Study of Growth Properties on the Basis of Gol’dberg Order of Composite Entire Functions of Several Variables

Sanjib Kumar Datta

Department of Mathematics, University of Kalyani
P.O.-Kalyani, Dist-Nadia, Pin-741235
West Bengal, India
Former Address:
(Department of Mathematics, University of North Bengal
P.O.-North Bengal University, Raja Rammohunpur
Dist-Darjeeling, Pin-734013
West Bengal, India)
sanjib_krdatta@yahoo.co.in
s_krdatta_ku@yahoo.co.in
sk_datta_nbu@yahoo.co.in

Meghlal Mallik

Panigha U.D.M. High School
P.O.-Paglachandi, Dist Nadia, Pin-741181
West Bengal, India
meghlal1982@yahoo.com
meghlal_mallik@yahoo.com

Abstract

In the paper we discuss about the comparative growth properties related to the Gol’dberg order of composition of two entire function of several complex variables.

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1 Introduction, Definitions and Notations.

We denote complex and real $n$-space by $C^n$ and $R^n$ respectively and indicate the point $(z_1, ..., z_n), (m_1, ..., m_n)$ of $C^n$ or $I^n$ by their corresponding unsuffixed symbols $z, m$ respectively where $I$ denotes the set of non-negative integers. The modulus of $z$, denoted by $|z|$, is defined as $|z| = (|z_1|^2 + ... + |z_n|^2)^{1/2}$. If the coordinates of the vector $m$ are non-negative integers, then $z^m$ will denote $(z_1^m, ..., z_n^m)$ and $||m|| = m_1 + ... + m_n$. Let $D \subset C^n$ be an arbitrary bounded complete $n$-circular domain with center at the origin of coordinates. Then for the analytic function $f$ and $R > 0$, $M_{f,D}(R) = \sup_{z \in DR} |f(z)|,$ where a point $z \in DR$ if and only if $\frac{z}{R} \in D$.

**Definition 1** ([2], p. 339) The Gol'dberg order (briefly $G$-order) $\rho_D^f$ of $f$ with respect to the domain $D$ is defined as

$$\rho_D^f (f) \equiv \rho_D^f = \limsup_{R \to \infty} \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

where $\log^k x = \log(\log^{k-1} x)$ for $k = 1, 2, 3, ...$ and $\log^0 x = x$.

The lower Gol’dberg order $\lambda_D^f$ of $f$ with respect to the domain $D$ is defined as

$$\lambda_D^f (f) \equiv \lambda_D^f = \liminf_{R \to \infty} \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

We say that $f$ is of regular growth if $\rho_D^f = \lambda_D^f$.

**Definition 2** [1] Let $f$ and $g$ be two entire functions of $n$ variables and $D$ be a bounded complete $n$-circular domain with centre at the origin in $C^n$. Then the relative $G$-order $\rho^f_{s,D}(f)$ of $f$ with respect to $g$ and the domain $D$ is defined by

$$\rho^f_{s,D}(f) = \inf \{ \mu > 0; M_{f,D}(R) < M_{g,D}(R^\mu), \text{ for all } R > R_0(\mu) > 0 \}.$$  

If $f$ is a non-constant entire function, then $M_{f,D}(R)$ is a strictly increasing and continuous function of $R$ and its inverse function $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists. It then easily follows that

$$\rho^f_{s,D}(f) = \limsup_{R \to \infty} \frac{\log M_{g,D}^{-1}(M_{f,D}(R))}{\log R}.$$
Similarly, the relative lower order \( \lambda_{g,D}(f) \) of \( f \) with respect to \( g \) and the domain \( D \) is defined by

\[
\lambda_{g,D}(f) = \liminf_{R \to \infty} \frac{\log M_{g,D}^{-1}(M_{f,D}(R))}{\log R}.
\]

Throughout this paper we shall measure the growth of entire functions relative to the entire function \( g \) and \( D \) will represent a bounded complete \( n \)-circular domain. Unless otherwise stated all the entire functions under consideration will be transcendental. We do not explain the standard definitions and notations in the theory of entire (and meromorphic) functions as those are available in [3]. Extending the notions of Definition 1 and Definition 2 we may give the following definitions:

