,E)$ is a multiplication operator. Then $M_\psi$ is invertible if

(i) for each $x \in B_X$, $\psi_x : E \to E$ is onto;

(ii) for each $p \in cs(E)$, there exists $q \in cs(E)$ such that $w(x)p(y) \leq \tilde{v}(x)q(\psi_x(y))$, for every $x \in B_X$ and $y \in E$.

**Proof.** Fix $x_0 \in B_X$. We first show that $\psi_{x_0} : E \to E$ is bounded below. Let $p \in cs(E)$. Then by condition (ii), there exists $q \in cs(E)$ such that $w(x_0)p(y) \leq \tilde{v}(x_0)q(\psi_{x_0}(y))$, for every $y \in E$. Let $\lambda = \frac{\tilde{v}(x_0)}{w(x_0)}$ and let $q_1 = \lambda q$. Then $q_1 \in cs(E)$ such that $p(y) \leq q_1(\psi_{x_0}(y))$, for every $y \in E$. This proves that $\psi_{x_0}$ is bounded below. Since $\psi_{x_0}$ is onto by Condition (i), Theorem 4.1 implies that $\psi_{x_0}$ is invertible in $B(E)$. We denote the inverse of $\psi_{x_0}$ by $\psi_{x_0}^{-1}$. Define $\psi^{-1} : B_X \to B(E)$ as $\psi^{-1}(x) = \psi_{x_0}^{-1}$, for every $x \in B_X$. Clearly $\psi^{-1}$ is an analytic map. Again, from Condition (ii), it follows that $w(x)p(\psi^{-1}(y)) \leq \tilde{v}(x)q(y)$, for every $x \in B_X$ and $y \in E$. Thus according to Theorem 3.1 $\psi^{-1}$ induces the multiplication operator $M_{\psi^{-1}} : H_v(B_X,E) \to H_w(B_X,E)$ such that $M_\psi M_{\psi^{-1}} = I$ and $M_{\psi^{-1}} M_\psi = I$. Hence $M_\psi$ is invertible.

**Corollary 4.1.** Let $v$ and $w$ be continuous weights on $B_X$ and let $\psi : B_X \to B(E)$ be a holomorphic mapping such that $M_\psi : H_w(B_X,E) \to H_v(B_X,E)$ is a multiplication operator. Then $M_\psi$ is invertible if

(i) for each $x \in B_X$, $\psi_x : E \to E$ is onto;

(ii) $\psi$ is bounded away from zero.

(iii) $w \leq v$.

**Proof.** From Condition (ii), it follows that for every $p \in cs(E)$, there exists $q \in cs(E)$ such that $p(y) \leq q(\psi_x(y))$, for every $x \in B_X$ and $y \in E$. Further,
it implies that \( w(x)p(y) \leq w(x)q(\psi_x(y)) \). Using condition (iii), it follows that \( w(x)p(y) \leq \tilde{v}(x)q(\psi_x(y)) \) for every \( x \in B_X \) and \( y \in E \), since \( v \leq \tilde{v} \). Hence by Proposition 4.1, it follows that \( M_\psi \) is invertible.

**Corollary 4.2.** Let \( V = \{ \lambda \chi_K : \lambda \geq 0, \text{ and } K \subseteq B_X, \text{ K compact set} \} \). Then every holomorphic map \( \psi : B_X \to B(E) \) induces an invertible multiplication operator \( M_\psi \) on \( HV(B_X,E) \) if \( \psi(x) \) is invertible for every \( x \in B_X \).

**Remark 4.1.** Corollary 4.2 makes it clear that if an analytic map \( \psi : B_X \to B(E) \) is not bounded away from zero, even then \( M_\psi \) is invertible on some of the weighted spaces. Also, we note that invertible multiplication operators on Bergman spaces of analytic functions \([3]\) and weighted Banach spaces of analytic function \([7]\) are generated only by the functions which are bounded away from zero. Thus in our case the invertible behaviour is very much influenced by different weights on \( B_X \).

**Theorem 4.2.** Let \( v \) and \( w \) be continuous weights on \( B_X \). Let \( \psi : B_X \to B(E) \) be a holomorphic mapping such that each \( \psi(x) \) is one-one and \( M_\psi : H_w(B_X,E) \to H_v(B_X,E) \) is a multiplication operator. Then \( M_\psi \) is invertible if and only if

(i) for each \( x \in B_X \), \( \psi_x : E \to E \) is onto;

(ii) for each \( p \in cs(E) \), there exists \( q \in cs(E) \) such that \( w(x)p(y) \leq \tilde{v}(x)q(\psi_x(y)) \), for every \( x \in B_X \) and \( y \in E \).