**Definition 3** The hyper Gol'dberg order (briefly hyper \( G \)-order) \( \overline{\rho}_D(f) \) of \( f \) with respect to the domain \( D \) is defined by

\[
\overline{\rho}_D(f) \equiv \overline{\rho}_D^f = \limsup_{R \to \infty} \frac{\log^{[3]} M_{f,D}(R)}{\log R}.
\]

The hyper lower Gol’dberg order (briefly hyper lower \( G \)-order) of \( f \) with respect to the domain \( D \) is defined by

\[
\overline{\lambda}_D(f) \equiv \overline{\lambda}_D^f = \liminf_{R \to \infty} \frac{\log^{[3]} M_{f,D}(R)}{\log R}.
\]

**Definition 4** Let \( f \) and \( g \) be two entire functions of \( n \) variables and \( D \) be a bounded complete \( n \)-circular domain with centre at origin in \( \mathbb{C}^n \). Then the relative hyper Gol’dberg order \( \overline{\rho}_{g,D}(f) \) of \( f \) with respect to \( g \) and the domain \( D \) is defined as

\[
\overline{\rho}_{g,D}(f) \equiv \overline{\rho}_{g,D}^f = \limsup_{R \to \infty} \frac{\log^{[2]} M_{g,D}^{-1}(M_{f,D}(R))}{\log R}.
\]

Similarly the relative hyper lower order \( \overline{\lambda}_{g,D}(f) \) of \( f \) with respect to \( g \) and the domain \( D \) is defined by

\[
\overline{\lambda}_{g,D}(f) \equiv \overline{\lambda}_{g,D}^f = \liminf_{R \to \infty} \frac{\log^{[2]} M_{g,D}^{-1}(M_{f,D}(R))}{\log R}.
\]

Generalising our notion we may get that
Definition 5 The generalised Gol’dberg order (briefly generalised G-order) \( \rho^{(k)}_D(f) \) of \( f \) with respect to the domain \( D \) is defined by

\[
\rho^{(k)}_D(f) = \limsup_{R \to \infty} \frac{\log[k] M_{f,D}(R)}{\log R} \text{ where } k = 1, 2, 3, \ldots
\]

The generalised lower Gol’dberg order (briefly generalised lower G-order) of \( f \) with respect to the domain \( D \) is defined by

\[
\lambda^{(k)}_D(f) = \liminf_{R \to \infty} \frac{\log[k] M_{f,D}(R)}{\log R} \text{ where } k = 1, 2, 3, \ldots
\]

Definition 6 Let \( f \) and \( g \) be two entire functions of \( n \) variables and \( D \) be a bounded complete \( n \)-circular domain with centre at origin in \( \mathbb{C}^n \). Then the generalised relative Gol’dberg order \( \rho^{(k)}_{g,D}(f) \) of \( f \) with respect to \( g \) and the domain \( D \) is defined by

\[
\rho^{(k)}_{g,D}(f) = \limsup_{R \to \infty} \frac{\log[k] M_{g^{-1},D}(M_{f,D}(R))}{\log R}, \text{ for } k = 1, 2, 3, \ldots
\]

Similarly, the generalised relative lower Gol’dberg order \( \lambda^{(k)}_{g,D}(f) \) of \( f \) with respect to \( g \) and the domain \( D \) is defined as

\[
\lambda^{(k)}_{g,D}(f) = \liminf_{R \to \infty} \frac{\log[k] M_{g^{-1},D}(M_{f,D}(R))}{\log R}, \text{ for } k = 1, 2, 3, \ldots
\]

In the paper we establish some results on the comparative growth properties related to Gol’dberg order (lower Gol’dberg order) and relative Gol’dberg order (relative lower Gol’dberg order).

2 Theorems.

In this section we present the main results of the paper.

Theorem 1 Let \( f \) and \( g \) be two entire functions of \( n \) variables and \( D \) be a bounded complete \( n \)-circular domain with centre at origin in \( \mathbb{C}^n \). Also let \( 0 < \lambda_D(fog) \leq \rho_D(fog) < \infty \) and \( 0 < \lambda_D(g) \leq \rho_D(g) < \infty \). Then

\[
\frac{\lambda_D(fog)}{\rho_D(g)} \leq \liminf_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{g,D}(R)} \leq \min \left\{ \frac{\lambda_D(fog)}{\rho_D(g)}, \frac{\rho_D(fog)}{\lambda_D(g)} \right\} \\
\leq \max \left\{ \frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\rho_D(fog)}{\rho_D(g)} \right\} \leq \limsup_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{g,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(g)}.
\]
Proof. From the definition of Gol’dberg order and lower Gol’dberg order of an entire function $g$, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $R$,

$$\log^2 M_{g,\varepsilon}(R) \leq (\rho_D(g) + \varepsilon) \log R \tag{1}$$

and

$$\log^2 M_{g,\varepsilon}(R) \geq (\lambda_D(g) - \varepsilon) \log R. \tag{2}$$

Also for a sequence of values of $R$, tending to infinity,

$$\log^2 M_{g,\varepsilon}(R) \leq (\lambda_D(g) + \varepsilon) \log R \tag{3}$$

and

$$\log^2 M_{g,\varepsilon}(R) \geq (\rho_D(g) - \varepsilon) \log R. \tag{4}$$

Again from the definition of Gol’dberg order and lower Gol’dberg order of the composite entire function $f \circ g$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $R$,

$$\log^2 M_{f \circ g,\varepsilon}(R) \leq (\rho_D(f \circ g) + \varepsilon) \log R \tag{5}$$

and

$$\log^2 M_{f \circ g,\varepsilon}(R) \geq (\lambda_D(f \circ g) - \varepsilon) \log R. \tag{6}$$

Again for a sequence of values of $R$ tending to infinity,

$$\log^2 M_{f \circ g,\varepsilon}(R) \leq (\lambda_D(f \circ g) + \varepsilon) \log R \tag{7}$$

and

$$\log^2 M_{f \circ g,\varepsilon}(R) \geq (\rho_D(f \circ g) - \varepsilon) \log R. \tag{8}$$

Now from (1) and (6) it follows for all sufficiently large values of $R$ that

$$\frac{\log^2 M_{f \circ g,\varepsilon}(R)}{\log^2 M_{g,\varepsilon}(R)} \leq \frac{\lambda_D(f \circ g) - \varepsilon}{\rho_D(g) + \varepsilon}. \tag{9}$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{R \to \infty} \frac{\log^2 M_{f \circ g,\varepsilon}(R)}{\log^2 M_{g,\varepsilon}(R)} \geq \frac{\lambda_D(f \circ g)}{\rho_D(g)}. \tag{9}$$

Again combining (2) and (7) we get for a sequence of values of $R$ tending to infinity,

$$\frac{\log^2 M_{f \circ g,\varepsilon}(R)}{\log^2 M_{g,\varepsilon}(R)} \leq \frac{\lambda_D(f \circ g) + \varepsilon}{\lambda_D(g) - \varepsilon}. \tag{9}$$
Since $\varepsilon(>0)$ is arbitrary, it follows that

$$
\liminf_{R \to \infty} \frac{\log^2 M_{fog, D}(R)}{\log^2 M_{g, D}(R)} \leq \frac{\lambda_D(fog)}{\lambda_D(g)}.
$$

(10)

Similarly from (4) and (5) it follows for a sequence of values of $R$ tending to infinity that

$$
\frac{\log^2 M_{fog, D}(R)}{\log^2 M_{g, D}(R)} \leq \frac{\rho_D(fog) + \varepsilon}{\rho_D(g) - \varepsilon}.
$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$
\liminf_{R \to \infty} \frac{\log^2 M_{fog, D}(R)}{\log^2 M_{g, D}(R)} \leq \frac{\rho_D(fog)}{\rho_D(g)}.
$$

(11)

Now combining (9), (10) and (11) we get that

$$
\frac{\lambda_D(fog)}{\rho_D(g)} \leq \liminf_{R \to \infty} \frac{\log^2 M_{fog, D}(R)}{\log^2 M_{g, D}(R)} \leq \min\left\{\frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\rho_D(fog)}{\rho_D(g)}\right\}.
$$

(12)

Now from (3) and (6) we obtain for a sequence of values of $R$ tending to infinity that

$$
\frac{\log^2 M_{fog, D}(R)}{\log^2 M_{g, D}(R)} \geq \frac{\lambda_D(fog) - \varepsilon}{\lambda_D(g) + \varepsilon}.
$$

Choosing $\varepsilon \to 0$ we get that

$$
\limsup_{R \to \infty} \frac{\log^2 M_{fog, D}(R)}{\log^2 M_{g, D}(R)} \geq \frac{\lambda_D(fog)}{\lambda_D(g)}.
$$

(13)