**Proof.** According to Proposition 4.1, conditions (i) and (ii) implies that \( M_\psi \) is invertible. Conversely, suppose that \( M_\psi \) is invertible. By Theorem 4.1, \( M_\psi \) is bounded below and onto. To prove condition (i), let \( x_0 \in B_X \) and \( f_{x_0} \in H_v(B_X) \) be such that \( f_{x_0}(x_0) = 1 \). Let \( 0 \neq y_0 \in E \). Define \( g_0 : B_X \to E \) as \( g_0(x) = f_{x_0}(x)y_0 \), for every \( x \in B_X \). Cleary \( g_0 \in H_v(B_X,E) \). Since \( M_\psi \) is onto, there exists \( h_0 \in H_w(B_X,E) \) such that \( M_\psi(h_0) = g_0 \). That is \( \psi_{x_0}(h_0(x_0)) = g_0(x_0) = y_0 \). This prove that \( \psi_{x_0} \) is onto. Now to prove condition (ii), we fix \( p \in cs(E) \). Since \( M_\psi \) is bounded below, there exists \( q \in cs(E) \) such that \( M_\psi(B_{w,p}^c) \subseteq B_{q,E}^c \). Now, we claim that \( w(x)p(\psi_{x_0}^{-1}(y)) \leq \tilde{v}(x)q(y) \), for every \( x \in B_X \) and \( y \in E \). Fix \( x_0 \in B_X \) and \( y_0 \in E \). We shall prove that \( w(x_0)p(\psi_{x_0}^{-1}(y_0)) \leq \tilde{v}(x_0)q(y_0) \).

Case-I: Let \( q(y_0) \neq 0 \). Then by Proposition 2.1 (iv), there exists \( f_{x_0} \in B_v \) such that \( f_{x_0}(x_0) = \frac{1}{\tilde{v}(x_0)} \). Define \( g_0 : B_X \to E \) as \( g_0(x) = \frac{f_{x_0}(x)y_0}{q(y_0)} \), for every \( x \in B_X \). Cleary \( g_0 \in B_{v,q} \). Since \( M_\psi \) is onto, there exists \( h_0 \in H_w(B_X,E) \) such that \( M_\psi(h_0) = g_0 \). That is \( \psi_{x_0}(h_0(x_0)) = g_0(x_0) \). Again, since \( M_\psi(h_0) = g_0 \notin B_{w,p}^c \), we conclude that \( h_0 \notin B_{w,p}^c \). That is, \( w(x)p(h_0(x)) \leq 1 \), for every \( x \in B_X \). For \( x = x_0 \), we have \( w(x_0)p(\psi_{x_0}^{-1}(g_0(x_0))) \leq 1 \). That is, \( w(x_0)p(\psi_{x_0}^{-1}(y_0)) \leq \tilde{v}(x_0)q(y_0) \).
Case-II: Let $q(y_0) = 0$. Then it is enough to show that $p(\psi_0^{-1}(y_0)) \neq 0$. On the contrary, we assume that $p(\psi_0^{-1}(y_0)) = 0$. Again, there exists $f_{x_0} \in B_v$ such that $f_{x_0}(x_0) = \frac{1}{\tilde{v}(x_0)}$. Let $r > 1$ and let $\alpha = \frac{r \tilde{v}(x_0)}{w(x_0) p(\psi_0^{-1}(y_0))}$. Define the function $g_0 : B_X \to E$ as $g_0(x) = \alpha f_{x_0}(x) y_0$, for every $x \in B_X$. Then clearly $g_0 \in B_{v,q}$. Since $M_\psi$ is onto, there exists $h_0 \in H_w(B_X,E)$ such that $M_\psi(h_0) = g_0$. That is, $h_0(x_0) = \psi_0^{-1}(g_0(x_0))$. Also, since $M_\psi(h_0) = g_0 \notin B_{w,p}$, we conclude that $h_0 \notin B_{w,p}$. That is, $w(x) p(h_0(x)) \leq 1$, for every $x \in B_X$. Further, it implies that $\alpha w(x_0) p(\psi_0^{-1}(y_0)) f_{x_0}(x_0) \leq 1$. That is, $r \leq 1$ which is a contradiction. This proves our claim that $w(x) p(\psi_0^{-1}(y)) \leq \tilde{v}(x) q(y)$, for every $x \in B_X$ and $y \in E$. From this, it readily follows that $w(x) p(y) \leq \tilde{v}(x) q(\psi_x(y))$, for every $x \in B_X$ and $y \in E$. This completes the proof. 

**Remark 4.2.** In Theorem 4.2, if $v$ is an essential weight, then $\tilde{v}$ will be replaced by $v$ and the proof follows on the same lines. Also, in Proposition 4.1 and Theorem 4.2 if we take $B_X = G$, an open connected subset of $\mathbb{C}^N$ and $E$ as a Banach space, then these results reduce to Manhas [18, Proposition 4.2 and Theorem 4.6].

5. Dynamical System Induced by Multiplication Operators

Let $E$ be a Banach space and let $g \in H^\infty(U_X,B(E))$. Let $\|g\|_\infty = \sup \left\{ \|g(x)\|_{B(E)} : x \in U_X \right\}$. Then for each $t \in \mathbb{R}$, we define $\psi_t : U_X \to B(E)$ as $\psi_t(x) = e^{tg(x)}$, for every $x \in U_X$. Clearly each $\psi_t$ is an operator-valued bounded analytic function and hence by Proposition 3.1, $M_{\psi_t} : HV(U_X,E) \to HV(U_X,E)$ is a multiplication operator.