Again from (2) and (5) it follows for all sufficiently large values of $R$ that

$$
\frac{\log^2 M_{fog, D}(R)}{\log^2 M_{g, D}(R)} \leq \frac{\rho_D(fog) + \varepsilon}{\lambda_D(g) - \varepsilon}.
$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$
\limsup_{R \to \infty} \frac{\log^2 M_{fog, D}(R)}{\log^2 M_{g, D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(g)}.
$$

(14)

Similarly combining (1) and (8) we get for a sequence of values of $R$ tending to infinity that

$$
\frac{\log^2 M_{fog, D}(R)}{\log^2 M_{g, D}(R)} \geq \frac{\rho_D(fog) - \varepsilon}{\rho_D(g) + \varepsilon}.
$$
Since \( \varepsilon(>0) \) is arbitrary, it follows that

\[
\limsup_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{g,D}(R)} \geq \frac{\rho_D(fog)}{\rho_D(g)}.
\]  

(15)

Therefore combining (13), (14) and (15) we get that

\[
\max\left\{\frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\rho_D(fog)}{\rho_D(g)}\right\} \leq \limsup_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{g,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(g)}.
\]  

(16)

Thus the theorem follows from (12) and (16). ■

**Remark 1** If we take \( 0 < \lambda_D(f) \leq \rho_D(f) < \infty \) instead of \( 0 < \lambda_D(g) \leq \rho_D(g) < \infty \) and the other conditions remain the same then also Theorem 1 holds with \( g \) replaced by \( f \) in the denominator as we see in the next theorem.

**Theorem 2** Let \( f \) and \( g \) be two entire functions of \( n \) variables and \( D \) be a bounded complete \( n \)-circular domain with centre at origin in \( \mathbb{C}^n \). Also let \( 0 < \lambda_D(fog) \leq \rho_D(fog) < \infty \) and \( 0 < \lambda_D(f) \leq \rho_D(f) < \infty \). Then

\[
\frac{\lambda_D(fog)}{\lambda_D(g)} \leq \liminf_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \leq \min\left\{\frac{\lambda_D(fog)}{\lambda_D(f)}, \frac{\rho_D(fog)}{\rho_D(f)}\right\}
\]

\[
\leq \max\left\{\frac{\lambda_D(fog)}{\lambda_D(f)}, \frac{\rho_D(fog)}{\rho_D(f)}\right\} \leq \limsup_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(f)}.
\]

**Proof.** From the definition of Gol'dberg order and lower Gol'dberg order of an entire function \( f \) we have for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( R \),

\[
\log^2 M_{f,D}(R) \leq (\rho_D(f) + \varepsilon) \log R
\]

(17)

and \( \log^2 M_{f,D}(R) \geq (\lambda_D(f) - \varepsilon) \log R. \)  

(18)

Also for a sequence of values of \( R \) tending to infinity,

\[
\log^2 M_{f,D}(R) \leq (\lambda_D(f) + \varepsilon) \log R
\]

(19)

and \( \log^2 M_{f,D}(R) \geq (\rho_D(f) - \varepsilon) \log R. \)  

(20)
Again from the definition of Gol’dberg order and lower Gol’dberg order of the composite entire function (or, composition of two entire functions $f$ and $g$) $fog$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $R$,

\[
\log^2 M_{fog,D}(R) \leq (\rho_D(fog) + \varepsilon) \log R
\]

(21)

and \[
\log^2 M_{fog,D}(R) \geq (\lambda_D(fog) - \varepsilon) \log R.
\]

(22)

Again for a sequence of values of $R$ tending to infinity,

\[
\log^2 M_{fog,D}(R) \leq (\lambda_D(fog) + \varepsilon) \log R
\]

(23)

and \[
\log^2 M_{fog,D}(R) \geq (\rho_D(fog) - \varepsilon) \log R.
\]

(24)

Now from (17) and (22) it follows for all sufficiently large values of $R$ that

\[
\frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \geq \frac{\lambda_D(fog) - \varepsilon}{\rho_D(f) + \varepsilon}.
\]

As $\varepsilon(>0)$ is arbitrary, we obtain that

\[
\liminf_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \geq \frac{\lambda_D(fog)}{\rho_D(f)}.
\]

(25)

Again combining (18) and (23) we get for a sequence of values of $R$ tending to infinity,

\[
\frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \geq \frac{\lambda_D(fog) + \varepsilon}{\lambda_D(f) - \varepsilon}.
\]

Since $\varepsilon(>0)$ is arbitrary, it follows that

\[
\liminf_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \leq \frac{\lambda_D(fog)}{\lambda_D(f)}.
\]

(26)

Similarly from (20) and (21) it follows for a sequence of values of $R$ tending to infinity that

\[
\frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \leq \frac{\rho_D(fog) + \varepsilon}{\rho_D(f) - \varepsilon}.
\]

As $\varepsilon(>0)$ is arbitrary, we obtain that

\[
\liminf_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \leq \frac{\rho_D(fog)}{\rho_D(f)}.
\]

(27)
Now combining (25), (26) and (27) we get that
\[
\frac{\lambda_D(fog)}{\rho_D(f)} \leq \liminf_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \leq \min \left\{ \frac{\lambda_D(fog)}{\lambda_D(f)}, \frac{\rho_D(fog)}{\rho_D(f)} \right\}. \tag{28}
\]

Now from (19) and (22) we obtain for a sequence of values of $R$ tending to infinity,
\[
\frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \geq \lambda_D(fog) - \varepsilon.
\]
Choosing $\varepsilon(> 0)$ we get that
\[
\limsup_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \geq \frac{\lambda_D(fog)}{\lambda_D(f)}. \tag{29}
\]

Again from (18) and (21) it follows for all sufficiently large values of $R$,
\[
\frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(f)} + \varepsilon.
\]
As $\varepsilon(> 0)$ is arbitrary, we obtain that
\[
\limsup_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(f)}. \tag{30}
\]

Similarly combining (17) and (24) we get for a sequence of values of $R$ tending to infinity
\[
\frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \geq \frac{\rho_D(fog) - \varepsilon}{\rho_D(f) + \varepsilon}.
\]
Since $\varepsilon(> 0)$ is arbitrary, it follows that
\[
\limsup_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \geq \frac{\rho_D(fog)}{\rho_D(f)}. \tag{31}
\]

Therefore combining (29), (30) and (31) we get that
\[
\max \left\{ \frac{\lambda_D(fog)}{\lambda_D(f)}, \frac{\rho_D(fog)}{\rho_D(f)} \right\} \leq \limsup_{R \to \infty} \frac{\log^2 M_{fog,D}(R)}{\log^2 M_{f,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(f)}. \tag{32}
\]
Thus the theorem follows from (28) and (32). Extending the notion we may prove the subsequent theorems using hyper Gol’dberg order. ■
Theorem 3 Let $f$ and $g$ be two entire functions of $n$ variables and $D$ be a bounded complete $n$-circular domain with centre at origin in $C^n$. Also let

$$0 < \lambda_D(fog) \leq \rho_D(fog) < \infty \quad \text{and} \quad 0 < \lambda_D(g) \leq \rho_D(g) < \infty.$$ 

Then

$$\frac{\lambda_D(fog)}{\lambda_D(g)} \leq \liminf_{R \to \infty} \frac{\log[3] M_{fog,D}(R)}{\log[3] M_{f,D}(R)} \leq \min \{ \frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\rho_D(fog)}{\rho_D(g)} \}$$

$$\leq \max \{ \frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\rho_D(fog)}{\rho_D(g)} \} \leq \limsup_{R \to \infty} \frac{\log[3] M_{fog,D}(R)}{\log[3] M_{g,D}(R)} \leq \frac{\rho_D(fog)}{\lambda_D(g)}.$$

Proof. From the definition of hyper Gol’dberg order and hyper Gol’dberg lower order of an entire function $g$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $R$,

$$\log[3] M_{g,D}(R) \leq (\rho_D(g) + \varepsilon) \log R \quad (33)$$

and $$\log[3] M_{g,D}(R) \geq (\lambda_D(g) - \varepsilon) \log R. \quad (34)$$

Also for the sequence of values of $R$ tending to infinity,

$$\log[3] M_{g,D}(R) \leq (\lambda_D(g) + \varepsilon) \log R \quad (35)$$

and $$\log[3] M_{g,D}(R) \geq (\rho_D(g) - \varepsilon) \log R. \quad (36)$$

Again from the definition of hyper Gol’dberg order and hyper lower Gol’dberg order of the composite entire function $fog$ we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $R$,

$$\log[3] M_{fog,D}(R) \leq (\rho_D(fog) + \varepsilon) \log R \quad (37)$$

and $$\log[3] M_{fog,D}(R) \geq (\lambda_D(fog) - \varepsilon) \log R. \quad (38)$$