**Proposition 5.1.** Let $h_n$ be a sequence converging to $h$ in $H^\infty(U_X,B(E))$ and let $f_n$ be a sequence converging to $f$ in $HV(U_X,E)$. Then the product $h_n f_n$ converges to $hf$ in $HV(U_X,E)$.

**Proof.** Let $v \in V$. Then

$$
\|h_n f_n - hf\|_{v,E} = \sup \{ v(x) \|h_n(x)(f_n(x)) - h(x)(f(x))\| : x \in U_X \} \\
= \sup \{ v(x) \|h_n(x)(f_n(x)) - h(x)(f_n(x)) + h(x)(f_n(x)) - h(x)(f(x))\| : x \in U_X \} \\
\leq \sup \{ v(x) \|h_n(x) - h(x)\|_{B(E)} \|f_n(x)\|_E : x \in U_X \} + \sup \{ v(x) \|h(x)\|_{B(E)} \|f_n(x) - f(x)\|_E : x \in U_X \} \\
\leq \|h_n - h\|_\infty \|f_n\|_{v,E} + \|h\|_\infty \|f_n - f\|_{v,E} \to 0,
$$

since $\|h_n - h\|_\infty \to 0$ and $\|f_n - f\|_{v,E} \to 0$. 

\[\square\]
Theorem 5.1. Let \( V \) be a countable family of continuous weights on \( U_X \). Let \( \Pi : \mathbb{R} \times HV(U_X, E) \to H(U_X, E) \) be defined as \( \Pi(t, f) = M_{\psi_t}f \), for every \( t \in \mathbb{R} \) and \( f \in HV(U_X, E) \). Then \( \Pi \) is a (linear) dynamical system on \( HV(U_X, E) \). Moreover, the family \( \mathcal{M} = \{ M_{\psi_t} : t \in \mathbb{R} \} \) is locally equicontinuous \( c_0 - \)group of multiplication operators in \( B(HV(U_X, E)) \).

Proof. We have already observed that for each \( t \in \mathbb{R} \), \( \psi_t : U_X \to B(E) \) induces the multiplication operator \( M_{\psi_t} \) on \( HV(U_X, E) \). Thus it follows that \( \Pi(t, f) \in HV(U_X, E) \), for every \( t \in \mathbb{R} \) and \( f \in HV(U_X, E) \). Hence \( \Pi \) is a function from \( \mathbb{R} \times HV(U_X, E) \to HV(U_X, E) \). Also, it is easy to see that \( \Pi \) is linear and \( \Pi(0, f) = f \), for every \( f \in HV(U_X, E) \). Further, it can be easily seen that \( \Pi(t + s, f) = \Pi(t, \Pi(s, f)) \), for every \( t, s \in \mathbb{R} \) and \( f \in HV(U_X, E) \). In order to show that \( \Pi \) is a dynamical system on \( HV(U_X, E) \), it is enough to show that \( \Pi \) is jointly continuous. Let \( \{ (t_n, f_n) \} \) be a sequence in \( \mathbb{R} \times HV(U_X, E) \) such that \( (t_n, f_n) \to (t, f) \) in \( \mathbb{R} \times HV(U_X, E) \). Then we shall show that \( \Pi(t_n, f_n) \to \Pi(t, f) \) in \( \mathbb{R} \times HV(U_X, E) \). That is, we have to prove that \( \psi_{t_n}f_n \to \psi_tf \) in \( HV(U_X, E) \). But this follows from Proposition 5.1. Thus \( \Pi \) is a (linear) dynamical system on \( HV(U_X, E) \). Further, it is straightforward that the family \( \mathcal{M} = \{ M_{\psi_t} : t \in \mathbb{R} \} \) is a \( c_0 \)-group of multiplication operators on the weighted spaces \( HV(U_X, E) \). Now, we shall show that the family \( \mathcal{M} \) is locally equicontinuous in \( B(HV(U_X, E)) \). That is, we have to prove that for any fixed \( s \in \mathbb{R} \), the subfamily \( \mathcal{M}_s = \{ M_{\psi_t} : -s \leq t \leq s \} \) is equicontinuous on \( HV(U_X, E) \). Further, it is easy to see that the subfamily \( \mathcal{M}_s \) is a bounded set in \( B(HV(U_X, E)) \), since the map \( t \to M_{\psi_t} \) is continuous in the strong-operator topology. Also, for each \( f \in HV(U_X, E) \), the set \( \mathcal{M}_s(f) = \{ M_{\psi_t}f : -s \leq t \leq s \} \) is bounded in \( HV(U_X, E) \). Thus according to a corollary of the Banach-Steinhaus Theorem [20], it follows that the family \( \mathcal{M} \) is locally equicontinuous.

References


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