Again for a sequence of values of $R$ tending to infinity,

$$\log[3] M_{fog,D}(R) \leq (\lambda_D(fog) + \varepsilon) \log R \quad (39)$$

and $$\log[3] M_{fog,D}(R) \geq (\rho_D(fog) - \varepsilon) \log R. \quad (40)$$
Now from (33) and (38) it follows for all sufficiently large values of $R$
\[\frac{\log[^3] M_{fog,D}(R)}{\log[^3] M_{g,D}(R)} \geq \frac{\lambda_D(fog) - \varepsilon}{\bar{\rho}_D(g) + \varepsilon}.\]

As $\varepsilon(> 0)$ is arbitrary we obtain that
\[\liminf_{R \to \infty} \frac{\log[^3] M_{fog,D}(R)}{\log[^3] M_{g,D}(R)} \leq \frac{\lambda_D(fog)}{\bar{\lambda}_D(g)}. \quad (41)\]

Again combining (34) and (39) we get for a sequence of values of $R$ tending to infinity,
\[\frac{\log[^3] M_{fog,D}(R)}{\log[^3] M_{g,D}(R)} \leq \frac{\lambda_D(fog) + \varepsilon}{\lambda_D(g) - \varepsilon}.\]

Since $\varepsilon(> 0)$ is arbitrary, it follows that
\[\liminf_{R \to \infty} \frac{\log[^3] M_{fog,D}(R)}{\log[^3] M_{g,D}(R)} \leq \frac{\lambda_D(fog)}{\lambda_D(g)}. \quad (42)\]

Similarly from (36) and (37) it follows for a sequence of values of $R$ tending to infinity that
\[\frac{\log[^3] M_{fog,D}(R)}{\log[^3] M_{g,D}(R)} \leq \frac{\bar{\rho}_D(fog) + \varepsilon}{\bar{\rho}_D(g) - \varepsilon}.\]

As $\varepsilon(> 0)$ is arbitrary, we obtain that
\[\liminf_{R \to \infty} \frac{\log[^3] M_{fog,D}(R)}{\log[^3] M_{g,D}(R)} \leq \frac{\bar{\rho}_D(fog)}{\bar{\rho}_D(g)}. \quad (43)\]

Now combining (41), (42) and (43) we get that
\[\frac{\lambda_D(fog)}{\bar{\rho}_D(g)} \leq \liminf_{R \to \infty} \frac{\log[^3] M_{fog,D}(R)}{\log[^3] M_{g,D}(R)} \leq \min\left\{\frac{\lambda_D(fog)}{\lambda_D(g)}, \frac{\bar{\rho}_D(fog)}{\bar{\rho}_D(g)}\right\}. \quad (44)\]

Now from (35) and (38) we obtain for a sequence of values of $R$ tending to infinity that
\[\frac{\log[^3] M_{fog,D}(R)}{\log[^3] M_{g,D}(R)} \geq \frac{\lambda_D(fog) - \varepsilon}{\lambda_D(g) + \varepsilon}.\]
Choosing $\varepsilon \to 0$ we get that
\[
\limsup_{R \to \infty} \frac{\log[^{[3]}] M_{fog, D}(R)}{\log[^{[3]}] M_{g, D}(R)} \geq \frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(g)}.
\] (45)

Again from (34) and (37) it follows for all sufficiently large values of $R$,
\[
\frac{\log[^{[3]}] M_{fog, D}(R)}{\log[^{[3]}] M_{g, D}(R)} \leq \frac{\overline{\rho}_D(fog) + \varepsilon}{\overline{\lambda}_D(g) - \varepsilon}.
\]

As $\varepsilon(>0)$ is arbitrary, we obtain that
\[
\limsup_{R \to \infty} \frac{\log[^{[3]}] M_{fog, D}(R)}{\log[^{[3]}] M_{g, D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\lambda}_D(g)}.
\] (46)

Similarly combining (33) and (40) we get for a sequence of values of $R$ tending to infinity that
\[
\frac{\log[^{[3]}] M_{fog, D}(R)}{\log[^{[3]}] M_{g, D}(R)} \geq \frac{\overline{\rho}_D(fog) - \varepsilon}{\overline{\rho}_D(g) + \varepsilon}.
\]

Since $\varepsilon(>0)$ is arbitrary, it follows that
\[
\limsup_{R \to \infty} \frac{\log[^{[3]}] M_{fog, D}(R)}{\log[^{[3]}] M_{g, D}(R)} \geq \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(g)}.
\] (47)

Therefore combining (45), (46) and (47) we get that
\[
\max \left\{ \frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(g)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(g)} \right\} \leq \limsup_{R \to \infty} \frac{\log[^{[3]}] M_{fog, D}(R)}{\log[^{[3]}] M_{g, D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\lambda}_D(g)}.
\] (48)

Thus the theorem follows from (44) and (48). ■

**Remark 2** If we take $0 < \overline{\lambda}_D(f) \leq \overline{\rho}_D(f) < \infty$ instead of $0 < \overline{\lambda}_D(g) \leq \overline{\rho}_D(g) < \infty$ and the other conditions remain the same then also Theorem 3 holds with $g$ replaced by $f$ in the denominator as we see in the next theorem.

**Theorem 4** Let $f$ and $g$ be two entire functions of $n$ variables and $D$ be a bounded complete $n$-circular domain with centre at origin an $C^n$. Also let $0 < \overline{\lambda}_D(fog) \leq \overline{\rho}_D(fog) < \infty$ and $0 < \overline{\lambda}_D(f) \leq \overline{\rho}_D(f) < \infty$. Then
\[
\frac{\overline{\lambda}_D(fog)}{\overline{\rho}_D(f)} \leq \liminf_{R \to \infty} \frac{\log[^{[3]}] M_{fog, D}(R)}{\log[^{[3]}] M_{g, D}(R)} \leq \min \left\{ \frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(f)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(f)} \right\}
\]
\[
\leq \max \left\{ \frac{\overline{\lambda}_D(fog)}{\overline{\lambda}_D(f)}, \frac{\overline{\rho}_D(fog)}{\overline{\rho}_D(f)} \right\} \leq \limsup_{R \to \infty} \frac{\log[^{[3]}] M_{fog, D}(R)}{\log[^{[3]}] M_{g, D}(R)} \leq \frac{\overline{\rho}_D(fog)}{\overline{\lambda}_D(f)}.
\]
Proof. From the definition of hyper Gol’dberg order and hyper lower Gol’dberg order of an entire function \( f \) we have for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( R \),

\[
\log^{[3]} M_{f,D}(R) \leq (\overline{\rho}_D(f) + \varepsilon) \log R \tag{49}
\]

and \( \log^{[3]} M_{f,D}(R) \geq (\overline{\lambda}_D(f) - \varepsilon) \log R. \tag{50} \]

Also for a sequence of values of \( R \) tending to infinity,

\[
\log^{[3]} M_{f,D}(R) \leq (\overline{\lambda}_D(f) + \varepsilon) \log R \tag{51}
\]

and \( \log^{[3]} M_{f,D}(R) \geq (\overline{\rho}_D(f) - \varepsilon) \log R. \tag{52} \]

Again from the definition of hyper Gol’dberg order and hyper lower Gol’dberg order of the composite entire function \( fog \) we have for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( R \),

\[
\log^{[3]} M_{fog,D}(R) \leq (\overline{\rho}_D(fog) + \varepsilon) \log R \tag{53}
\]

and \( \log^{[3]} M_{fog,D}(R) \geq (\overline{\lambda}_D(fog) - \varepsilon) \log R. \tag{54} \]

Again for a sequence of values of \( R \) tending to infinity,

\[
\log^{[3]} M_{fog,D}(R) \leq (\overline{\lambda}_D(fog) + \varepsilon) \log R \tag{55}
\]

and \( \log^{[3]} M_{fog,D}(R) \geq (\overline{\rho}_D(fog) - \varepsilon) \log R. \tag{56} \]

Now from (49) and (54) it follows for all sufficiently large values of \( R \) that

\[
\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \geq \frac{\overline{\lambda}_D(fog) - \varepsilon}{\overline{\rho}_D(f) + \varepsilon}.
\]

As \( \varepsilon(>0) \) is arbitrary we obtain that

\[
\liminf_{R \to \infty} \frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \geq \frac{\overline{\lambda}_D(fog)}{\overline{\rho}_D(f)}. \tag{57}
\]

Again combining (50) and (55) we get for a sequence of values of \( R \) tending to infinity,

\[
\frac{\log^{[3]} M_{fog,D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{\lambda}_D(fog) + \varepsilon}{\overline{\lambda}_D(f) - \varepsilon}.
\]
Since \( \varepsilon > 0 \) is arbitrary, it follows that
\[
\liminf_{R \to \infty} \frac{\log^3 M_{f \circ g, D}(R)}{\log^3 M_{f, D}(R)} \leq \frac{\overline{\lambda}_D(f \circ g)}{\overline{\lambda}_D(f)}.
\] (58)

Similarly from (52) and (53) it follows for a sequence of values of \( R \) tending to infinity that
\[
\frac{\log^3 M_{f \circ g, D}(R)}{\log^3 M_{f, D}(R)} \leq \frac{\overline{\rho}_D(f \circ g) + \varepsilon}{\overline{\rho}_D(f) - \varepsilon}.
\]

As \( \varepsilon > 0 \) is arbitrary, we obtain that
\[
\liminf_{R \to \infty} \frac{\log^3 M_{f \circ g, D}(R)}{\log^3 M_{f, D}(R)} \leq \frac{\overline{\rho}_D(f \circ g)}{\overline{\rho}_D(f)}.
\] (59)

Now combining (57), (58) and (59) we get that
\[
\frac{\overline{\lambda}_D(f \circ g)}{\overline{\rho}_D(f)} \leq \liminf_{R \to \infty} \frac{\log^3 M_{f \circ g, D}(R)}{\log^3 M_{f, D}(R)} \leq \min\left\{ \frac{\overline{\lambda}_D(f \circ g)}{\overline{\lambda}_D(f)}, \frac{\overline{\rho}_D(f \circ g)}{\overline{\rho}_D(f)} \right\}
\] (60)

Now from (51) and (54) we obtain for a sequence of values of \( R \) tending to infinity,
\[
\frac{\log^3 M_{f \circ g, D}(R)}{\log^3 M_{f, D}(R)} \geq \frac{\overline{\lambda}_D(f \circ g) - \varepsilon}{\overline{\lambda}_D(f) + \varepsilon}.
\]

Choosing \( \varepsilon \to 0 \) we get that
\[
\limsup_{R \to \infty} \frac{\log^3 M_{f \circ g, D}(R)}{\log^3 M_{f, D}(R)} \geq \frac{\overline{\lambda}_D(f \circ g)}{\overline{\lambda}_D(f)}.
\] (61)

Again from (50) and (53) it follows for all sufficiently large values of \( R \),
\[
\frac{\log^3 M_{f \circ g, D}(R)}{\log^3 M_{f, D}(R)} \leq \frac{\overline{\rho}_D(f \circ g) + \varepsilon}{\overline{\lambda}_D(f) - \varepsilon}.
\]

As \( \varepsilon > 0 \) is arbitrary, we obtain that
\[
\limsup_{R \to \infty} \frac{\log^3 M_{f \circ g, D}(R)}{\log^3 M_{f, D}(R)} \leq \frac{\overline{\rho}_D(f \circ g)}{\overline{\lambda}_D(f)}.
\] (62)

Similarly combining (49) and (56) we get for a sequence of values of \( R \) tending to infinity that
\[
\frac{\log^3 M_{f \circ g, D}(R)}{\log^3 M_{f, D}(R)} \geq \frac{\overline{\rho}_D(f \circ g) - \varepsilon}{\overline{\rho}_D(f) + \varepsilon}.
\]
Since \( \varepsilon(> 0) \) is arbitrary, it follows that

\[
\limsup_{R \to \infty} \frac{\log^{[3]} M_{f \circ D}(R)}{\log^{[3]} M_{f,D}(R)} \geq \frac{\overline{p}_D(f \circ g)}{\overline{p}_D(f)}.
\]  

(63)

Therefore combining (61), (62) and (63) we get that

\[
\max \{ \frac{\overline{\lambda}_D(f \circ g)}{\overline{\lambda}_D(f)}, \frac{\overline{p}_D(f \circ g)}{\overline{p}_D(f)} \} \leq \limsup_{R \to \infty} \frac{\log^{[3]} M_{f \circ D}(R)}{\log^{[3]} M_{f,D}(R)} \leq \frac{\overline{p}_D(f \circ g)}{\overline{\lambda}_D(f)}.
\]  

(64)

Thus the theorem follows from (60) and (64).

References


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